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by Hiroshi SAITO

0. - Let F be a totally real algebraic number field with the degree $[F:\mathbb{Q}] = \ell$, and $\mathcal{O}$ its maximal order. For the sake of simplicity, we assume that the class number of F is one, and $\mathcal{O}$ has a unit with arbitary signature distribution. For an even positive integer k and for the subgroup $\Gamma = GL_2(\mathcal{O})^+$ of $GL_2(\mathcal{O})$ consisting of all elements with totally positive determinants, we denote by $S_k(\Gamma)$ the space of Hilbert cusp forms of weight k with respect to Γ, namely the set of all holomorphic functions f on the ℓ-fold product of the complex upper half plane H, which satisfy

1° $f(\gamma^{(1)} Z_1, \gamma^{(2)} Z_2, \ldots, \gamma^{(\ell)} Z_\ell) = \prod_i (c^{(i)} Z_i + d^{(i)})^k f(Z_1, \ldots, Z_\ell)$ for $\gamma \in \Gamma$,

2° f vanishes at every cusp,

where $\gamma^{(i)} = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix}$ are all distinct conjugates of γ over $\mathbb{Q}$. It is known that an element f of $S_k(\Gamma)$ has a Fourier expansion of the form

$$f(Z_1, \ldots, Z_\ell) = \sum_{\mathfrak{A}} C(\mathfrak{A}) \sum_{(\nu)=\mathfrak{A}/\mathfrak{D}, \nu \gg 0} \exp 2\pi i (\nu^{(1)} Z_1 + \ldots + \nu^{(\ell)} Z_\ell),$$

where $\mathfrak{A}$ runs through all integral ideals of F, and $\mathfrak{D}$ is the different of the extension $F/\mathbb{Q}$. We denote by Φ_f the associated Dirichlet series of f, that is,

$$\Phi_f(s) = \sum_{\mathfrak{A}} C(\mathfrak{A}) N\mathfrak{A}^{-s} .$$

For a place (archimedean or non-archimedean) of F, we denote by F_v the completion of F at v, and for a non-archimedean prime $v = \mathfrak{p}$, we denote by $\mathcal{O}_\mathfrak{p}$ the ring of all $\mathfrak{p}$-adic integers of $F_\mathfrak{p}$. Let F_A be the adele ring of F, and $\mathfrak{U}_F$ be the open subgroup of $GL_2(F_A)$ given by

$$\prod_{\mathfrak{p}:\text{non-archimedean}} GL_2(\mathcal{O}_\mathfrak{p}) \times \prod_{v:\text{archimedean}} GL_2(F_v) .$$

Then we can consider the Hecke ring $R(\mathfrak{U}_F, GL_2(F_A))$ with respect to $GL_2(F_A)$ and $\mathfrak{U}_F$, and its action T on $S_k(\Gamma)$ as in G. SHIMURA [9]. It is known that $S_k(\Gamma)$ has a basis consisting of common eigen functions for all Hecke operators and that if f is a common eigen function for all Hecke operators with $C(\mathcal{O}) = 1$, then the associated Dirichlet series Φ_f has an Euler product of the form

$$\Phi_f(s) = \prod_\mathfrak{p} (1 - C(\mathfrak{p})N\mathfrak{p}^{-s} + N\mathfrak{p}^{k-1-s})^{-1} ,$$

where $\mathfrak{p}$ runs through all prime ideals of F.

1. - On the following, we assume that F is a totally real algebraic number field which satisfies

1° F is a cyclic extension of $\mathbb{Q}$ with a prime degree ℓ ,

2° F is a tamely ramified extension of $\mathbb{Q}$,

3° The class number of F is one,

4° The index $[E:E_+]$ is 2^{ℓ} ,

where E is the group of all units of $\mathcal{O}$ and E_+ is its subgroup consisting of all totally positive units. It follows from these conditions that the conductor of the extension $F/\mathbb{Q}$ is a prime q with $q \equiv 1 \mod \ell$.

