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SPECTRAL PROPERTIES OF ARITHMETIC FUNCTIONS

BY Teturo KAMAE

The purpose of this talk is to investigate the spectral properties of arithmetic functions, particularly of functions on digits to some integral base $q \geqslant 2$. There are preceding works on this subject by S. KAKUTANI [5], [6], M. MENDÈS FRANCE [8], [9], H. DABOUSSI and M. MENDÈS FRANCE [4], J. COQUET and M. MENDÈS FRANCE [2] and J. BÉSINEAU [1]. Some parts of results here are proved in a joint work [3] of the lecturer and M. MENDÈS FRANCE and J. COQUET.

1. General theory.

For an arithmetic function $\alpha \in \underset{\sim}{C}^{N}$, the Besicovitch norm is defined as follows

$$\|\alpha\| = (\overline{\lim}_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |\alpha(j)|^2)^{\frac{1}{2}} .$$

By $\mathfrak{B}$ and $\mathfrak{N}$, we denote the class of $\alpha \in \underset{\sim}{C}^{N}$ such that $\|\alpha\| < \infty$ and $\|\alpha\| = 0$, respectively. It is known that the normed linear space $(\mathfrak{B}/\mathfrak{N} , \| \ \|)$, which is denoted by $\tilde{\mathfrak{B}}$ and called the Besicovitch space, is complete and hence a Banach space.

T denotes a shift on $\underset{\sim}{C}^{N}$. Note that $\underset{\sim}{T}$ on $\tilde{\mathfrak{B}}$ is a continuous linear operator which is bijective.

By $\mathfrak{D}$, we denote the class of $\alpha \in \mathfrak{B}$ for which the correlations

$$\gamma_\alpha(m) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \alpha(j+m)\, \overline{\alpha(j)}$$

exist for all $m \in \underset{\sim}{Z}$. In this case, there exists a unique measure Λ_α on $\underset{\sim}{T} = \underset{\sim}{R}/\underset{\sim}{Z}$, which we call the spectral measure of α, such that

$$\gamma_\alpha(m) = \int_{\underset{\sim}{T}} e(m\lambda)\, d\Lambda_\alpha(\lambda)$$

for any $m \in \underset{\sim}{Z}$, where $e(x) = \exp(2\pi i x)$. We can construct Λ_α directly without using correlations. Let Λ_α^n be the measure on $\underset{\sim}{T}$ such that

$$d\Lambda_\alpha^n(\lambda) = \frac{1}{n} |\sum_{j=0}^{n-1} \alpha(j)\, e(-j\lambda)|^2\, d\lambda .$$

An easy computation shows that

$$\lim_{n\to\infty} \int_{\underset{\sim}{T}} e(m\lambda)\, d\Lambda_\alpha^n(\lambda) = \gamma_\alpha(m)$$

for any $m \in \underset{\sim}{Z}$. Hence, Λ_α^n converges weakly to Λ_α as $n \to \infty$. For two measures (by measures, we mean positive finite Borel measures unless mentioned otherwise) P and Q on $\underset{\sim}{T}$, the affinity $\rho(P , Q)$ is defined by

$$\rho(P , Q) = \int_{\underset{\sim}{T}} \sqrt{\frac{dP}{dR}} \sqrt{\frac{dQ}{dR}}\, dR ,$$

where R is a measure with respect to which P and Q are absolutely continuous.

It is clear that this definition does not depend on the selection of R . It is known that if $P_n \to P$ and $Q_n \to Q$ weakly, then we have

$$\overline{\lim}_{n\to\infty} \rho(P_n , Q_n) \leq \rho(P , Q) .$$

Thus, if $\alpha , \beta \in \mathfrak{D}$, then

$$\begin{aligned}\rho(\Lambda_\alpha , \Lambda_\beta) &\geq \overline{\lim}_{n\to\infty} \rho(\Lambda_\alpha^n , \Lambda_\beta^n)\\ &= \overline{\lim}_{n\to\infty} \frac{1}{n} \int_{\underset{\sim}{T}} |\textstyle\sum_{j=0}^{n-1} \alpha(j)\ e(-\ j\lambda)|\,|\sum_{k=0}^{n-1} \beta(j)\ e(-\ k\lambda)|\ d\lambda\\ &\geq \overline{\lim}_{n\to\infty} \frac{1}{n} |\int_{\underset{\sim}{T}} \textstyle\sum_{j=0}^{n-1} \alpha(j)\ e(-\ j\lambda) \sum_{k=0}^{n-1} \overline{\beta(k)}\ e(k\lambda)\ d\Lambda|\\ &= \overline{\lim}_{n\to\infty} \frac{1}{n} |\sum_{j=0}^{n-1} \alpha(j)\ \overline{\beta(j)}| .\end{aligned}$$

