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SOME CONSEQUENCES OF A KIND OF HAHN-BANACH'S THEOREM

by Richard BECKER

Abstract. - The aim of this work is to give some consequences of a theorem of H. DINGES used by M. F. SAINTE-BEUVE.

Preliminaries

1. THEOREM. - Let X be an ordered vector space, and p an extended sub-linear functional on X, such that $p(x) \in R \cup (+\infty)$ for each $x \in X$, and $p(x) \leqslant 0$ for each $x \leqslant 0$. Let Y a linear subspace of X, and f a linear form on Y majorized by p. There exists a linear form on $Z = \{x \; ; \; \exists \, x_1 , x_2 \in Y$ with $x_1 \leqslant x \leqslant x_2\}$ which extends f and is majorized by p on Z ([5], [11]).

What is needed concerning conical measures can be found in [3] (§ 30, 38, 40). Notation not included in [3]. In this paper, $\mathcal{C}$ will be the class of weakly complete convex cones, not necessarily proper.

2. Summary. - Part I is devoted to conical measures. We generalize specially (proposition 12) the theorem of Cartier-Fell-Meyer ([10] p. 112) concerning dilatations of measures on a metrizable convex compact set. Positive measures on a metrizable convex compact set can be considered as conical measures on a proper convex closed cone of R^N. Here, we will consider arbitrary conical measures on R^N.

Part II (A) extends a result of STRASSEN ([8], p. 300-301), from which the theorem of Cartier-Fell-Meyer can be derived. We weaken, here, a condition of compactness (proposition 21). Part II (B) extends some results about "theory of balayage" ([8], p. 294, 297). This theory studies cones of continuous functions on a compact set containing a strictly positive function. We weaken this condition.

Part I : The case of conical measures.

I (A). Conical measures on an arbitrary weak space.

Recall the following proposition which enlightens the definition of the order $\prec$.

3. PROPOSITION. - Let E a complete weak space, and $\Gamma \subset E$ a convex cone of $\mathcal{C}$. For each $f \in h(E)$, such that $f_{|\Gamma}$ is sub-linear, there exist $\ell_1 , \ell_2 , \ldots , \ell_n \in E'$, such that we have on Γ, $f = \text{lub}(\ell_1 , \ell_2 , \ldots , \ell_n)$.

Proof. - We can suppose E of finite dimension.

There exist $u_1, \dots, u_p \in E'$ such that, for each $x \in E$, $f(x)$ is equal to one of the $u_p(x)$. Hence, for each pair $x, y \in \Gamma$, there exist $p_{x,y}$, an integer $\leqslant p$, such that

$$f(x) = u_{p_{x,y}}(x) \text{ , and } f(y) \geqslant u_{p_{x,y}}(y) .$$

For each $x \in \Gamma$, let $v_x = g\ell b_{y\in\Gamma}(u_{p_{x,y}})$. The family $(v_x)_{x\in\Gamma}$, is finite, and we have on Γ $f = \text{lub}_{x\in\Gamma}(v_x)$; as $(-v_x) \in S(E)$, we can conclude with the help of the elementary form of the theorem of Hahn-Banach because dimension of $E < \infty$.

4. PROPOSITION. - If E is a complete weak space, and $\mu \in M^+(E)$, then, for each $\ell \in E'$ with $\ell \neq 0$, the two following properties are equivalent.

1° $\forall f \in h^+(E)$, we have $\mu(f) = \lim(\mu(f \wedge n\ell^+))$ when $n \longrightarrow \infty$.

2° $\exists\, m$, σ-additive and positive functional on the tribe on $e = \ell^{-1}(1)$ generated by $h(E)|_e$, such that $\mu(f) = m(f|_e)$, for each $f \in h(E)$.

If μ satisfies to 1° and 2°, then each $\lambda \in M^+(E)$, with $\lambda < \mu$, satisfies also to 1° and 2°.

