William J. Ellison

Differences between prime numbers

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Séminaire DELANGE-PISOT-POITOU
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DIFFERENCES BETWEEN PRIME NUMBERS

par William J. ELLISON

1. Generalities.

We denote by $\{p_n\}$ the sequence pf prime numbers, and by $\{d_n\}$ the sequence of differences between consecutive prime numbers, i. e. $d_n = p_{n+1} - p_n$. The sequence $\{d_n\}$ is very irregular ; one knows virtually nothing about its members. For example, the celebrated "twin primes" conjecture is equivalent to the assertion that $\{d_n\}$ contains an infinity of 2' s .

A trivial remark is that d_n is "in mean asymptotic to $\log n$ ", for by the prime number theorem we have

$$\sum_{n\leq N} d_n = p_{N+1} \sim N \log N$$

and so

$$\sum_{N\leq n\leq 2N} d_n \sim N \log N .$$

Thus, the mean value of d_n for $n \in [N , 2N]$ is asymptotic to $\log N$. Because of this observation, it became customary to consider the "normalised" sequence $\{d_n/\log n\}$. It is unreasonable to expect answers to detailed questions about the elements of the sequence, however, considered simply as a sequence of real numbers it is sensible to ask :

(1) What is $\liminf_{n\to\infty} (d_n/\log n)$?

(2) What is $\limsup_{n\to\infty} (d_n/\log n)$?

(3) Does there exist a "simple" function $f(n)$ such that $(d_n/\log n) \leq f(n)$ for all n ?

As for the first question we know that

$$0 \leq \liminf_{n\to\infty} (d_n/\log n) \leq 0,46\ldots ,$$

a result of BOMBIERI and DAVENPORT [1]. It is probable that the upper limit can be reduced slightly, but naturally one conjectures that the true answer is 0 .

For the second problem it was shown by WESTZYNTHUIS [12] that

$$\lim \inf (d_n/\log n) = \infty$$

and later RANKIN [9] proved that there exists a constant $c > 0$ and an infinity of integers n such that

$$\frac{d_n}{\log n} > \frac{c(\log\log n)(\log\log\log\log n)}{(\log\log\log n)^2} .$$

Nothing more precise on questions (1) and (2) seems to be known. However, there is a curious problem which has been open for twenty years. RICCI [10] and ERDÖS proved that the set of limit points of the sequence $\{d_n/\log n\}$ has a positive measure, but no specific non-negative real number is known to be a limit point and the precise measure of the set is unknown.

Today I would like to discuss the third question in some detail. As usual in prime number theory there are conjectures which are almost certainly true and then there are the considerably weaker results which one can prove. The third question is more or less equivalent to the following.

(4) Does there exist a "simple" function $F(n)$ such that for each integer n the interval $(n\ ,\ n + F(n)]$ contains a prime number ?

Denote by $\pi(x)$ the number of primes less than x. If we can show that

$$\pi(x + h) - \pi(x) > 0\ ,$$

then it follows that there is a prime number in the interval $(x\ ,\ x + h]$. However, for technical reasons, it is better to work with the function $\theta(x)$ defined by

$$\theta(x) = \sum_{p\leq x} \log P$$

and then use the observation that $\theta(x + h) - \theta(x) > 0$ is equivalent to the fact that the interval $(x\ ,\ x + h]$ contains a prime number.

2. Hoheisel's theorem.

The prime number theorem asserts that $\theta(x) = x + O(x)$, thus

$$\theta(x + h) - \theta(x) = h + O(x + h)\ .$$

If $\varepsilon > 0$ and we choose $h = \varepsilon x$, then

$$\theta(x + h) - \theta(x) = \varepsilon x + O(x)\ ,$$

and for all $x > x_0(\varepsilon)$ the right hand side is positive. Thus for any $\varepsilon > 0$ and all $x > x_0(\varepsilon)$ the interval $(x\ ,\ x + \varepsilon x]$ contains a prime number. With a better error term in the prime number theorem one can obtain a stronger conclusion. For example, if the Riemann's hypothesis is true one can prove that $\theta(x) = x + O(x^{\frac{1}{2}+\varepsilon})$ from which it follows that for any $\varepsilon > 0$ and all $x > x_0(\varepsilon)$ the interval $(x\ ,\ x + x^{\frac{1}{2}+\varepsilon}]$ contains a prime number.

