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MATTI JUTILA

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RECENT PROGRESS IN THE THEORY OF L-FUNCTIONS

by Matti JUTILA [Turku]

The rapid development in the theory of L-functions during the last ten years has to a great extent been stimulated by some important discoveries by E. BOMBIERI, G. HALÁSZ and P. X. GALLAGHER (the "three basic principles" in Montgomery's book [23]). In the monographs of H. L. MONTGOMERY [23] and M. N. HUXLEY [4], the mentioned new fruitful ideas and their applications are discussed in detail, so that I will in this talk tell about some more recent (partly as yet unpublished) work in this field. Furthermore, I confine myself to two topics :

1. The mean-value estimates,

2. The zero-density estimates.

Since the mean-value estimates play an important rôle in proving zero-density estimates, the two problems are closely related. On the other hand, the zeros of L-functions are of interest because of many arithmetical applications.

1. Mean-value estimates.

Let us consider the following problem. Given a set K of Dirichlet characters, we want to know for which k (a positive integer) the estimate

$$(1.1) \qquad \sum_{\chi\in K} \int_{-T}^{T} |L(\tfrac{1}{2} + it , \chi)|^k \, dt \ll ((\# K)T)^{1+\varepsilon}$$

is true for any $\varepsilon > 0$. Instead of (1.1), we may also consider stronger statements, with the ε'th power replaced by a power of the logarithm, or with the symbol $\ll$ replaced by asymptotic equality.

We consider four types of sets K :

(i) K consists of the principal character mod 1 ;

(ii) K = the set of all characters mod q ;

(iii) K = the set of all characters $\chi\chi_1$ with χ_1 a fixed character of modulus $\leqslant Q$ and χ running over the primitive characters of modulus $\leqslant Q$;

(iv) K = the set of all real non-principal characters of modulus $\leqslant Q$.

Case (i) : k = 4 by a classical result of INGHAM (actually an asymptotic equality with an error term). INGHAM used the approximate functional equation for $\zeta^2(s)$ due to HARDY and LITTLEWOOD. Now it is of interest that quite recently K. RAMACHANDRA [29] has given a new proof for Ingham's theorem without appealing to the approximate functional equation. RAMACHANDRA's proof is based on the "ordinary" functional equation for $\zeta(s)$ and a recent nice mean-value theorem for Dirichlet series due to MONTGOMERY and VAUGHAN [25]. I will return to RAMACHANDRA's method in the next case (ii) ; his ideas proved to be very fruitful also in zero-density questions.

Case (ii) : It is not easy to generalize Ingham's theorem to L-functions, even if asymptotic equality is weakened to an inequality, because the approximate functional equation for $\zeta(s)$ does not immediately generalize to L-functions. LAVRIK [19] has proved an approximate functional equation for $L(s\,,\chi)$ which can be used to settle the case $k = 4$ (see [20], or [23], Theorem 10.1). The calculation is, however, somewhat tedious because the coefficients are not constants (as in the case of the zeta-function). However, RAMACHANDRA [28] has proved the same result without using the approximate functional equation at all (and even in a simpler way). Paradoxically, the approximate functional equation seems to be a too "deep" result for mean-value problems and so difficult to handle.

The starting-point in Ramachandra's argument is the identity

$$(1.2)\qquad \sum_{n=1}^{\infty} \chi(n)\, e^{-n/U}\, n^{-s} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} L(s+w\,,\chi)\,\Gamma(w)\,U^{w}\,dw \qquad (\sigma \geq 0)$$

which is verified by Mellin's transformation formula. Let χ be a non-principal primitive character and $s = \frac{1}{2} + it$, and move the integration to the line $\mathrm{Re}(s + w) = - b$ $(0 < b < 1)$, acquiring at $w = 0$ the residue $L(s\,,\chi)$. (If χ is the principal character, the pole of $\zeta(s)$ at $s = 1$ gives an additional residue which is, however, small if t is large enough.) Next use the functional equation in the form $L(s\,,\chi) = \psi(s\,,\chi)\,L(1-s\,,\bar\chi)$. Now $\mathrm{Re}(1-s-w) = 1 + b > 1$, so that

$$L(1-s-w\,,\bar\chi) = \sum_{n=1}^{\infty} \chi(n)\, n^{s+w-1} = \sum_{n\leq U} + \sum_{n>U} .$$

So we get for $L(s\,,\chi)$ the following expression :

$$(1.3)\qquad L(s\,,\chi) = \sum_{n=1}^{\infty} \chi(n)\, e^{-n/U}\, n^{-s} + I_1 + I_2 ,$$

where I_1 and I_2 are the integrals corresponding to the partial sum $\sum_{n\leq U}$ and the remainder $\sum_{n>U}$. In proving the fourth power mean-value theorem, choose $U = (qT)^{\frac{1}{2}}$, $b = (\log qT)^{-1}$, and use a standard mean-value theorem to estimate I_2 . The integral I_1 is estimated similarly, but the integration is first moved back to the line $\mathrm{Re}\, w = 0$.

Case (iii) : $k = 4$ by similar arguments (using mean value estimates of large sieve type).