Proof. 1° and 2° are equivalent on account of ([3], 38.13).
Proof that λ satisfies to 1°. Note that $h(E) = S^+(E) - S^+(E)$. Let $f \in S^+(E)$, we have

$$0 \leqslant \lambda(f - f \wedge n\ell^+) \leqslant \lambda((f - n\ell)^+) \leqslant \mu((f - n\ell)^+) \longrightarrow 0 \quad \text{when} \quad n \longrightarrow \infty ,$$

hence

$$\lambda(f) = \lim(\lambda(f \wedge \ell^+)) \quad \text{when} \quad n \longrightarrow \infty .$$

5. PROPOSITION. - Suppose E is a weak space, and $\lambda, \mu \in M^+(E)$. If $\lambda < \mu$, then, for each sequence $\lambda_1, \lambda_2, \dots, \lambda_n$ of $M^+(E)$ such that $\lambda = \sum_1^n \lambda_i$, there exists a sequence $\mu_1, \mu_2, \dots, \mu_n$ of $M^+(E)$ such that $\mu = \sum_1^n \mu_i$, and $\lambda_i < \mu_i$ for $i = 1, 2, \dots, n$.

Proof. - For each $f \in h(E)$, let $\hat{f}$ such that :

1° $\hat{f} = g\ell b(\ell ;\ \ell \in E'$ and $\ell \geqslant f)$ if f is majorized by an element of E'. (In fact, on account of [1] (chap. II, § 7, exercice 24), we have $-\hat{f} \in S(E)$.)

2° Otherwise, $\hat{f} \equiv +\infty$ on E.
For each $\nu \in M^+(E)$, let p_ν such that :

1° If $\hat{f} \neq \infty$, $p_\nu(f) = g\ell b(\nu(g) ;\ -g \in S(E) ,\ g \geqslant f)$. We have $p_\nu(f) \in R$.

2° If $\hat{f} \equiv +\infty$, $p_\nu(\hat{f}) = +\infty$.
For $i = 1, 2, \dots, n$, let $p_i = p_{\nu_i}$.
On the space $(h(E))^n$, let us consider the functional p, such that

$$(f_i)_{1\leqslant i\leqslant n} \longmapsto p((f_i)) = \sum_1^n p_i(f_i) .$$

p is sub-linear with values in $R \cup (+\infty)$, and

$$(f_i \leq 0 \text{ , for } i = 1 , 2 , \dots , n) \Longrightarrow (p((f_i)) \leq 0) .$$

Let $\bar{\mu}$ the linear form on the diagonal of $(h(E))^n$, such that

$$\bar{\mu}((f , f , \dots , f)) = \mu(f) .$$

$\bar{\mu}$ is majorized by p . As each element of $(h(E))^n$ is majorized by an element of the diagonal, we can apply the version of the theorem of Hahn-Banach recalled in 1. $\bar{\mu}$ has an extension $\tilde{\mu} \in (h(E)^n)^*_+$ with $\tilde{\mu} \leq p$. We can write $\tilde{\mu} = (\mu_i)_{1 \leq i \leq n}$ with $\mu_i \in h(E)^*_+$, for $i = 1 , 2 , \dots , n$.
The μ_i are convenient.

6. PROPOSITION. - Suppose E is a complete weak space, and λ , μ $M^+(E)$. The two following properties are equivalent.

1° $\lambda \prec \mu$.

2° There exists a conical measure $\pi \in M^+(M^+(E) \times M^+(E))$ carried by the cone $B = \{(\varepsilon_x , \nu) ; x \in E \text{ and } \varepsilon_x \prec \nu\}$, such that $r(\pi) = (\lambda , \mu)$.

Proof. - For simplification, we will write sometimes M instead of $M(E)$ and M^+ instead of $M^+(E)$.
1° $\Longrightarrow$ 2° : For each sequence $\lambda_1 , \lambda_2 , \dots , \lambda_n$ satisfying the hypothesis of proposition 5, let us choose a sequence $\mu_1 , \mu_2 , \dots , \mu_n$ satisfying the conclusion of 5.
We say that a sequence $s' = \lambda'_1 , \lambda'_2 , \dots , \lambda'_m$ is finer than a sequence $s = \lambda_1 , \lambda_2 , \dots , \lambda_n$ if, and only if, there exists a partition of $\{1,2,\dots,m\}$ into n subsets $p_1 , p_2 , \dots , p_n$, such that $\lambda_i = \sum_{j \in p_i} \lambda'_j$ for $i = 1,2,\dots,n$.
Let U_s the set consisting of all the (finite) sequences finer than s . The family of sets U_s is a filter basis over $U_{(\lambda)}$ where (λ) means the sequence λ .
Let φ be the application

$$s \longmapsto \varphi(s) = \pi_s = \sum_1^n \varepsilon(\varepsilon_{r(\lambda_i)} , \mu_i) ,$$

we have $\pi_s \in M^+(M^+ \times M^+)$. The family of sets $\varphi(U_s)$ is a filter basis over $M^+(M^+ \times M^+)$. We have $r(\pi_s) = \sum_1^n (\varepsilon_{r(\lambda_i)} , \mu_i)$.
Each element of $h(E)^+$ is majorized by an element of $S(E)^+$, and for each $f \in S(E)^+$, we have

$$\sum_1^n (\varepsilon_{r(\lambda_i)})(f) = \sum_1^n f(r(\lambda_i)) \leq \sum_1^n \lambda_i(f) = \lambda(f) .$$