We fix an embedding of F into the real number field $\mathbb{R}$, and consider F as a subfield of $\mathbb{R}$. We fix a generator σ of the Galois group $\mathrm{Gal}(F/\mathbb{Q})$. With this σ , we consider $GL_2(F)$ as a subgroup of $GL_2(\mathbb{R})^{\ell}$ by

$$\gamma \longmapsto (\gamma\ ,\ {}^{\sigma}\gamma\ ,\ \dots\ ,\ {}^{\sigma^{\ell-1}}\gamma) \quad \text{for } \gamma \in GL_2(F) .$$

For this fixed generator σ , we define a linear operator T_σ on $S_k(\Gamma)$ by

$$T_\sigma\, f(Z_1\ ,\ Z_2\ ,\ \dots\ ,\ Z_\ell) = f(Z_2\ ,\ \dots\ ,\ Z_\ell\ ,\ Z_1) .$$

Using this T_σ and Hecke operators, we define a subspace $SS_k(\Gamma)$ of $S_k(\Gamma)$ as follows

$$SS_k(\Gamma) = \{f \in S_k(\Gamma)\ ;\ T_\sigma\, T(e)\, f = T(e)\, T_\sigma\, f \ \text{ for any } \ e \in R(\mathfrak{U}_F\ ,\ GL_2(F_A))\} .$$

It is easy to see that this subspace is stable under the action of Hecke operators, and that if f is a common eigen function for all Hecke operators, then

$$f \in SS_k(\Gamma) \Longleftrightarrow C(\mathfrak{A}) = C({}^{\sigma}\mathfrak{A}) = \dots = C({}^{\sigma^{\ell-1}}\mathfrak{A}) \quad \text{for any integral ideal } \mathfrak{A} .$$

Our purpose is to show that this subspace $SS_k(\Gamma)$ is closely related with spaces of cusp forms of one variable, in fact, this subspace can be lifted from spaces of cusp forms of one variable.

Let $S_k(SL_2(\mathbb{Z}))$ be the space of cusp forms of weight k with respect to $SL_2(\mathbb{Z})$. Let us introduce other spaces of cusp forms of one variable. From the condition on F , it follows that there exist $\ell - 1$ characters mod q of order ℓ corresponding to the extension $F/\mathbb{Q}$ in the sense of class field theory. We denote them by χ_i , $1 \leqslant i \leqslant \ell - 1$. For each character χ_i , we denote by $S_k(\Gamma_0(q)\ ,\ \chi_i)$ the space of cusp forms g which satisfy

$$g(\gamma Z) = (cZ + d)^k\, \chi_i(d)\, g(Z) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q) ,$$

where

$$\Gamma_0(q) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})\ ,\quad c \equiv 0 \mod q\} .$$

The Hecke ring $R(\mathfrak{U}_Q\ ,\ GL_2(\mathbb{Q}_A))$ acts on these spaces of cusp forms. On $S_k(SL_2/\mathbb{Z}))$, it acts in the usual manner. On the other spaces, we make it act in the following way. For a prime p , let $T(p)$ and $T(p\ ,\ p)$ be the elements of $R(\mathfrak{U}_Q\ ,\ GL_2(\mathbb{Q}_A))$

given in the next section. For $p \neq q$, $T(p)$ and $T(p , p)$ acts in the usual manner. For $p = q$, we define the action of $T(q)$ and $T(q , q)$ on $S_k(\Gamma_0(q), \chi_i)$ by

$$T(q)\, g = g|[\Gamma_0(q) \begin{pmatrix}1 & 0\\ 0 & q\end{pmatrix} \Gamma_0(q)]_{k,\chi_i} + g|[\Gamma_0(q) \begin{pmatrix}1 & 0\\ 0 & q\end{pmatrix} \Gamma_0(q)]^{*}_{k,\chi_i} ,$$