With these observations in mind one might hope to give an unconditional proof of the following result.

THEOREM 1. – *There exists a real number* $\theta < 1$ *such that for all* $x > x_0(\theta)$ *the interval* $(x\ ,\ x + x^{\theta}]$ *contains a prime number.*

It was long thought that in order to prove this theorem one must make progress in proving Riemann's hypothesis, however a remarkable breakthrough was made by HOHEISEL

[6] who proved the theorem with $\theta = 29999/30000$. In recent years Hoheisel's result has been improved upon and smaller values of θ have been bound. As they represent the best known unconditional response to question (4), I will give an account of them before discussing the probable true state of affairs.

The objective is to prouve a result slightly stronger than is asserted in theorem 1, namely that if $h > x^\theta$, then $\theta(x+h) - \theta(x) \sim h$. Our starting point is the well known "explicit formula" for $\theta(x)$, namely

$$\theta(x) = x + \sum_{\substack{\rho=\beta+i\gamma \\ \zeta(\rho)=0 \\ 0<|\gamma|\leq T}} \frac{x^\rho}{\rho} + O(x^{\frac{1}{2}} + \frac{x}{T}\log^2 x),$$

valid for $x \geq 2$ and $2 \leq T \leq x$. A simple calculation yields

$$\frac{\theta(x+h) - \theta(x)}{h} - 1 = O(\sum_{0<|\gamma|\leq T} |\frac{(x+h)^\rho - x^\rho}{h\rho}| + \frac{x^{\frac{1}{2}}}{h} + \frac{x}{hT}\log^2 x).$$

If it is possible to choose h and T as functions of x such that the "O" term tends to zero when x tends to infinity, then we will have $\theta(x+h) - \theta(x) \sim h$, which implies theorem 1.

It is trivial that

$$|\frac{(x+h)^\rho - x^\rho}{\rho h}| = |\frac{1}{h}\int_x^{x+h} u^{\rho-1}\,du| \leq \frac{1}{h}\int_x^{x+h} u^{\beta-1}\,du \leq x^{\beta-1}$$

with the consequence

$$\frac{\theta(x+h) - \theta(x)}{h} - 1 = O(\sum_{\substack{0<|\gamma|<T \\ \rho=\beta+i\gamma}} x^{\beta-1} + \frac{x^{\frac{1}{2}}}{h} + \frac{x}{hT}\log^2 x).$$

To estimate the sum $\sum x^{\beta-1}$, we first note that

$$x^{\beta-1} = \frac{1}{\sqrt{x}} + (\log x)\int_{\frac{1}{2}}^{\beta} x^{\sigma-1}\,d\sigma$$

and so

$$\sum_{\substack{0<|\gamma|\leq T \\ \rho}} x^{\beta-1} = \frac{1}{\sqrt{x}} \sum_{0<|\gamma|\leq T} 1 + \log x \sum_{\substack{0<|\gamma|\leq T \\ \rho}} \int_{\frac{1}{2}}^{\beta} x^{\sigma-1}\,d\sigma.$$

If we introduce the function $\ell(\rho, \sigma)$ defined by

$$\ell(\rho, \sigma) = \begin{cases} 1 & \text{if } \sigma \leq \beta \\ 0 & \text{if } \sigma > \beta, \end{cases}$$

then

$$\sum_{\substack{0<|\gamma|\leq T \\ \rho}} \int_{\frac{1}{2}}^{\beta} x^{\sigma-1}\,d\sigma = \sum_{\substack{0<|\gamma|\leq T \\ \rho}} \int_{\frac{1}{2}}^{1} \ell(\rho, \sigma)\, x^{\sigma-1}\,d\sigma$$

$$= \int_{\frac{1}{2}}^{1} \{\sum_{\substack{0<|\gamma|\leq T \\ \rho}} \ell(\rho, \sigma)\}\, x^{\sigma-1}\,d\sigma = \int_{\frac{1}{2}}^{1} N(\sigma, T)\, x^{\sigma-1}\,d\sigma,$$

where $N(\sigma, T)$ is defined by

$$N(\sigma, T) = \#\{\rho : \zeta(\rho) = 0, \ \rho = \beta + i\gamma \text{ with } \beta \geq \sigma, \ |\gamma| \leq T\}.$$