Hence the filter basis $\varphi(U_s)$ has at least a cluster point, let π . The element π answers the question, since each π_s is carried by B , and we have

$$r(\pi) = \lim(r(\pi_s)) = (\lambda , \mu) .$$

2° $\Longrightarrow$ 1° : If $\pi \in M^+(M^+ \times M^+)$ with $r(\pi) = (\lambda , \mu)$, and if π is carried by B , then for each $f \in S(E)$, we have $\pi((- f , f)) \geq 0$, since the element $(- f , f)$

of $h(E) \times h(E)$ is $\geqslant 0$ on B . Hence we have $\lambda \prec \mu$.

7. Remark. - We can prove 6 with the method of [10] (p. 108) (and without the theorem of § 1) by looking at the convex closure of the set

$$\{(\varepsilon_x , \nu) \; ; \; x \in E \text{ and } r(\nu) = x\}$$

in $M^+ . M^+$. Then § 5 can be obtained for R^n as in [10] (p. 112) and in the general case by a projective limit argument.

8. Definition (of a pure pair and a pure measure). - Suppose $\lambda , \mu \in M^+(E)$. We say the pair (λ , μ) is pure if, ans only if,

$$(\mu' \in M^+(E) \text{ and } \mu' \leqslant \mu , \; \lambda \prec \mu') \text{ involves } (\mu' = \mu) .$$

Suppose $\lambda \in M^+(E)$. We say that λ is pure, when the two following equivalent condition are fulfilled.

1° $(\varepsilon_{r(\lambda)} , \lambda)$ is a pure pair.

2° K_λ admits 0 as an extremal point.

Proof.

1° $\Longrightarrow$ 2°: Suppose 2° is false. Let $\lambda_1 \leqslant \lambda$, and $\lambda_2 \leqslant \lambda$ with $r(\lambda_1) = - r(\lambda_2) \neq 0$. If $\lambda_0 = \lambda - (\lambda_1 + \lambda_2)/2$, we have $0 \leqslant \lambda_0 \leqslant \lambda$, $\lambda_0 \neq \lambda$, and $r(\lambda_0) = r(\lambda)$, then $(\varepsilon_{r(\lambda)} , \lambda)$ is not a pure pair.
2° $\Longrightarrow$ 1°: Suppose $\mu \leqslant \lambda$ with $\mu \geqslant 0$ and $r(\mu) = 0$. Let $\mu = \mu_1 + \mu_2 + \dots + \mu_n$ be any decomposition of μ where $\mu_i \geqslant 0$. We have $\mu_i \in K_\lambda$, and $\sum_{1 \leqslant i \leqslant n} r(\mu_i) = 0$, hence $r(\mu_i) = 0$ for $i = 1 , 2 , \dots , n$. Then $\mu = 0$.

9. Example. - In the cartesian product R^2 , let a , b , c , d be the consecutive vertices of a square of center 0 . If

$\lambda = \varepsilon_a + \varepsilon_b + \varepsilon_c + \varepsilon_d + \varepsilon_{-(c+d)}$,

$\lambda_1 = \varepsilon_a + \varepsilon_b + \varepsilon_c + \varepsilon_d$,

$\lambda_2 = \varepsilon_c + \varepsilon_d + \varepsilon_{-(c+d)}$,

we have $r(\lambda_1) = r(\lambda_2) = 0$, and $(\lambda - \lambda_1)$, $(\lambda - \lambda_2)$ are pure.

10. PROPOSITION. - Suppose E is a complete weak space, and $\lambda , \mu \in M^+(E)$ with $\lambda \prec \mu$. Then, the three following properties are equivalent.

1° The pair (λ , μ) is pure.