THEOREM 1 [3]. - <u>For</u> $\alpha , \beta \in \mathfrak{D}$, <u>we have</u>

$$\overline{\lim}_{n\to\infty} \frac{1}{n} |\textstyle\sum_{j=0}^{n-1} \alpha(j + m)\ \overline{\beta(j)}| \leq \rho(\Lambda_\alpha , \Lambda_\beta) \quad \underline{\text{for any}} \quad m \in \underset{\sim}{Z} .$$

In this theorem, let $\beta(n) = e(\lambda n)$ $(\forall\ n \in \underset{\sim}{N})$. Then, since $\Lambda_\beta = \delta_\lambda$, we have the following corollary.

COROLLARY [3]. - <u>For</u> $\alpha \in \mathfrak{D}$ <u>and</u> $\lambda \in \underset{\sim}{T}$, <u>we have</u>

$$\overline{\lim}_{n\to\infty} \frac{1}{n} |\textstyle\sum_{j=0}^{n-1} \alpha(j)\ e(-\ j\lambda)| \leq \Lambda_\alpha(\{\lambda\})^{\frac{1}{2}} .$$

Note that $\rho(P , Q) = 0$ is equivalent to that P and Q are singular to each other. For $\alpha \in \mathfrak{D}$, let $H(\alpha)$ be the closed subspace of $\tilde{\mathfrak{B}}$ generated by $\{T^m \alpha ;\ m \in \underset{\sim}{Z}\}$. It is easy to see that $H(\alpha)$ is a separable Hilbert space with the inner product

$$(\eta , \zeta) = \lim_{n\to\infty} \frac{1}{n} \textstyle\sum_{j=0}^{n-1} \eta(j)\ \overline{\zeta(j)} ,$$

which always exists as long as $\eta , \zeta \in H(\alpha)$.

Suppose that Λ_α and Λ_β are singular to each other. Then by the above theorem, we have $(T^m \alpha , T^\ell \beta) = 0$ $(\forall\ m , \ell \in \underset{\sim}{Z})$. Hence, $H(\alpha) \perp H(\beta)$.

Moreover, there is a complete characterization of the property that Λ_α and Λ_β are singular to each other.

THEOREM 2 (A. N. KOLMOGOROV [7]). - <u>Let</u> $\alpha , \beta \in \mathfrak{D}$. <u>Then,</u> Λ_α <u>and</u> Λ_β <u>are singular to each other if, and only if,</u> $H(\alpha) \perp H(\beta)$ <u>and</u> $\alpha \in H(\alpha + \beta)$.

For a signed measure P , we denote its total variance by $\|P\|$. Note that $\|\Lambda_\alpha\| = \|\alpha\|^2$ for any $\alpha \in \mathfrak{D}$. Let $\alpha , \beta \in \mathfrak{D}$. Then,

$$\|\Lambda_\alpha - \Lambda_\beta\| \leq \underline{\lim}_{n\to\infty} \|\Lambda_\alpha^n - \Lambda_\beta^n\| = \underline{\lim}_{n\to\infty} \frac{1}{n} \int_{\underset{\sim}{T}} \Big|\,|\sum_{j=0}^{n-1} \alpha(j)\ e(-\ j\lambda)|^2 - |\sum_{j=0}^{n-1} \beta(j)\ e(-\ j\lambda)|^2\Big|\ d\lambda$$

$$\leq \underline{\lim}_{n\to\infty} \frac{1}{n} \int_{\underset{\sim}{T}} |\sum_{j=0}^{n-1} (\alpha(j) - \beta(j))\ e(-\ j\lambda)|\,(|\sum_{j=0}^{n-1} \alpha(j)\ e(-\ j\lambda)| + |\sum_{j=0}^{n-1} \beta(j)\ e(-\ j\lambda)|)\ d\lambda$$