$$T(q , q)\, g = q^{k-2}\, g ,$$

where $[\Gamma_0(q) \begin{pmatrix}1 & 0\\ 0 & q\end{pmatrix} \Gamma_0(q)]_{k,\chi_i}$ is the action of the double coset $\Gamma_0(q) \left(\begin{smallmatrix}1 & 0\\ 0 & q\end{smallmatrix}\right)\Gamma_0(q)$ defined in G. SHIMURA [10], and $[\ ]^{*}_{k,\chi_i}$ means the adjoint operator of $[\ ]_{k,\chi_i}$ with respect to the Petersson inner product. To compare the above two kinds of representations of Hecke rings, namely the representation of $R(\mathfrak{U}_F , GL_2(F_A))$ on $SS_k(\Gamma)$ and those of $R(\mathfrak{U}_{\underset{\sim}{Q}} , GL_2(\underset{\sim}{Q}_A))$ on $S_k(SL_2(\underset{\sim}{Z}))$ and $S_k(\Gamma_0(q) , \chi_i)$, we define a natural homomorphism λ from $R(\mathfrak{U}_F , GL_2(F_A))$ to $R(\mathfrak{U}_{\underset{\sim}{Q}} , GL_2(\underset{\sim}{Q}_A))$ in the next section. First assuming this λ , we will state our theorem. By means of λ , the spaces $S_k(SL_2(\underset{\sim}{Z}))$ and $S_k(\Gamma_0(q) , \chi_i)$, $1 \leqslant i \leqslant \ell - 1$, can be regarded as $R(\mathfrak{U}_F , GL_2(F_A))$-modules. On these notations, we can prove [7], the following theorem.

THEOREM. - If $k \geqslant 4$, there exists a subspace S of $\bigoplus_{i=1}^{\ell-1} S_k(\Gamma_0(q) , \chi_i)$ such that

$$SS_k(\Gamma) \simeq S_k(SL_2(\underset{\sim}{Z})) \oplus S ,$$

(and $\bigoplus_{i=1}^{\ell-1} S_k(\Gamma_0(q) , \chi_i) \simeq S \oplus S$) as $R(\mathfrak{U}_F , GL_2(F_A))$-modules.

Let $g \in S_k(SL_2(\underset{\sim}{Z}))$ be a common eigen function for all Hecke operators and let $f \in SS_k(\Gamma)$ be a common eigen function for all Hecke operators which corresponds to g in the above isomorphism, then it holds the following relation between the associated Dirichlet series φ_g of g and Φ_f of f , namely,

$$\Phi_f(s) = \varphi_g(s) \prod_{i=1}^{\ell-1} \varphi_g(s , \chi_i) ,$$

where

$$\varphi_g(s , \chi_i) = \sum_{n=1}^{\infty} a_n\, \chi_i(n)\, n^{-s} \quad \text{for} \quad \varphi_g(s) = \sum_{n=1}^{\infty} a_n\, n^{-s} .$$

This theorem can be considered an analogue for automorphic forms of the decomposition theorem of Dedekind zeta-functions.

The above theorem can be derived easily from the following theorem on trace of Hecke operators.

THEOREM. - If $k \geqslant 4$,

$$\mathrm{tr}\ T(e)/SS_k(\Gamma) = \mathrm{tr}\ T(\lambda(e))/S_k(SL_2(\underset{\sim}{Z})) + \frac{1}{2} \sum_{i=1}^{\ell-1} \mathrm{tr}\ T(\lambda(e))/S_k(\Gamma_0(q) , \chi_i)$$

for any $e \in R(\mathfrak{U}_F , GL_2(F_A))$, where $\mathrm{tr}\ T(e^{*})/^{*}$ is the trace of $T(e^{*})$ on the space $*$.

Remark. - The above theorem is a generalization and a refinement of the result of

K. DOI and H. NAGANUMA ([2] , [6]), which treated the lifting for quadratic extensions. H. JACQUET [3] studied the lifting for quadratic extensions from the view point of representation theory. Alternative proofs for K. DOI and H. NAGANUMA's result are given by D. ZAGIER [12] and S. KUDLA [4]. T. ASAI [1] treated the lifting in the case of imaginary quadratic extensions over $\mathbb{Q}$.