Case (iv) : $k = 2$ (JUTILA [16]). The proof is based on the following mean-value theorem for real character sums [14] :

$$(1.4)\qquad {\sum_{|D|\leq Q}}' \Big|\sum_{1\leq n\leq N} \left(\frac{D}{n}\right)\Big|^{2} \ll QN \log^{8} Q ,$$

where D runs over the non-square numbers $\equiv 0\,,1 \pmod 4$. Using Gallagher's lemma ([23], Lemma 1.10), (1.4) implies a mean-value estimate for Dirichlet polynomials involving real characters (see [15]). The desired mean square estimate is now easy to prove using Ramachandra's expression (1.3) for $L(s, \chi)$. It seems difficult to manage the case $k = 4$, even if only the primitive characters are taken into account.

2. Zero-density estimates.

We use the standard notation: $N(\alpha, T, \chi)$ denotes the number of zeros of the function $L(s, \chi)$ in the rectangle $\alpha \leqslant \sigma \leqslant 1$, $|t| \leqslant T$; $N(\alpha, T) = N(\alpha, T, \chi_0)$. Let the numbers λ_i be such that for $\frac{1}{2} \leqslant \alpha \leqslant 1$, $T \geqslant 2$ and for any $\varepsilon > 0$

(2.1) $$N(\alpha, T) \ll T^{\lambda_1(1-\alpha)+\varepsilon},$$

(2.2) $$\sum_{\chi \bmod q} N(\alpha, T, \chi) \ll (qT)^{\lambda_2(1-\alpha)+\varepsilon};$$

(2.3) $$\sum_{q\leqslant Q} \sum^{*}_{\chi \bmod q} N(\alpha, T, \chi) \ll (Q^2 T)^{\lambda_3(1-\alpha)+\varepsilon},$$

(2.4) $$\sum_{q\leqslant Q} \sum^{*}_{\chi \bmod q} N(\alpha, T, \chi) \ll (Q^2 T^2)^{\lambda_4(1-\alpha)+\varepsilon},$$

where the star denotes primitivity, and the constants implied by the notation depend on ε.

It can be conjectured that $\lambda_i = 2$ in each case. Owing to the work of HUXLEY [5], [9], we know that anyway the above estimates hold with $\lambda_i = 12/5$. In particular, the result $\lambda_1 = 12/5$ implies the estimate $p_{n+1} - p_n \ll p_n^{7/12+\varepsilon}$ for the difference between consecutive primes.

The sharpest zero-density results for α between $3/4$ and 1 have been obtained by the Halász-Montgomery method via estimates for the frequency with which a Dirichlet polynomial

$$f(s, \chi) = \sum_{N+1}^{2N} a_n \chi(n) n^{-s}$$

can be "large". Even the simplest variant of this method gave remarkable results as regards the size of λ_i and the density hypotheses. Let the numbers α_i be such that for $\alpha \geqslant \alpha_i$ the respective density hypothesis holds, i. e. we have (2.i) with $\lambda_i = 2$. MONTGOMERY [22] proved that $\alpha_1 < 9/10$, and, by a simple additional argument, HUXLEY [5] improved the bound to $5/6$.

The main problem in the Halász-Montgomery method is to estimate a double sum over values of a "modified L-series".

(2.5) $$H(s, \chi) = \sum_{n=1}^{\infty} b_n \chi(n) n^{-s},$$

where $b_n \geqslant 0$ for all n, $b_n \geqslant 1$ for $n = N+1, \dots, 2N$, and $\sum_n b_n$ converges. HUXLEY ([7]-[9]) introduced the so-called "reflection argument" which is closely related to the approximate functional equation for $L(s, \chi)$. The principle is to relate $H(s, \chi)$ to a "conjugate" Dirichlet polynomial which is a partial sum for $L(s, \chi)$ of length about $q|t|/N$. This method gave the mentioned result $\lambda_i = 12/5$, and also that $\alpha_1 \leqslant 4/5$ [9]; $\alpha_2, \alpha_3 \leqslant 5/6$ [7]; $\alpha_4 \leqslant 11/14$ [6].

The same bound for α_2 and α_3 was established by different methods also by BALASUBRAMANIAN and RAMACHANDRA [1] and myself [13].

In [17] and [18], I found, using Ramachandra's ideas (see [27]), a simplified version of Huxley's reflection argument. Let us choose

$$b_n = \exp(-(n/2N)^h) - \exp(-(n/N)^h) ,$$

where $h \geqslant 1$ will be specified below (striktly speaking, we should multiply b_n by a constant in order that $b_n \geqslant 1$ for $n = N+1 , \ldots , 2N$). Then we have the following lemma ([18], Lemma 1) :

LEMMA 1. - Let χ be a character (mod q) , $\varepsilon > 0$, $T \geqslant 2$, $N \leqslant qT$, $h = \log^2 qT$, $s = \sigma + it$, $0 \leqslant \sigma \leqslant 1$, $|t| \leqslant T$; also $|t| \geqslant h^2$ if $\chi = \chi_0$. Let $q(T+h^3)(\pi N)^{-1} \leqslant M \leqslant (qT)^2$. Then

$$H(s , \chi) \ll_\varepsilon N^{\frac{1}{2}} q^\varepsilon \int_{-h^2}^{h^2} |\textstyle\sum_1^M \overline{\chi}(n)\, n^{-\frac{1}{2}+i(t+u)}|\,du + 1 .$$