$$\leqslant \underline{\lim}_{n\to\infty}\Big(\frac{1}{n}\int_{\underset{\sim}{T}}\Big|\sum_{j=0}^{n-1}(\alpha(j)-\beta(j))e(-j\lambda)\Big|^2 d\lambda\Big)^{\frac{1}{2}} \times \Big(\frac{1}{n}\int_{\underset{\sim}{T}}\Big(\Big|\sum_{j=0}^{n-1}\alpha(j)e(-j\lambda)\Big|+\Big|\sum_{j=0}^{n-1}\beta(j)e(-j\lambda)\Big|\Big)^2 d\lambda\Big)^{\frac{1}{2}}$$

$$\leqslant \underline{\lim}_{n\to\infty}\|\alpha-\beta\|\Big(\frac{2}{n}\int_{\underset{\sim}{T}}\Big(\Big|\sum_{j=0}^{n-1}\alpha(j)\ e(-j\lambda)\Big|^2+\Big|\sum_{j=0}^{n-1}\beta(j)\ e(-j\lambda)\Big|^2\Big)d\lambda\Big)^{\frac{1}{2}} = \|\alpha-\beta\|\sqrt{2\|\alpha\|^2+2\|\beta\|^2}\ .$$

THEOREM 3. - <u>Let</u> $\alpha_n \in \mathbb{Q}$ $(n = 1, 2, \ldots)$, $\alpha \in \mathbb{B}$ <u>and</u> $\lim_{n\to\infty}\|\alpha - \alpha_n\| = 0$. <u>Then, we have</u> $\alpha \in \mathbb{Q}$ <u>and</u> $\lim_{n\to\infty}\|\Lambda_\alpha - \Lambda_{\alpha_n}\| = 0$.

2. q-multiplicative functions.

Let $q \geqslant 2$ be an integer. Let $e_k^q(n)$ $(k = 0, 1, 2, \ldots)$ be the digits in q-adic representation of $n \in \underset{\sim}{N}$:

$n = \sum_{k=0}^{\infty} e_k^q(n)\ q^k \qquad (e_k^q(n) \in \{0, 1, \ldots, q-1\}\ ;\ k = 0, 1, 2, \ldots)$.

Let $c = (c_0, c_1, c_2, \ldots)$ be a sequence of real numbers. Let

$$\zeta_c(n) = e\Big(\sum_{k=0}^{\infty} c_k\ e_k^q(n)\Big) \qquad (\forall\ n \in \underset{\sim}{N})\ .$$

Then, it is known [2] that $\zeta_c \in \mathbb{Q}$. Using the relation that $\zeta_c(nq) = \zeta_{\tau c}(n)$, where $\tau c = (c_1, c_2, \ldots)$, it holds for any continuous function f on $\underset{\sim}{T}$ that

$$\int_{\underset{\sim}{T}} f\ d\Lambda_{\zeta_c} = \lim_{n\to\infty}\int_{\underset{\sim}{T}} f\ d\Lambda^{q^n}_{\zeta_c}$$

$$= \lim_{n\to\infty}\int_{\underset{\sim}{T}} f(\lambda)\ \frac{1}{q^n}\ \Big|\sum_{j=0}^{q^n-1}\zeta_c(j)\ e(-\ j\lambda)\Big|^2\ d\lambda$$

$$= \lim_{n\to\infty}\int_{\underset{\sim}{T}} f(\lambda)\ \frac{1}{q}\ \Big|\sum_{j=0}^{q-1} e(j(c_0-\lambda))\Big|^2\ \frac{1}{q^{n-1}}\ \Big|\sum_{j=0}^{q^{n-1}-1}\zeta_{\tau c}(j)\ e(-jq\lambda)\Big|^2\ d\lambda$$

$$= \lim_{n\to\infty}\int_{\underset{\sim}{T}} f(\lambda)\ \frac{1}{q}\ \Big|\sum_{j=0}^{q-1} e(j(c_0-\lambda))\Big|^2\ \frac{1}{q}\ d\Lambda^{q^{n-1}}_{\zeta_{\tau c}}(q\lambda)$$

$$= \lim_{n\to\infty}\int_{\underset{\sim}{T}} f(\lambda)\ \frac{1}{q^2}\ \Big|\sum_{j=0}^{q-1} e(j(c_0-\lambda))\Big|^2\ d\Lambda_{\zeta_{\tau c}}(q\lambda)\ .$$