2° For each $\pi \in M^+(M^+ \times M^+)$, representing (λ , μ) according to § 6 and carried by the cone B , then the restriction π_0 of π to the cone $A = \{(0 , \nu) \; ; \; \nu \in M^+(E) \text{ and } r(\nu) = 0\}$ is equal to zero.

3° Each $\pi \in M^+(M^+ \times M^+)$ representing (λ , μ) , and carried by the cone B , is carried by the cone $B_p = \{(\varepsilon_x , \nu) \; ; \; x \in E , \; r(\nu) = x , \; \nu \text{ is pure}\}$.

Proof.

$1° \Longrightarrow 2°$: Suppose 2° is false. If π represents (λ , μ) with $\pi_0 \neq 0$ (for the definition of π_0 , see [3], 30.8), we have $r(\pi_0) = (0 , \nu)$ with $\nu \neq 0$ and $r(\nu) = 0$.
For each $f \in S(E)$, we have $(- f , f) \geq 0$ on B . Hence

$$\nu(f) = \pi_0((- f , f)) \leq \pi((- f , f)) = - \lambda(f) + \mu(f)$$

Therefore $\lambda < \mu - \nu$, and (λ , μ) is not pure.
$2° \Longrightarrow 3°$: We can write $\pi = \lim_{\mathcal{U}} \sum \varepsilon_{(\varepsilon_x , \nu)}$ with $(\varepsilon_x , \nu) \in B$ where $\mathcal{U}$ is an ultrafilter.
For each $\nu \in M^+(E)$, let us choose $p_\nu \in M^+(E)$ such that :

(a) p_ν is pure ,

(b) $p_\nu \leq \nu$,

(c) $k.p_\nu = p_{k.\nu}$ for any $k \geq 0$.

We will prove that $\pi = \lim_{\mathcal{U}}(\sum \varepsilon_{(\varepsilon_x , p_\nu)})$.
We have

$$(\lambda , \mu) = (\lambda , \lim_{\mathcal{U}}(\sum_{p_\nu})) + (0 , \lim ((\nu - p_\nu)) .$$

On account of the hypothesis, we have $\lim_{\mathcal{U}} \sum(\nu - p_\nu) = 0$, hence $\lim_{\mathcal{U}}(\sum \varepsilon_{(0,\nu-p_\nu)}) = 0$. For each $f \in S(M \times M)$, we have

$$f(\varepsilon_x , p_\nu) - f(0 , p_\nu - \nu) \leq f(\varepsilon_x , \nu) \leq f(0 , \nu - p_\nu) + f(\varepsilon_x , p_\nu) .$$

As we have $\lim_{\mathcal{U}}(\sum \varepsilon_{(0,\nu-p_\nu)}) = 0$, then $\pi(f) = \lim_{\mathcal{U}}(\sum f(\varepsilon_x , p_\nu))$. Therefore π is carried by B_p .
$3° \Longrightarrow 1°$: Suppose 1° is false. We have $(\lambda , \pi) = r(\varepsilon_{(\lambda,\alpha)} + \varepsilon_{(0,\beta)})$ with (λ , α) pure, and $(0 , \beta) \in A$ with $\beta \neq 0$. Therefore $\varepsilon_{(0,\beta)}$ is not carried by B_p .

11. <u>Example</u> (G. CHOQUET). - In R^2 suppose C_1 and C_ρ are the circles (for the classical distance) of center 0 with radius 1 and $\rho > 1$. For each $x \in C_1$, let $x_1 , x_2 \in C_\rho$ so that (x_1 , x_2) is tangent to C_1 at x . Let dx be the Haar measure on C_1 . We have

$$\int_{C_1} (\varepsilon_{x_1} + \varepsilon_{x_2}) \, dx = \rho' \int_{C_1} \varepsilon_x \, dx \quad (\text{with } \rho' > 1)$$

as conical measures.
The pair $(\varepsilon_x , \varepsilon_{x_1} + \varepsilon_{x_2})$ is pure for each $x \in C_1$, but the resultant of $\int_{C_1} \varepsilon_{(\varepsilon_x , \varepsilon_{x_1}+\varepsilon_{x_2})} \, dx$ is the pair $(\int_{C_1} \varepsilon_x \, dx , \rho' \int_{C_1} \varepsilon_x \, dx)$ which is not pure since $\rho' > 1$.

I (B). <u>Conical measures on</u> R^N <u>or</u> R^n .