Thus we have shown that

$$\sum_{\substack{0<|\gamma|\leq T \\ \rho}} x^{\beta-1} = \frac{N(\frac{1}{2}, T)}{\sqrt{x}} + (\log x) \int_{\frac{1}{2}}^{1} N(\sigma, T)\, x^{\sigma-1}\, d\sigma ,$$

which implies

$$(*) \quad \frac{\theta(x+h)-\theta(x)}{h} - 1 = O\left(\frac{N(\frac{1}{2},T)}{\sqrt{x}} + (\log x) \int_{\frac{1}{2}}^{1} N(\sigma,T)\, x^{\sigma-1}\, d\sigma + \frac{x^{\frac{1}{2}}}{h} + \frac{x}{hT} \log^2 x\right) .$$

To make further progress, we need to have some information about the function $N(\sigma, T)$. The following results are known but are rather technical to prove :

(a) $N(\frac{1}{2}, T) = O(T \log T)$,

(b) $N(\sigma, T) = O(T^{c(1-\sigma)} \log^A T)$ for $\frac{1}{2} \leq \sigma \leq 1$, where $c \geq 2$ and $A \geq 1$ are absolute constants.

(c) $N(\sigma, T) = 0$ for $\sigma > 1 - a/(\log T)^{\alpha}$ with $0 < \alpha < 1$ and $a > 0$ an absolute constant..

For proofs of (a) and (c), see ELLISON [3] (chapters 5 and 11), and for (b), see MONTGOMERY [8] (chapter 12).

Let us write $z = a/(\log T)^{\alpha}$, $T = x^{\delta}$, $h = x^{\theta}$, where δ and θ will be chosen later. For any $\varepsilon > 0$, we have, upon using (a), (b), (c), the following estimate for the "O" term in (*) :

$$O\{x^{\delta-\frac{1}{2}} \log x + (\log x)^{A+1} \int_{\frac{1}{2}}^{1-z} \exp\{(\sigma-1)(1-\delta(c+\varepsilon))\log x\}\, d\sigma + x^{\frac{1}{2}-\theta} + x^{1-\delta-\theta} \log^2 x\}$$

$$= O\{x^{\delta-\frac{1}{2}} \log x + (\log x)^{A} \exp\{-z(\delta(c+\varepsilon)-1)\log x\} + x^{\frac{1}{2}-\theta} + x^{1-\delta-\theta} \log^2 x\} .$$

Since $c \geq 2$, we can choose $\delta < \frac{1}{2}$ so that $\delta(c+\varepsilon) - 1 > 0$, say

$$\delta = (1 + \frac{1}{4}\varepsilon)/(c+\varepsilon) .$$

Finally, we choose $\theta > 1 - \delta$, and since ε can be as small as we like this means that we can choose $\theta > 1 - (1/c) \geq 1/2$. Thus, with this choice of δ and θ , the "O" term in (*) tends to zero when x tends to infinity. Hence, we have shown that if $\theta > 1 - 1/c$, then for all $x > x_0(\theta)$ the interval $(x , x + x^{\theta}]$ contains a prime number. As for a numerical value of c , HUXLEY [7] proved that we can take $c = 12/5$ which gives $\theta > 7/12$. This is quite close to the exponent $1/2 + \varepsilon$ which we gave earlier as a consequence of the Riemann's hypothesis.

3. Cramér's conjecture and Selberg's theorem.

Now I would like to consider a conjecture, due to H. CRAMÉR, which is almost certainly true, but seems impossibly difficult to prove. The conjecture is

$$\limsup_{n\to\infty} \frac{p_{n+1} - p_n}{(\log n)^2} = 1 .$$

Obviously the conjecture implies that if $K > 1$ and $x > x_0(K)$, then the interval $(x , x + K \log^2 x]$ contains a prime number. This is vastly superior to the type of result given by Hoheisel's theorem.