Hence, we have

$$\frac{d\Lambda_{\zeta_c}(\lambda)}{d\Lambda_{\zeta_{\tau c}}(q\lambda)} = \frac{1}{q^2}\ \Big|\sum_{j=0}^{q-1} e(j(c_0-\lambda))\Big|^2\ .$$

Thus,

$$\Lambda_{\zeta_c}(\{\lambda\}) = \prod_{k=0}^{n-1}\frac{1}{q^2}\ \Big|\sum_{j=0}^{q-1} e(j(c_k-\lambda q^k))\Big|^2\ \Lambda_{\zeta_{\tau^n c}}(\{q^n\lambda\})$$

for any $n = 1, 2, \ldots$ Since

$$\Lambda_{\zeta_{\tau^n c}}(\{\lambda\}) \leqslant \Lambda_{\zeta_{\tau^n c}}(\underset{\sim}{T}) = \|\zeta_{\tau^n c}\|^2 = 1\ ,$$

we have

$$\Lambda_{\zeta_c}(\{\lambda\})^{\frac{1}{2}} \leqslant \underline{\lim}_{n\to\infty}\prod_{k=0}^{n-1}\frac{1}{q}\ \Big|\sum_{j=0}^{q-1} e(j(c_k-\lambda q^k))\Big| = \underline{\lim}_{n\to\infty}\frac{1}{q^n}\ \Big|\sum_{j=0}^{q^n-1}\zeta_c(j)\ e(-\ j\lambda)\Big|\ .$$

Considering the corollary, we have the following theorem :

THEOREM 4 [3].

$$\Lambda_{\zeta_c}(\{\lambda\})^{\frac{1}{2}} = \prod_{k=0}^{\infty} \frac{1}{q} \left|\sum_{j=0}^{q-1} e(j(c_k - \lambda q^k))\right| = \prod_{k=0}^{\infty} \left|\frac{\sin \pi q(c_k - \lambda q^k)}{q \sin \pi(c_k - \lambda q^k)}\right| .$$

Suppose that $\sum_{k=0}^{\infty} \|c_k - \lambda q^k\|^2 = \infty$ for any $\lambda \in \underset{\sim}{T}$, where for $x \in \underset{\sim}{R}$, $\|x\| = \min_{n\in \underset{\sim}{Z}} |x - n|$. Then it follows that

$$\Lambda_{\zeta_c}(\{\lambda\})^{\frac{1}{2}} = \prod_{k=0}^{\infty} \left|\frac{\sin \pi q(c_k - \lambda q^k)}{q \sin \pi(c_k - \lambda q^k)}\right| = 0 .$$

for any $\lambda \in \underset{\sim}{T}$. Thus, Λ_{ζ_c} is continuous.

Moreover, it can be proved [3] that Λ_{ζ_c} is singular with respect to the Lebesgue measure.

Now suppose that $\sum_{k=0}^{\infty} \|c_k - \lambda_0 q^k\|^2 < \infty$ for some $\lambda_0 \in \underset{\sim}{T}$. It holds that

$$\sum_{\lambda\in \underset{\sim}{T}} \Lambda_{\zeta_c}(\{\lambda\}) = \sum_{\lambda\in[0,1/q)} \sum_{j=0}^{q-1} \Lambda_{\zeta_c}(\{\lambda + \tfrac{j}{q}\})$$

$$= \sum_{\lambda\in[0,1/q)} \left(\sum_{j=0}^{q-1} \left|\frac{\sin \pi q(c_0 - \lambda - \frac{j}{q})}{q \sin \pi(c_0 - \lambda - \frac{j}{q})}\right|\right) \prod_{k=1}^{\infty} \left|\frac{\sin \pi q(c_k - \lambda q^k)}{q \sin \pi(c_k - \lambda q^k)}\right|^2$$

$$= \sum_{\lambda\in[0,1/q)} \prod_{k=1}^{\infty} \left|\frac{\sin \pi q(c_k - \lambda q^k)}{q \sin \pi(c_k - \lambda q^k)}\right|^2$$

$$= \sum_{\lambda\in[0,1/q)} \Lambda_{\zeta_{\tau c}}(\{\lambda q\}) = \sum_{\lambda\in \underset{\sim}{T}} \Lambda_{\zeta_{\tau c}}(\{\lambda\}) .$$