12. PROPOSITION. - <u>Suppose</u> $\lambda , \mu \in M^+(R^N)$ <u>with</u> $\lambda < \mu$, <u>and the pair</u> (λ , μ) <u>is pure. Then, there exist</u> :

1° a K_σ of $(R^N \backslash 0)$, let X such that each half-line issued from 0 intersects X into at most one point,

2° a Radon measure Λ on X,

3° a Borel application $x \longmapsto \mu_x$ defined on X where μ_x is a Radon measure on X such that $r(\mu_x) = x$.

And we have :

(a) Λ is a localization of λ (Note that Λ is unique when X is given).

(b) $\mu = \int_X \mu_x \, d\Lambda(x)$.

Proof (with the notations of the proof of § 6). – We had

$$\pi_s = \sum_1^n \varepsilon_{(\varepsilon_{r(\lambda_i)}, \mu_i)} .$$

For each $n \in N$, let x_n be the function n-th coordinate on R^N. We have

$$\pi_s((|x_p| , |x_p|)) \leq \lambda(|x_p|) + \mu(|x_p|) \leq 2\mu(|x_p|) .$$

Let ℓ be the affine l. s. c. function defined on $M^+ \times M^+$ by

$$\ell(\alpha , \beta) = \sum_p((\alpha + \beta)|x_p|)/2^{p+1} \mu(|x_p|) .$$

We have

$$\ell(r(\pi_s)) = \sum_p \pi_s(|x_p| , |x_p|)/2^{p+1} \mu(|x_p|) \leq \sum 1/2^p \leq 1 .$$

π has a localization by a Radon measure m on a cap K of $M^+ \times M^+$, with $K = \{(\alpha , \beta) ; \alpha , \beta \in M^+ , \text{ and } \ell(\alpha , \beta) \leq 1\}$. Moreover m can be assumed to be carried by the cone B.

Let Ψ be the l. s. c. function defined on R^N by $\Psi(x) = \ell(\varepsilon_x , \varepsilon_x)$. For each $n \in N$, let $K_n = \{(\varepsilon_x , \alpha) ; (\varepsilon_x , \alpha) \in K , \text{ and } 1/(n + 1) < \Psi(x) \leq 1/n\}$. Let m_n be the restriction of m to K_n. We have $m = \sum m_n$ on account of § 10, since (λ , μ) is a pure pair. Let π_n be the conical measure on $M^+ \times M^+$ localized by m_n.

Let m_n' be the Radon measure on $(n + 1)K$ such that, for each continuous function f on $(n + 1)K$, we have

$$m_n'(f) = \int_K \Psi(x) \, f(\varepsilon_x/\Psi(x) , \alpha/\Psi(x)) \, dm_n(\varepsilon_x , \alpha) ,$$

then m_n' localizes π_n.

Suppose p is the projection on the first factor of the product $M^+ \times M^+$, then $p(m_n')$ is carried by $\mathcal{K} = \{\varepsilon_x ; x \in R^N \text{ with } \Psi(x) = 1\}$. $\mathcal{K}$ is a Borel set because Ψ is l. s. c., moreover $\mathcal{K}$ intersects each half-line issued from 0 in at most one point.

Suppose $x \longmapsto m_x^n$ is a disintegration of m_n' with respect to p ([2], p. 58). Then each m_x^n has a resultant which is a conical measure ν_x^n on R^N, and we have $\mu(\nu_x^n) = x$.

Now $\Lambda = \sum_n p(m_n')$ can be seen as a Radon measure on a K_σ subset X_λ of $\tilde{\mathcal{K}}$. We

can write, for each $n \in N$, $p(m'_n) = u_n \Lambda$ where u_n is a Borel function on X_λ. We have $\sum_n u_n = 1$, Λ-a.e.
Recall that Λ represents λ, and that $\mu = \sum_n \int \nu_x^n \, d(p(m'_n))$ (equality of conical measures), then we have $\mu = \int (\sum_n u_n \nu_x^n) \, d\Lambda$. Therefore $\mu_x = \sum_n u_n \nu_x^n$ exists as a conical measure Λ-a.e., and we have $r(\mu_x) = x$.