2. – In this section, we give the definition of λ . Since it is known that

$$R(\mathfrak{U}_F , GL_2(F_A)) = \bigotimes_{\mathfrak{p}} R(GL_2(\mathcal{O}_{\mathfrak{p}}) , GL_2(F_{\mathfrak{p}})) ,$$
$$R(\mathfrak{U}_{\mathbb{Q}} , GL_2(\mathbb{Q}_A)) = \bigotimes_p R(GL_2(\mathbb{Z}_p) , GL_2(\mathbb{Q}_p)) ,$$

it is enough to define a homomorphism $\lambda_{\mathfrak{p}}$ from $R_{\mathfrak{p}} = R(GL_2(\mathcal{O}_{\mathfrak{p}}) , GL_2(F_{\mathfrak{p}}))$ to $R_p = R(GL_2(\mathbb{Z}_p) , GL_2(\mathbb{Q}_p))$ for each prime ideal $\mathfrak{p}$ of F , where p is a prime such as $\mathfrak{p}|p$. Let $T(\mathfrak{p})$ and $T(\mathfrak{p} , \mathfrak{p})$ (resp. $T(p)$ and $T(p , p)$) be the elements of $R_{\mathfrak{p}}$ (resp. R_p) given by the double cosets

$$GL_2(\mathcal{O}_{\mathfrak{p}}) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} GL_2(\mathcal{O}_{\mathfrak{p}}) \quad \text{and} \quad GL_2(\mathcal{O}_{\mathfrak{p}}) \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix} GL_2(\mathcal{O}_{\mathfrak{p}})$$

(resp. $GL_2(\mathbb{Z}_p) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} GL_2(\mathbb{Z}_p)$ and $GL_2(\mathbb{Z}_p) \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} GL_2(\mathbb{Z}_p)$) respectively, where π is a prime element of $\mathcal{O}_{\mathfrak{p}}$. We denote by $R_{\mathfrak{p}}^I$ (resp. R_p^I) the subring of $R_{\mathfrak{p}}$ (resp. R_p) generated by $T(\mathfrak{p})$ and $T(\mathfrak{p} , \mathfrak{p})$ (resp. $T(p)$ and $T(p , p)$). If we put

$$\begin{matrix} T(\mathfrak{p}) = X + Y \\ T(\mathfrak{p} , \mathfrak{p}) = N\mathfrak{p}XY \end{matrix} \quad \left(\text{resp.} \quad \begin{matrix} T(p) = x + y \\ T(p , p) = pxy \end{matrix} \right) ,$$

we can embed $R_{\mathfrak{p}}$ (resp. R_p) into the polynomial ring $\mathbb{Q}[X , Y]$ (resp. $\mathbb{Q}[x, y]$) of two variables over $\mathbb{Q}$. Now, consider the mapping from $\mathbb{Q}[X , Y]$ to $\mathbb{Q}[x , y]$ given by

$$X \longmapsto x^f ,$$
$$Y \longmapsto y^f ,$$

where f is an integer such that $N\mathfrak{p} = p^f$. Then we see easily that this mapping can be extended to a homomorphism from $R_{\mathfrak{p}}$ to R_p .

3. – On this section, we give a numerical example of our theorem. We take as F the maximal real subfield of 7-th root of unity , then $[F:\mathbb{Q}] = 3$, and F satisfies the condition in § 1 . Let χ be the character mod 7 of order 3 given by $\chi(3) = \omega$, $\omega = (-1 + \sqrt{-3})/2$. For $k = 4$, we have $\dim S_4(\Gamma) = 1$ and $\dim S_4(\Gamma_0(7) , \chi) = 1$. In this case, the subspace $SS_4(\Gamma)$ coincides with $S_4(\Gamma)$, hence $S_4(\Gamma)$ is isomorphic to $S_4(\Gamma_0(7) , \chi)$ as $R(\mathfrak{U}_F , GL_2(F_A))$-modules. Let f (resp. g) be an element of $S_4(\Gamma)$ (resp. $S_4(\Gamma_0(7) , \chi)$) with the associated Dirichlet series

$$\Phi_f(s) = \sum C(\mathfrak{A}) N\mathfrak{A}^{-s} \quad (\text{resp.} \quad \varphi_g(s) = \sum_{n=1}^{\infty} a_n n^{-s}).$$