Since

$$1 = \lim_{n\to\infty} \prod_{k=n}^{\infty} \left|\frac{\sin \pi q(c_k - \lambda_0\, q^k)}{q \sin \pi(c_k - \lambda_0\, q^k)}\right|^2$$

$$= \lim_{n\to\infty} \Lambda_{\zeta_{\tau^n c}}(\{\lambda_0\, q^n\})$$

$$\leqslant \underline{\lim}_{n\to\infty} \sum_{\lambda\in \underset{\sim}{T}} \Lambda_{\zeta_{\tau^n c}}(\{\lambda\})$$

$$\leqslant \overline{\lim}_{n\to\infty} \sum_{\lambda\in \underset{\sim}{T}} \Lambda_{\zeta_{\tau^n c}}(\{\lambda\}) \leqslant 1 ,$$

we have

$$\sum_{\lambda\in \underset{\sim}{T}} \Lambda_{\zeta_c}(\{\lambda\}) = \lim_{n\to\infty} \sum_{\lambda\in \underset{\sim}{T}} \Lambda_{\zeta_{\tau^n c}}(\{\lambda\}) = 1 .$$

Since $\Lambda_{\zeta_c}(\underset{\sim}{T}) = 1$, this implies that Λ_{ζ_c} is discrete.

THEOREM 5 [3]. - Λ_{ζ_c} _is either discrete or continuous and singular corresponding as_ $\sum_{k=0}^{\infty} \|c_k - \lambda q^k\|^2 < \infty$ _for some_ $\lambda \in \underset{\sim}{T}$ _or not._

3. Mutual singularity.

Let $S_q(n) = \sum_{k=0}^{\infty} e_k^q(n)$ be the sum of digits. Denote $f_\lambda(n) = e(\lambda n)$ for $\lambda \in \mathbb{T}$ and $n \in \mathbb{N}$. It is known [1] that $H(f_\lambda \circ S_p) \perp H(f_\eta \circ S_q)$ if $(p , q) = 1$, $(p - 1)\lambda \notin \mathbb{Z}$ and $(q - 1)\eta \notin \mathbb{Z}$. We can prove further the following theorem.

THEOREM 6. - *If* $(p , q) = 1$, $(p - 1)\lambda \notin \mathbb{Z}$ *and* $(q - 1)\eta \notin \mathbb{Z}$, *then* $\Lambda_{f_\lambda \circ S_p}$ *and* $\Lambda_{f_\lambda \circ S_q}$ *are singular to each other.*

Sketch of the proof of theorem 6. - Let

$$\Gamma_\gamma(n) = \prod_{k=0}^{\infty} \cos \pi n \gamma^{-k} .$$

Then, it is known (H. G. SENGE and E. G. STRAUS [10]) that $\Gamma_{p^2}(n)\, \Gamma_q(n) \to 0$ $(n \to \infty)$. We can assume that q is odd. By $\tau_q(n)$, we denote the greatest number j such that there exist integers $0 \leqslant k_1 < k_2 < \dots < k_{2j}$ satisfying $e^q_{k_{2i-1}}(n) > 0$ and $e^q_{k_{2i}}(n) < q - 1$ for $i = 1 , 2 , \dots , j$. Since

$$\inf_{m > \ell} |\Gamma_{p^2}(p^{2m} - p^{2\ell})| > 0 ,$$

we have

$$\lim_{m\to\infty, \ell\to\infty, m>\ell} \Gamma_q(p^{2m} - p^{2\ell}) = 0 .$$

It follows from this that

$$\lim_{m\to\infty, \ell\to\infty, m>\ell} \tau_q(p^{2m} - p^{2\ell}) = \infty .$$