On account of [3] (38.8), there exists a compact subset H of R^N with $H = \prod_1^\infty(-k_n, k_n)$ where $k_n > 0$, such that μ is localizable on H by a Radon measure.
For simplification we shall use the same notation for μ, and its unique ([7], prop. 2.13) localization on the set $E(H) = \{x ; x \in H, \forall k > 1, kx \notin H\}$.
As μ_x is a Daniell integral on $h(R^N)$ ([3] 38.13), and since

$$E(H) = \{x ; \text{lub}(|x_n|/k_n) = 1\},$$

then ([9] prop. II.7.1) μ_x can be extended to a σ-additive measure, called also μ_x for simplification, on the tribe $\mathcal{T}$ of $E(H)$ generated by the closed half-spaces containing 0. Recall we know that, for each $f \in h(R^N)$, the map $x \longmapsto \mu_x(f)$ is Borel-measurable. Then, for each $e \in \mathcal{T}$, we have $\mu(e) = \int \mu_x(e) \, d\Lambda$.
Let X_μ be a K_σ subset of $E(H)$ which bears μ. In order to show that μ_x lives on X_μ for Λ-a.e.x, it is sufficient to prove the following lemma.

13. LEMMA. - Each compact subset A of $E(H)$ is a member of $\mathcal{T}$.

Proof. - Let us suppose the sequence $(\omega_n)_{n\in N}$ is a basis of open subsets of R^N.
Let Σ be the subset of N such that $n \in \Sigma$ if, and only if, there exists $h \in h^+(E)$, with $h = 0$ on A, and $h > 0$ on ω_n. For each $n \in \Sigma$, we choose $h_n \in h^+(E)$, with $h_n = 0$ on A, and $h_n > 0$ on ω_n.
Let us show that, for each $x \notin R^+ A$, we have $h_n(x) > 0$ for at least one $n \in \Sigma$.
For each $y \in A$, there exists $h_y \in E'$ with $h_y(x) > 0$, and $h_y(y) < 0$. By compacity, there exists $h_x \in h^+(E)$, with $h_x(x) > 0$, and $h_x = 0$ on A. As the set $\{z ; h_x(z) > 0\}$ is open, then there exists $n \in N$ such that $x \in \omega_n$ and $h_x > 0$ on ω_n. Therefore, we have $n \in \Sigma$ and $h_n(x) > 0$.
Now, if we let $h = \text{lub}_{n\in\Sigma}(h_n)$, then we have $h = 0$ on A, and $h(z) > 0$ for each $z \notin R^+ A$. Hence $A \in \mathcal{T}$.
Now, it is easy to complete the proof of § 12 by a mixture of X_λ and X_μ.

14. Remark (M. F. SAINTE BEUVE [11], theorem 3). - In the case of R^n, we can take the unit sphere of R^n (for the usual distance) for X.

15. Example (Answer to a question of G. CHOQUET). - Let $\mathfrak{M}$ be the set of Radon measures on $[0, 1]$, and $\mathfrak{M}_1^+$ the subset of probability measures.
Let E the vector subspace of $\mathfrak{M}$ generated by the Dirac probabilities, E is equiped with the weak*-topology.
Suppose μ is the maximal measure on $\mathfrak{M}_1^+$ which represents the element $dx \in \mathfrak{M}_1^+$.

The measure μ and dx induce, in a canonical way, elements of $M^+(E)$, $\tilde{\mu}$ and ε_{dx} , since $E \cap \mathcal{M}_1^+$ is dense in $\mathcal{M}_1^+$.

Let φ be the canonical injection from $[0 , 1]$ into $\mathcal{M}$, and $X = \varphi([0 , 1])$. We have $\varepsilon_{dx} < \tilde{\mu}$ (in fact, $\varepsilon_{dx} = \varepsilon_{r(\tilde{\mu})}$ in the weak completion of E), however $\tilde{\mu}$ has a localization on the compact subset X of E , while ε_{dx} does not have such a localization.

Part II : Extension of a result of STRASSEN and "theory of balayage".

II (A). Extension of a result of STRASSEN.

16. Notations and definitions. - Suppose X and Y are two compacts(HAUSDORFF) spaces and $x \longmapsto M_x$ is a mapping of X in the set of closed convex subsets of $\mathcal{M}^+(Y)$ (positive Radon measures on Y).
For each $f \in \mathcal{C}(Y)$ (continuous real functions on Y), we let

$$\forall\ x \in X\ ,\quad \hat{f}(x) = \text{lub}_{\nu \in M_x}(\nu(f))\ ,$$

we have $\hat{f}(x) \in \bar{R}$, and $(\hat{f}(x) = -\infty) \Longleftrightarrow (M_x = \emptyset)$.
The map $f \longmapsto \hat{f}$ has been previously considered by P.-A. MEYER ([8], p. 301).
Suppose $\lambda \in \mathcal{M}^+(X)$. For each function φ on X with values in $\bar{R}$, we let

$$\lambda^*(\varphi) = g\ell b(\lambda(u)\ ;\ \ u \geqslant \varphi\ ,\ \ u\ \text{l. s. c. on}\ X\ ,\ \text{with values in}\ R \cup (+\infty)).$$

