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by Wladyslaw NARKIEWICZ

1. - F. FOGELS (1943, [2]) has shown that in $Q(\sqrt{-5})$, the simplest quadratic field with non-trivial class-group, almost no algebraic integer has unique factorization, that is to say, if $F(x)$ denotes the number of non-associated integers α with $|N(\alpha)| \leqslant x$, which have unique factorization, then $F(x)/x$ tends to zero.

He proved also, that the same holds for the number $H(x)$ of natural numbers $n \leqslant x$ with unique factorization in $Q(\sqrt{-5})$.

In fact, analogous results are true for all fields with non-trivial class-groups, as shown in [4],[6]. For $F(x)$, one gets evaluations of the form

$$F(x) \ll x/\log^{\alpha} x \ , \quad \alpha > 0 \ ,$$

whereas similar evaluations for $H(x)$ are obtained only for K/Q normal.

So we get the first question.

QUESTION I. - Is the evaluation $H(x) \ll x/\log^{\alpha} x$ $(\alpha = \alpha(K) > 0)$, true for all fields K with non-trivial class-group ?

2. - In 1960, L. CARLITZ [1] observed, that in an algebraic number field K with the class-number $h(K) \geqslant 3$, one can find integers α which have factorizations of different lengths, i. e. $\alpha = \pi_1 \dots \pi_r = \rho_1 \dots \rho_s$, with π_j , ρ_i irreductible, and $r \neq s$.

If $G(x)$ is the number of non-associated integers α with $|N(\alpha)| \leqslant x$, whose all factorizations are of the same length, then again (see [4]) one has

$$G(x) \ll x/\log^{\beta} x \quad (\beta = \beta(K) > 0) \ ,$$

and the analogue for natural numbers holds, provided K/Q is normal [5]. For non-normal K/Q , only $o(x)$ is proved at this moment [6].

So we have the second question.

QUESTION II. - Obtain the evaluation $O(x/\log^{\beta} x)$ $(\beta > 0)$ for natural numbers having all factorizations of the same length, in a given field K with $h(K) \geqslant 3$.

3. - We shall now indicate the main points of the proof of the following theorem.

THEOREM. - If $h = h(K) \geqslant 2$, then

$$F(x) = (C + o(1)) \frac{x(\log \log x)^M}{(\log x)^{1-(1/h)}} \ ,$$

where M is the maximal number of non-principal prime ideals, which can occur in a factorization of a number counted by $F(x)$ with the exponent one.

Let $X_1, \ldots, X_{h-1}$ be the non-principal ideal classes in K. If I is any ideal, without principal prime ideal factors, write it in the form

$$I = \prod_i (\mathfrak{p}_{i1}^{\alpha_{i1}} \cdots \mathfrak{p}_{ik_i}^{\alpha_{ik_i}}).Q \ , \quad 1 \leq i \leq h-1 \ ,$$

with $\mathfrak{p}_{ij} \in X_i$, $\alpha_{ij} > 0$.

We say, that the system

$$\tau = \tau(I) = \langle \{\alpha_{11}, \ldots, \alpha_{1k_1}\}, \ldots, \{\alpha_{h-1,1}, \ldots, \alpha_{h-1,k_{h-1}}\} \rangle$$

is the type of I. If I has no prime divisors from a class X_i say, then we write $\emptyset$ in the place of $\{\alpha_{i1}, \ldots\}$.

For a given type τ, let $d(\tau) = \#\{\alpha_{ij} = 1\}$ be its depth.

The proof of the theorem is based on the following result.

PROPOSITION. - Let $\mathfrak{A}$ be any set of principal ideals subject to the following conditions :

(i) $I \in \mathfrak{A}$, $\tau(I) = \tau(J) \Longrightarrow J \in A$;

(ii) If all prime ideal factors of I are principal, then $I \in \mathfrak{A}$;

(iii) $\exists B$, $I \in \mathfrak{A} \Longrightarrow d(\tau(I)) \in B$, whenever $\tau(I)$ is defined.

Then

$$\#\{I = N(I) \leq x \ ; \ I \in \mathfrak{A}\} = (\underset{\sim}{C} + o(1)) \frac{x(\log\log x)^M}{(\log x)^{1-(1/h)}}$$

with $\underset{\sim}{C} = \underset{\sim}{C}(\mathfrak{A}) > 0$ and $M = \max\{d(\tau(I)) \ ; \ I \in \mathfrak{A}\} \leq B$.

This implies immediately the theorem, as if α has a unique factorization, then α has at most $2h-1$ different prime ideal factors from a given class. Indeed, if it has $\geqslant 2h$, say $\mathfrak{p}_1, \ldots, \mathfrak{p}_{2h}, \ldots$ then

$$(\mathfrak{p}_1 \cdots \mathfrak{p}_h)(\mathfrak{p}_{1+h} \cdots \mathfrak{p}_{2h}) = (\mathfrak{p}_{1+h}\,\mathfrak{p}_2 \cdots)(\mathfrak{p}_1\,\mathfrak{p}_{2+h} \cdots) .$$

The proof of the proposition is based on the tauberian theorem of DELANGE.

One starts with the following lemma.

LEMMA. - If $X \in \mathfrak{X}(K)$, and $F_1(t), \ldots, F_n(t)$ are real and

$$0 < F_i(t) \ll t^{-2} \ , \quad d \geqslant 1 \ ,$$

for $\operatorname{Re} s > 1$ write

$$S(s) = \sum_{\mathfrak{p}_1,\ldots,\mathfrak{p}_\alpha} 1/(N\mathfrak{p}_1^s \ldots N\mathfrak{p}_\alpha^s) \sum_{\mathfrak{q}_1,\ldots,\mathfrak{q}_n} F_1(N\mathfrak{q}_1^s) \ldots F_n(N\mathfrak{q}_n^s) ,$$

with $\mathfrak{p}_1 , \ldots , \mathfrak{p}_\alpha \in X$, and distinct ; $\mathfrak{q}_1 , \ldots , \mathfrak{q}_n \in X$, distinct, and $\mathfrak{q}_i \neq \mathfrak{p}_j$. then $S(s) = P(\log(1/s - 1))$, $P \in \Omega[X]$, Ω ring of funtions regular in $\mathrm{Re}\ s \geqslant 1$, $\deg P = d$, and the leading coefficient is positive at $s = 1$.

Proof. - Induction in d . This allows to show, that if τ is given, and

$$S_\tau = \{I : I = I_1 I_2 , \tau(I_1) = \tau , I_2 \text{ has all prime factors principal}\} ,$$

then

$$\sum_{I \in S_\tau} N(I)^{-s} = (b(\log \frac{1}{s-1}))/(s-1)^{1/h} ,$$

$b \in \Omega[X]$, $\deg \delta = d(\tau)$, leading coefficient of b positive at 1 , and in fact the same result holds if we sum up this equality over any set of τ with $d(\tau)$ fixed.

There are also another applications of our proposition. Using it in the case $K = Q$, one regains a theorem of L. MIRSKY [3] :

$$\pi\{n \leqslant x : d(n) = k\} = (\underset{\sim}{C} + o(1)) \frac{x^{1/(p-1)}(\log \log x)^m}{\log x} ,$$

whose $p = \min\{p : p | k\}$, $p^m \| k$.

As well as analogues of this for all multiplicative functions with $f(p^t) = a_k$.

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