We can prove, using this fact, that

$$\lim_{m\to\infty, \ell\to\infty, m>\ell} \Gamma_{f_\eta \circ S_q}(p^{2m} - p^{2\ell}) = 0 .$$

Then it follows that

$$\lim_{N\to\infty} \left\|\frac{1}{N} \sum_{n=1}^{N} T^{p^{2n}}(f_\eta \circ S_q)\right\| = 0 .$$

On the other hand, we can prove that

$$\lim_{N\to\infty} \left\|\frac{1}{N} \sum_{n=1}^{N} T^{p^{2n}}(f_\lambda \circ S_p) - K\, f_\lambda \circ S_p\right\| = 0 ,$$

where

$$K = \frac{(p - 1)\, e(p\lambda)}{pe((p - 1)\lambda) - 1} \neq 0 .$$

Thus we have

$$\lim_{N\to\infty} \left\|\frac{1}{NK} \sum_{n=1}^{N} T^{p^{2n}}(f_\lambda \circ S_p + f_\eta \circ S_q) - f_\lambda \circ S_p\right\| = 0 ,$$

hence $f_\lambda \circ S_p \in H(f_\lambda \circ S_p + f_\eta \circ S_q)$. By theorem 2 and the fact that $H(f_\lambda \circ S_p) \perp H(f_\eta \circ S_q)$, we conclude that $\Lambda_{f_\lambda \circ S_p}$ and $\Lambda_{f_\eta \circ S_q}$ are singular to each other.

Problem. - Let

$$\zeta_c(n) = e\Big(\sum_{k=0}^{\infty} c_k\, e_k^q(n)\Big)$$

and

$$\zeta_d(n) = e(\sum_{k=0}^{\infty} d_k\ e_k^q(n)) .$$

We can prove that $H(\zeta_c) \perp H(\zeta_d)$ if, and only if, $\sum_{k=0}^{\infty} \|c_k - d_k\|^2 < \infty$. But we do not know whether this condition is sufficient for the mutual singularity of Λ_{ζ_c} and Λ_{ζ_d} or not.

4. Almost periodic functions.

Recall that $\alpha \in \underset{\sim}{C}^{\mathbb{N}}$ is called an almost periodic function in the sense of Besicovitch if it belongs to the closed subspace $\mathbb{P}$ of $\tilde{\mathbb{B}}$ generated by $\{f_\lambda ; \lambda \in \underset{\sim}{T}\}$. By theorem 3, if $\alpha \in \mathbb{P}$, then $\alpha \in \mathbb{Q}$ and Λ_α is discrete.

Let

$$\zeta_c(n) = e(\sum_{k=0}^{\infty} c_k\ e_k^q(n)) .$$

THEOREM 7. - ζ_c is almost periodic in the sense of Besicovitch if, and only if, there exists $\lambda \in \underset{\sim}{T}$ such that

$$\sum_{k=0}^{\infty} \|c_k - \lambda q^k\|^2 < \infty$$

and that

$$\sum_{k=0}^{N} (c_k - \lambda q^k)$$

converge modulo 1 when $N \longrightarrow \infty$.

Proof. - Suppose that ζ_c is almost periodic. Then Λ_{ζ_c} is discrete and by theorem 5, there exists $\lambda \in \underset{\sim}{T}$ such that

$$\sum_{k=0}^{\infty} \|c_k - \lambda q^k\|^2 < \infty .$$

Here, we may assume without loss of generality that $\|c_k - \lambda q^k\| = |c_k - \lambda q^k|$ for $k = 0 , 1 , 2 , \dots$

Take $\ell \in \underset{\sim}{N}$ such that $|c_k - \lambda q^k| < 1/q$ for any $k = \ell , \ell + 1 , \dots$ Since $\zeta_{\tau^\ell c}(n) = \zeta_c(nq^\ell)$ $(\forall\ n \in \underset{\sim}{N})$, $\zeta_{\tau^\ell c}$ is also almost periodic.