We have $\lambda^*(\varphi) \in \bar{R}$.
If $\lambda \in \mathcal{M}^+(X)$ and $\mu \in \mathcal{M}^+(Y)$, we write $\lambda < \mu$ if, and only if, for each $f \in \mathcal{C}(Y)$, we have $\mu(f) \leqslant \lambda^*(\hat{f})$. We let $p_\lambda(f) = \lambda^*(\hat{f})$.

17. PROPOSITION. - Suppose $\lambda \in \mathcal{M}^+(X)$ and $\mu \in \mathcal{M}^+(Y)$ with $\lambda < \mu$. For each sequence $\lambda_1 , \dots , \lambda_n$, such that $\lambda = \lambda_1 + \dots + \lambda_n$ with $\lambda_i \geqslant 0$, there exists a sequence $\mu_1 , \dots , \mu_n$ with $\mu_i \geqslant 0$, such that $\mu = \mu_1 + \dots + \mu_n$, and $\lambda_i < \mu_i$ for $i = 1 , 2 , \dots , n$.

Proof. - Let 1 be the constant function equal to 1 on Y .
We have $\lambda^*(\widehat{-1}) \geqslant \mu(-1) > -\infty$. Hence, for each $f \in \mathcal{C}(Y)$, we have

$$\lambda_i(\hat{f}) \in R \cup (+\infty) \quad \text{for } i = 1 , 2 , \dots , n ,$$

then we can use the same proof than in proposition 5.

18. PROPOSITION. - We let $H = \{(\varepsilon_x , \nu)\ ;\ \ x \in X\ ,\ \ \nu \in M_x\}$. If $\lambda \in \mathcal{M}^+(X)$ and $\mu \in \mathcal{M}^+(Y)$, the two following properties are equivalent

1° $(\lambda , \mu) \in \overline{\text{conv}(R^+ H)}$ in $\mathcal{M}^+(X).\mathcal{M}^+(Y)$ equiped with the weak-$*$-topology.

2° For each $f \in \mathcal{C}(Y)$, we have

$$\mu(f) \leqslant g\ell b(\lambda(g)\ ;\ \ g \in \mathcal{C}(X)\ \text{and}\ \hat{f} \leqslant g)$$

Proof. - We apply the theorem of Hahn-Banach.
Suppose $g \in \mathcal{C}(X)$ and $f \in \mathcal{C}(Y)$. Then $(g , - f)$ is in the polar of H if, and only if, $\hat{f} \leqslant g$.

1° $\Longrightarrow$ 2°: If $f \in \mathcal{C}(Y)$, we have $\lambda(g) \geqslant \mu(f)$ for each $g \in \mathcal{C}(X)$ with $\hat{f} \leqslant g$, hence 2° is fulfilled.
2° $\Longrightarrow$ 1°: For each $g \in \mathcal{C}(X)$ and each $f \in \mathcal{C}(Y)$ with $\hat{f} \leqslant g$, we have, on account of 2°, $\mu(f) \leqslant \lambda(g)$. Hence 1° is fulfilled on account of the bipolar theorem.

19. Definition of the relation $\ll$. - If $\lambda \in \mathfrak{M}^+(X)$, proposition 18 invites us to let, for each $f \in \mathcal{C}(Y)$

$$q_\lambda(f) = g\ell b(\lambda(g) \ ; \ g \in \mathcal{C}(X) \text{ and } g \geqslant \hat{f}) \ .$$

Note we have $p_\lambda \leqslant q_\lambda$. Moreover, if H is a closed subset of $\mathfrak{M}^+(X) \times \mathfrak{M}^+(Y)$, then we have $p_\lambda(- 1(y)) = q_\lambda(- 1(y))$ because $- \hat{1}(y)$ is negative, and u. s. c.