We may assume $C(\mathcal{O}) = 1$ and $a_1 = 1$. Then our theorem asserts that it holds the following relation between $C(\mathfrak{p})$ and a_p , namely

$$C(\mathfrak{p}) = \begin{cases} a_p & (p) = \mathfrak{p}\mathfrak{p}'\mathfrak{p}'' \\ a_p^3 - 3\chi(p)\, p^3\, a_p & (p) = \mathfrak{p} \\ a_p + \overline{a}_p & (p) = \mathfrak{p}^3 , \end{cases}$$

where $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{p}''$ are the distinct prime divisors of (p) . This relation can be checked for several $\mathfrak{p}$ and p . The coefficients a_p can be calculated by Eichler-Selberg's trace formula using the class numbers of imaginary quadratic fields. On the other hand, $C(\mathfrak{p})$ can be obtained by Shimizu's trace formula [8] using the class numbers of totally imaginary quadratic extensions of F . For example, to calculate $C((2))$, we need the following class numbers.

$$h(F(\sqrt{-8})) = 1 , \quad h(F(\sqrt{-7})) = 1 , \quad h(F(\sqrt{\alpha^2 - 8})) = 1 , \quad h(F(\sqrt{\alpha^2 + 2\alpha - 7})) = 1 .$$

Here $h(K)$ is the class number of K and α is a root of the equation

$$X^3 + X^2 - 2X - 1 = 0 .$$

On this way, we have the following table.

p	$\chi(p)$	a_p	$\mathfrak{p}$	$C(\mathfrak{p})$
2	ω^2	2ω	(2)	-40
3	ω	$7\omega^2$	(3)	-224
7	0	$7-14\omega$	$(2-\alpha)$	28
13	1	-14	(α^2+1)	-14
29	1	58	$(3-\alpha)$	58

4. 1. - Let F be as in § 1, and $\mathfrak{A}$ an integral ideal of F such as ${}^{\sigma}\mathfrak{A} = \mathfrak{A}$, then we can define a subspace $SS_k(\Gamma_0(\mathfrak{A}))$ of $S_k(\Gamma_0(\mathfrak{A}))$ in the same way as in § 1, where $\Gamma_0(\mathfrak{A}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma , \ c \equiv 0 \mod \mathfrak{A}\}$, and we can prove a similar but more complicated result on this $SS_k(\Gamma_0(\mathfrak{A}))$. Also, in the case of definite quaternion algebras, we can consider a similar problem, and the case where $\ell \geqslant 3$ has been treated by H. HIJIKATA. In the case of quadratic extensions, we can prove the following. Let F be $\mathbb{Q}(\sqrt{q})$ with a prime q , $q \equiv 1 \mod 4$, and B a definite quaternion algebra over $\mathbb{Q}$ which ramifies at q and at the archimedean prime. Let R be a maximal order of $B \otimes F$ which satisfies ${}^{\sigma}R = R$, where σ is the generator of $\mathrm{Gal}(F/\mathbb{Q})$. For a non-negative even integer k , let $M(\mathrm{id} , \{k , k\})$ be the space of continuous functions on $(B \otimes F)_A^{\times}$ defined in H. SHIMIZU [8] with respect to the open subgroup $\prod_p R_p^{\times} \times \prod_v (B \otimes F)_v^{\times}$ of $(B \otimes F)_A^{\times}$. We can define the action of T_σ on $M(\mathrm{id} , \{k , k\})$ by means of the action of σ on $(B \otimes F)_A$, and in these notations we can prove the following theorem.

THEOREM. - <u>For any</u> $e \in \bigotimes_{p \neq q} R(R_p^{\times} , (B \otimes F)_A^{\times})$, <u>it holds</u>

$$\operatorname{tr} T(e)\, T_\sigma / M(\mathrm{id}\ ,\ \{k\ ,\ k\})$$
$$= - \operatorname{tr} T(\lambda(e))/S_{k+2}(SL_2(\underset{\sim}{Z})) + \frac{1}{2} \operatorname{tr} T(\lambda(e))/S_{k+2}(\Gamma_0(q)\ ,\ (\overline{q}))\ ,$$

where $\mathfrak{q}$ *is the prime ideal such as* $\mathfrak{q}^2 = (q)$ *and* $(\overline{q})$ *is the quadratic residue symbol* mod q .

4. 2. - The theorem in § 1 has been generalized from the view point of representation theory by T. SHINTANI [11] and R. P. LANGLANDS [5] by a similar method as ours. Especially, R. P. LANGLANDS found an important application of his generalization of our theorem to the conjecture of Artin on the poles of Artin's L-functions.

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