The reasoning which lead Cramér to make his conjecture is extremely interesting. It was based upon considerations of probability. Denote by P the sequence $\{0, 1, 1, 0, 1, \ldots, S_n, \ldots\}$, where

$$S_n = \begin{cases} 1 & \text{if } n \text{ is a prime} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly the study of the differences between prime numbers is equivalent to the study of chains of consecutive zeros in the sequence P. Arguing very naively the prime number theorem shows that the probability that $S_n = 1$ is asymptotic to $1/\log n$. Cramér's idea was to consider the set of sequences $\{0, 1\}^{\underset{\sim}{N}}$ and to define a probability measure on this set which picks out the sequences "like" P. Then one can prove that with respect to this measure almost all sequences satisfy Cramér's conjecture. One is then tempted to suppose that the sequence P also satisfies Cramér's conjecture. This technique is extremely useful in number theory, especially for proving the existence of integer sequences with specified properties. For an excellent introduction to this method, we refer the reader to HALBERSTAM and ROTH [5].

The principal tool from probability theory is the following "Borel-Cantelli" lemma :

<u>Let</u> $\{D_m\}$ <u>be a sequence of mutually independant events. If</u> $\sum \mu(D_m) < \infty$, <u>then the probability that an infinity of the events occurs is zero. However, if</u> $\sum \mu(D_m) = \infty$, <u>then the probability that an infinity of the events occurs is one.</u>

The precise mathematical structure for our particular problem is as follows. For $n = 2, 3, \ldots$, let $\mathcal{E}_n$ denote the finite probability space

$\{0,1\}$; $\{\emptyset, \{1\}, \{0\}, \{1,0\}$; μ_n, where $\mu_n(1) = 1/\log n$, $\mu_n(0) = 1 - 1 \log n\}$,

and let $\mathcal{C}$ be the product of the spaces $\mathcal{E}_n$ with the induced product measure μ.

In order to mimic the sequence of differences between the primes we define for $\underset{\sim}{a} \in \mathcal{C}$ first the sequence $\{r_n(\underset{\sim}{a})\}$ by

$$\{r_n(\underset{\sim}{a})\} = \{\nu \text{ in ascending order such that } a_\nu = 1\},$$

then the sequence $\{d_n(\underset{\sim}{a})\}$ where $d_n(\underset{\sim}{a}) = r_{n+1}(\underset{\sim}{a}) - r_n(\underset{\sim}{a})$.

Let $c > 0$ be a fixed real number and, for each positive integer m, we denote by D_m the event

$$D_m = \{\{a_n\} \in \mathcal{C} \text{ with } a_{m+\nu} = 0 \text{ for } 1 \leq \nu \leq c(\log m)^2\}.$$

It is clear that the following two events have the same probability :

(i) $\{\underset{\sim}{a} \in \mathcal{C} : d_n(\underset{\sim}{a}) > c \log^2 n$ for an infinity of integers $n\}$,

(ii) An infinite number of the events D_m are realised.

We shall show that if $c > 1$ then the above probability is 0, and that for

$c < 1$ the probability is 1.

We obviously have

$$\mu(D_m) = \prod_{\nu=1}^{c \log^2 m} \left(1 - \frac{1}{\log(m+\nu)}\right)$$

and an elementary calculation shows that for suitable absolute constants A and B

$$\frac{A}{m^c} < \mu(D_m) < \frac{B}{m^c} .$$

The events $\{D_m\}$ are mutually independent, so if $c > 1$, then $\sum \mu(D_m) < \infty$ and the probability that an infinity of the events occur is 0.

Suppose now that $c < 1$. Consider the sequence $\{D_{m_i}\}$ where $m_2 = 2$,

$$m_{r+1} = m_r + [c \log^2 m] + 1 .$$

An elementary calculation shows that for a suitable $K > 0$ and all large r

$$m_r < Kr(\log r)^2 .$$

Thus, since $c < 1$, we have $\sum \mu(D_m) = \infty$. Hence with probability 1 an infinity of the events $\{D_m\}$ occurs.