We quote another lemma ([18], Lemma 2 in a slightly modified form) which is useful in dealing with the double sum arising in the Halász-Montgomery method :

LEMMA 2. - If $|a_n| \leqslant A$ for $n = 1 , \ldots , N$, then

$$\sum_{r,s=1}^R |\sum_{n=1}^N a_n \overline{\chi}_r(n)\, \chi_s(n)\, n^{-\frac{1}{2}+i(t_r-t_s)}|^2$$

$$\leqslant A^2 e^2 \sum_{r,s=1}^R |\sum_{n=1}^\infty \overline{\chi}_r(n)\, \chi_s(n) \exp(-(n/N))\, n^{-\frac{1}{2}+i(t_r-t_s)}|^2$$

$$= A^2 e^2 (2\pi)^{-2} \sum_{r,s=1}^R |\int_{\mathrm{Re}\, w=2} L(w+\tfrac{1}{2}-i(t_r-t_s) , \overline{\chi}_r \chi_s)\, N^w\, \Gamma(w)\, dw|^2 .$$

Actually, lemma 2 is a special case of a more general inequality for complex numbers.

Using lemmas 1 and 2, as well as mean fourth power estimates for L-functions, I could prove that $\alpha_1 \leqslant 11/14$, α_2 , $\alpha_3 \leqslant 21/26$, $\alpha_4 \leqslant 7/9$. In a joint paper by HUXLEY and myself [10], the two last results are improved, the new bounds being resp. $4/5$ and $557/718 = 0.7759 \ldots$

Next I wish to mention a few words about the zeros of L-functions with real characters. Writing $\chi_D(n) = (\frac{D}{n})$, we have [15]

$$\sum'_{|D| \leqslant Q} N(\alpha , T , \chi_D) \ll (QT)^{(7-6\alpha)/(6-4\alpha)+\varepsilon}$$

(the numbers D are restricted as in Case (iv) in §1). Because the exponent is < 1 for $\alpha > \frac{1}{2}$, the estimate is non-trivial for all $\alpha > \frac{1}{2}$, but becomes weak near $\alpha = 1$. The reason is that because the imprimitive characters are not excluded, the zeros are, in general, counted several times. It would be nice to have $(QT)^{3/2-\alpha+\varepsilon}$ on the right side of (2.6).

Finally, some remarks about Linnik's density theorem. This theorem reads as follows :

$$(2.7) \qquad \sum_{\chi \bmod q} N(\alpha , T , \chi) \ll (qT)^{c(1-\alpha)} ;$$

actually LINNIK proved it for bounded T in connection with his famous theorem on the least prime in an arithmetic progression. The generalization is due to FOGELS [2]. Still more generally, GALLAGHER [3] proved the estimate

$$(2.8) \qquad \sum_{q \leq Q} \sum^{*}_{\chi \bmod q} N(\alpha , T , \chi) \ll (Q^2 T)^{c(1-\alpha)} .$$

TURÁN [30] has given a relatively simple proof for Linnik's theorem, using his power sum method, and the papers of FOGELS and GALLAGHER were based on Turán's work. Also, the sieve method of BRUN-SELBERG has been indispensable in all known proofs of Linnik's theorem.

Quite recently, SELBERG and MONTGOMERY [24] have given new proofs for (2.7)-(2.8) with remarkably good constants in the exponent. The new estimates are respectively

$$\ll_{\varepsilon} (qT)^{(3+\varepsilon)(1-\alpha)}$$

and

$$\ll_{\varepsilon} (Q^5 T^3)^{(1+\varepsilon)(1-\alpha)} .$$

Methodically it is interesting that the new proofs use ideas from the large sieve and Selberg sieve methods, but Turán's method is not applied. Further, MOTOHASHI [26] has refined the above estimates, replacing them by

$$\ll_{\varepsilon} (q^2 T^3)^{(1+\varepsilon)(1-\alpha)}$$

and

$$\ll_{\varepsilon} (Q^4 T^3)^{(1+\varepsilon)(1-\alpha)}$$

for $\alpha \geq 4/5$.

Linnik's famous theorem states that the least prime $p \equiv k \pmod q$ is $\ll k^L$ if $(k , q) = 1$; here L is a numerical constant (Linnik's constant). Calculating certain constants in Linnik's density theorem and in Linnik's second main theorem concerning the exceptional zero (see [11]) I have found for Linnik's constant the estimate $L \leq 550$ (see [12]). Now it would be of interest to calculate the constants implied by the symbols $\ll_{\varepsilon}$ in the above estimates (for $\varepsilon = 1$, say). If the constant were not too large, and if the second theorem of Linnik could also be proved by the new method with good constants, then the estimate for Linnik's constant could be significantly improved.

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Matti JUTILA
Department of Mathematics
University of Turku
TURKU (Finlande)