Hence, the inner product $(\zeta_{\tau^\ell c} , f\lambda)$ exists. Therefore,

$$(\zeta_{\tau^\ell c} , f\lambda) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \zeta_{\tau^\ell c}(j)\ e(- j\lambda)$$

$$= \prod_{k=\ell}^{\infty} \frac{1}{q} \sum_{j=0}^{q-1} e(j(c_k - \lambda q^k))$$

$$= \prod_{k=\ell}^{\infty} e(\frac{q-1}{2}(c_k - \lambda q^k))\ \frac{\sin \pi q(c_k - \lambda q^k)}{q \sin \pi(c_k - \lambda q^k)}$$

exists. Since

$$\prod_{k=\ell}^{\infty} \frac{\sin \pi q(c_k - \lambda q^k)}{q \sin \pi(c_k - \lambda q^k)}$$

exists and is not 0,

$$\prod_{k=\ell}^{\infty} e\left(\frac{q-1}{2}(c_k - \lambda q^k)\right) = e\left(\sum_{k=\ell}^{\infty} \frac{q-1}{2}(c_k - \lambda q^k)\right)$$

should exist. That is to say that

$$\sum_{k=0}^{N} \frac{q-1}{2}(c_k - \lambda q^k) \quad \text{converge modulo 1 when } N \to \infty .$$

Since $c_k - \lambda q^k \to 0$ $(k \to \infty)$, this implies that $\sum_{k=0}^{N} (c_k - \lambda q^k)$ converges modulo 1 when $N \to \infty$. Conversely, suppose that there exists $\lambda \in \underline{T}$ such that

$$\sum_{k=0}^{\infty} \|c_k - \lambda q^k\| < \infty$$

and

$$\sum_{k=0}^{N} (c_k - \lambda q^k) \quad \text{converges modulo 1 when } N \to \infty .$$

Here, we assume without loss of generality that

$$\|c_k - \lambda q^k\| = |c_k - \lambda q^k| \qquad (k = 0 , 1 , 2 , \ldots) .$$

Let

$$\alpha_N(n) = e\left(\lambda n + \sum_{k=0}^{N-1} (c_k - \lambda q^k)\, e_k^q(n)\right) \qquad (\forall\, n \in \underline{N}) .$$

Since α_N is almost periodic, it is sufficient to prove that $\|\alpha - \alpha_N\| \to 0$ $(N \to \infty)$. We have

$$\|\alpha - \alpha_N\|^2 \leqslant q\, \overline{\lim}_{M\to\infty} \frac{1}{q^M} \sum_{j=0}^{q^M-1} \left|1 - e\left(\sum_{k=N}^{M-1} (c_k - \lambda q^k)\, e_k^q(j)\right)\right|^2$$

$$\leqslant 2\pi q\, \overline{\lim}_{M\to\infty} \frac{1}{q^M} \sum_{j=0}^{q^M-1} \left\|\sum_{k=N}^{M-1} (c_k - \lambda q^k)\, e_k^q(j)\right\|^2$$

$$\leqslant 4\pi q\, \overline{\lim}_{M\to\infty} \frac{1}{q^M} \sum_{j=0}^{q^M-1} \left(\left|\sum_{k=N}^{M-1} (c_k - \lambda q^k)\left(e_k^q(j) - \frac{q-1}{2}\right)\right|^2 + \left\|\frac{q-1}{2} \sum_{k=N}^{M-1} (c_k - \lambda q^k)\right\|^2\right)$$

$$\leqslant 4\pi \sum_{j=0}^{q-1} \left(j - \frac{q-1}{2}\right)^2 \sum_{k=N}^{\infty} |c_k - \lambda q^k|^2 + 4\pi q\, \overline{\lim}_{M\to\infty} \left\|\sum_{k=N}^{M-1} \frac{q-1}{2} (c_k - \lambda q^k)\right\|^2 .$$

Since $\sum_{k=0}^{n} (c_k - \lambda q^k)$ converges modulo 1 when $n \to \infty$, and $c_k - \lambda q^k \to 0$ $(k \to \infty)$,

$$\sum_{k=0}^{n} \frac{q-1}{2} (c_k - \lambda q^k) \quad \text{converges modulo 1 when } n \to \infty .$$

Hence, $\|\alpha - \alpha_N\| \to 0$ when $N \to \infty$.

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