Combining the above results it follows that with probability 1

$$\limsup_{n\to\infty} \frac{d_n(\underset{\sim}{a})}{(\log n)^2} = 1 .$$

It is then reasonable to suppose that the same result holds for the special sequence P. One can prove other results about $\mathcal{C}$ and so obtain suggestive conjectures about P. For example, if we define the random variable π by

$$\pi(\underset{\sim}{a}, x) = \sum_{n \leq x} a_n ,$$

then the mean value of $\pi(\underset{\sim}{a}, x)$ is asymptotic to $\ell i(x)$ and one can even prove that with probability 1

$$\limsup_{x\to\infty} \frac{|\pi(\underset{\sim}{a}, x) - \ell i(x)|}{\sqrt{2x}\sqrt{(\log\log x)/\log x}} = 1 .$$

Thus, one could conjecture that

$$\pi(P, x) = \ell i(x) + O\left(\sqrt{\frac{x \log\log x}{\log x}}\right) .$$

It seems hopeless to expect any proof of Cramér's conjecture in the immediate (or distant !) future, so it is worth-while to try and see how far one can go towards deciding its status by assuming, say, the Riemann's hypothesis. CRAMÉR [2] himself did this, and later SELBERG [11] improved upon Cramér's investigations. The most interesting conclusion of this work is the following theorem and its corollaries, wich provide some moral support for a believe in Cramér's conjecture. For they imply that if the Riemann's hypothesis is true, then the number of primes for which $(P_{n+1} - P_n)$ is larger than $(\log n)^2$ is "small".

Let us introduce the following notation

$$\ell_h(X) = \sum_{\substack{X < P_n \leq 2X \\ d_n \geq h}} d_n , \quad N_h(X) = \sum_{\substack{P_n \leq X \\ d_n \geq h}} 1 .$$

We can now state the principal result.

THEOREM 2. – <u>If the Riemann hypothesis is true, then</u>

$$\ell_h(X) = O\left(\frac{X}{h}\log^2 X\right) , \quad N_h(X) = O\left(\frac{X}{h^2}\log^2 X\right) .$$

COROLLARY. – <u>If the Riemann hypothesis is true, then</u>

(i) $d_n = O(\sqrt{P_n}\log P_n)$,

(ii) $\sum_{X<P_n\leq 2X} d_n^2 = O(X\log^3 X)$,

(iii) <u>If</u> $\lambda > 4$, <u>then</u> $\sum_{n=1}^{\infty} \frac{d_n^2}{P_n}(\log P_n)^{-\lambda} < \infty$.

The above theorem is an elementary consequence of the following result.

THEOREM 3. – <u>Suppose that the Riemann hypothesis is true. If</u> $\varepsilon > 0$ <u>and</u> ω <u>is a function of</u> X <u>such that</u> $0 < \omega < X^{-\varepsilon}$, <u>then as</u> X <u>tends to infinity</u>

$$\int_0^X \left\{\frac{\theta(x+\omega x)-\theta(x)}{\omega x} - 1\right\}^2 dx = O\left(\frac{\log^2 X}{\omega}\right) .$$

<u>Deduction of theorem 2 from theorem 3.</u> – Let $\varepsilon > 0$ be a fixed real number to be chosen later. We shall consider two cases :

(i) $0 < h \leq X^{1-\varepsilon}$ and

(ii) $X^{1-\varepsilon} < h \leq X$.

In case (i), we choose $\omega = h/4X$ and so $0 < \omega < X^{-\varepsilon}$. Now suppose that

$$(p_n , p_{n+1}] \subseteq (X , 2X]$$

and that $d_n > h$. If x satisfies

$$p_n < x \leq p_n + \frac{1}{2} d_n ,$$

then

$$x + \omega x < p_n + \frac{1}{2} d_n + 2\omega X \leq p_n + d_n = p_{n+1}$$

with the consequence

$$\theta(x+\omega x) - \theta(x) = 0 .$$

Hence we have

$$\frac{1}{2} d_n = \int_{p_n}^{p_n+\frac{1}{2}d_n} \left\{\frac{\theta(x+\omega x)-\theta(x)}{\omega x} - 1\right\}^2 dx .$$

From theorem 3, we conclude that

$$\sum_{\substack{X<p_n\leq 2X \\ d_n\geq h}} d_n = 2\sum_{\substack{X<p_n\leq 2X \\ d_n\geq h}} \int_{p_n}^{p_n+\frac{1}{2}d_n} \left\{\frac{\theta(x+\omega x)-\theta(x)}{\omega x} - 1\right\}^2 dx$$

$$\leq 2\int_X^{2X} \left\{\frac{\theta(x+\omega x)-\theta(x)}{\omega x} - \cancel{1}\right\}^2 \cancel{dx = O\left(\frac{\log^2 X}{\omega}\right)}$$

$$= O\left(\frac{X}{h}\log^2 X\right) .$$

Thus if $0 < h \leqslant X^{1-\varepsilon}$, we have proved that $\ell_h(X) = O(\frac{X}{h} \log^2 X)$. However, if we take $\varepsilon < 1/2$ and choose $h = X^{\alpha}$ with $1/2 < \alpha < 1 - \varepsilon$, then $\ell_h(X) = 0$ for $X > X_0$. For if $\ell_h(X) \neq 0$, then $\ell_h(X) \geqslant h$, and since we are still in case (i), we also have

$$h = O(\frac{X}{h} \log^2 X) ,$$

which leads to a contradiction if X is sufficiently large. Hence $\ell_h(X) = 0$ for $X^{\alpha} \leq h \leq X$. Thus for all h satisfying $0 < h \leqslant X$ we have $\ell_n(X) = O(\frac{X}{h} \log^2 X)$.

From the definition of $N_h(X)$, it is trivial that

$$\ell_h(X) \geqslant h\{N_h(X) - N_h(\tfrac{1}{2} X)\} .$$

Upon replacing X by $X/2^r$, $r = 1 , 2 , \ldots$ and adding we deduce that

$$N_h(X) = O(\frac{1}{h} \ell_h(X)) = O(\frac{X}{h^2} \log^2 X) .$$

Proof of the corollaries.

(i) If we take $h = c\sqrt{X} \log X$ with c sufficiently large, it follows that $N_h(X) < 1$ and so $N_h(X) = 0$.

(ii) We have

$$\sum_{1 \leqslant h \leqslant X} \sum_{\substack{X < p_n \leqslant 2X \\ d_n > h}} d_n = \sum_{X < p_n \leqslant 2X} d_n \sum_{h \leqslant d_n} 1 = \sum_{X < p_n \leqslant 2X} d_n^2$$

and from theorem 2, we also have

$$\sum_{1 \leqslant h \leqslant X} \sum_{\substack{X < p_n \leqslant 2X \\ d_n > h}} d_n = O\{\sum_{h \leqslant X} \frac{X}{h} \log^2 X\} = O(X \log^3 X) .$$

(iii) From (i), it follows that

$$\sum_{X < p_n \leqslant 2X} \frac{d_n^2}{p_n} (\log p_n)^{-\lambda} \leqslant \frac{1}{X(\log X)^{\lambda}} \sum_{X < p_n \leqslant 2X} d_n^2 \leqslant \frac{A}{(\log X)^{\lambda-3}} .$$

Upon replacing X by $2^r X$ for $r = 1 , 2 , \ldots$ and adding, we obtain

$$\sum_{p_n > X} \frac{d_n^2}{p_n} (\log p_n)^{-\lambda} \leqslant A \sum_{r=1}^{\infty} (\log 2^r X)^{3-\lambda} = O(\sum_{r=1}^{\infty} r^{3-\lambda})$$

and this latter series is convergent if $\lambda > 4$.

Proof of theorem 3. - In this "exposé" lack of space prevents me from giving all technical details, so I shall only outline the proof and refer the reader to ELLISON [4] for a complete account. The basic idea behind the proof is quite simple. One starts from the well known formula :

$$\theta(x) = \frac{1}{2\pi i} \int_{(c)} \frac{Z^*(S)}{S} x^S \, dS ,$$

where $Z^*(S) = \sum_p (\log p) p^{-S}$, and (c) denotes the line $c + it$, $c > 1$.

Now, being completely formal, we move the line of integration to $1/2 + z + it$, where z will be chosen later, and encounter a pole at $S = 1$ with residue x. Taking a difference, we have

$$\theta(x + \omega x) - \theta(x) - \omega x = \frac{1}{2\pi i} \int_{(\frac{1}{2}+z)} \frac{Z^*(S)}{S} \{(1 + \omega)^S - 1\} x^S \, dS ,$$

thus

$$\frac{\theta(x + \omega x) - \theta(x) - \omega x}{x^{\frac{1}{2}+z}} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{Z^*(\frac{1}{2} + z + it)}{(\frac{1}{2} + it)} \{(1 + \omega)^{\frac{1}{2}+z+it} - 1\} x^{it} \, dt .$$

We now observe that the L. H. S. of the above equation is the formal Fourier transform of the R. H. S. From the Parseval inequality, we have

$$\int_0^\infty \{\frac{\theta(x+\omega x)-\theta(x)-\omega x}{x^{\frac{1}{2}+z}}\}^2 \frac{dx}{x} \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\frac{Z^*(\frac{1}{2}+z+it)}{(\frac{1}{2}+z+it)} \{(1+\omega)^{\frac{1}{2}+z+it} - 1\}|^2 \, dt .$$

In fact, the above inequality does hold, but the rigorous argument, which closely parallels the above formal manipulations, starts not with $\theta(x)$ but with a more artificial function which approximates to $\theta(x)$. However, assuming that the inequality has been proved, we see

$$\int_0^\infty \{\frac{\theta(x+\omega x)-\theta(x)-\omega x}{x^{\frac{1}{2}+z}}\}^2 \frac{dx}{x} \geq \omega^2 \int_0^X \{\frac{\theta(x+\omega x)-\theta(x)}{\omega x} - 1\}^2 \frac{dx}{x^{2z}} \geq \frac{\omega^2}{X^{2z}} \int_0^X \{\frac{\theta(x+\omega x)-\theta(x)}{\omega x} - 1\}^2 \, dx$$

and so

$$\int_0^X \{\frac{\theta(x+\omega x)-\theta(x)}{\omega x} - 1\}^2 \, dx \leq \frac{X^{2z}}{2\pi\omega^2} \int_{-\infty}^{+\infty} |\frac{Z^*(\frac{1}{2}+z+it)}{(\frac{1}{2}+z+it)} \{(1+\omega)^{\frac{1}{2}+z+it} - 1\}|^2 \, dt .$$

Now we consider the integral on the R. H. S. of the above inequality. First of all, we note that

$$|(1 + \omega)^S - 1| = |\int_1^{1+\omega} Su^{S-1} \, du| \leq |S| \, \omega$$

and

$$|(1 + \omega)^S - 1| \leq (1 + \omega)^\sigma + 1 \leq 3 ,$$

since $\omega < 1$ and $\sigma < 1$. Thus, upon spliting the range of integration $(-\infty , +\infty)$ to the three parts $(-\infty , -T]$, $[-T , +T]$, (T , ∞) and using the first estimate in the middle range and the second estimate in the end ranges we obtain as an upper bound for the integral :

$$18 \int_T^\infty |\frac{Z^*(\frac{1}{2} + z + it)}{\frac{1}{2} + z + it}|^2 \, dt + 2\omega^2 \int_0^T |Z^*(\tfrac{1}{2} + z + it)|^2 \, dt .$$

It is now a relatively straightforward technical lemma to show that

$$\int_T^\infty |\frac{Z^*(\frac{1}{2} + z + it)}{\frac{1}{2} + z + it}|^2 \, dt = O(\frac{1}{Tz^2})$$

and

$$\int_0^T |Z^*(\tfrac{1}{2} + z + it)|^2 \, dt = O(\frac{T}{z^2}) .$$

Thus we now have

$$\int_0^X \{\frac{\theta(x + \omega x) - \theta(x)}{\omega x} - 1\}^2 \, dx = O(\frac{X^{2z}}{\omega^2 Tz^2} + \frac{X^{2z} T}{z^2}) ,$$

and if we choose $T = 3/\omega$ and $z = 4/\varepsilon \log X$, the upper bound becomes $O(\log^2 X/\omega)$, which completes our outline of the proof of theorem 3 .

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William J. ELLISON
Université de Bordeaux-I
U. E. R. de Mathématiques
351 cours de la Libération
33405 TALENCE