If $\mu \in \mathfrak{M}^+(Y)$, we write $\lambda \ll \mu$ if, and only if, $\mu < q_\lambda$. We have

$$(\lambda < \mu) \Longrightarrow (\lambda \ll \mu) \ .$$

Of course, we can prove the analogous of proposition 17 for the relation $\ll$. Note, in the case, study by P.-A. MEYER ([8] p. 302) (i. e. H is compact), $\hat{f}$ is u. s. c. so that $\hat{f} = g\ell b(g \ ; \ g \in \mathcal{C}(X) \ , \ g \geqslant \hat{f})$. Hence $p_\lambda = q_\lambda$.

20. PROPOSITION. - Suppose $\bar{K}$ is the closure, in $\mathfrak{M}^+(X) \times \mathfrak{M}^+(Y)$, equiped with the weak-*-topology, of the set

$$K = \{(\varepsilon_x/(1 + \nu_x(1)) \ , \ \nu_x/(1 + \nu_x(1))) \ ; \ x \in X \ , \ \nu_x \in M_x\}$$

If $\lambda \in \mathfrak{M}^+(X)$ and $\mu \in \mathfrak{M}^+(Y)$, the two following properties are equivalent :

1° $\lambda \ll \mu$,

2° There exists a positive Radon measure π on the compact set $\bar{K}$ such that $r(\pi) = (\lambda \ , \ \mu)$.

Proof.
1° $\Longrightarrow$ 2°: Each element u of $\mathrm{conv}(R^+ H)$ can be written $u = \sum_{x\in X} k_x^u(\varepsilon_x \ , \ \nu_x^u)$ where the k_x^u are unique, positive, and equal to 0 except for a finite number of $x \in X$. We have $\nu_x^u \in M_x$.
On account of § 18, there exists an ultrafilter $\mathcal{U}$ on $\mathrm{conv}(R^+ H)$ such that $\lim_{\mathcal{U}}(u) = (\lambda \ , \ \mu)$.
u is the resultant of the following conical measure π_u on $\mathfrak{M}^+(X) \times \mathfrak{M}^+(Y)$ with $\pi_u = \sum_{x\in X} a_x^u \ \varepsilon_{((b_x^u, c_x^u))}$ where

$$a_x^u = (1 + \nu_x^u(1))k_x^u \ , \quad b_x^u = \varepsilon_x/(1 + \nu_x^u(1))$$

and

$$c_x^u = \nu_x^u/(1 + \nu_x^u(1)) \ .$$

π_u can be also seen as a positive Radon measure on $\bar{K}$.
We have $\lim_{\mathcal{U}}(\pi_u(1)) = \lambda(1) + \mu(1)$. Hence $\lim_{\mathcal{U}}(\pi_u)$ exists as a positive Radon measure π on $\bar{K}$ and $r(\pi) = (\lambda , \mu)$.
2° $\Longrightarrow$ 1°: Each discrete positive Radon measure on K can be written

$$m = \sum_{p\in K} a_p^m \, \varepsilon_{((b_p^m, c_p^m))} \quad \text{where} \quad (b_p^m , c_p^m) \in K , \quad a_p^m \geqslant 0 \quad \text{and} \quad a_p^m = 0 ,$$

except for a finite number of $p \in K$.
There exists an ultrafilter $\mathcal{U}$ on the discrete positive measures on K such that $\lim_{\mathcal{U}}(m) = \pi$.
If $g \in \mathcal{C}(X)$ and $f \in \mathcal{C}(Y)$ with $g \geqslant \hat{f}$, we have

$$\lambda(g) = \lim_{\mathcal{U}}(\sum_{p\in K} a_p^m \, b_p^m(1) \, g(b_p^m/b_p^m(1)))$$

and

$$\mu(f) = \lim_{\mathcal{U}}(\sum_{p\in K} a_p^m \, c_p^m(f) .$$

As $g \geqslant \hat{f}$, we have $b_p^m(1) \, g(b_p^m/b_p^m(1)) \geqslant c_p^m(f)$, hence $\lambda(g) > \mu(f)$.

21. PROPOSITION (Extension of a result of STRASSEN [8], p. 302). - Suppose moreover that X and Y are metrizable, and that H is a closed subset of $\mathcal{M}^+(X) \times \mathcal{M}^+(Y)$ equiped with the weak-*-topology. If $\lambda \in \mathcal{M}^+(X)$ and $\mu \in \mathcal{M}^+(Y)$ with $\lambda \ll \mu$, then, there exists a Borel mapping $x \longmapsto \nu_x$ defined on X such that $\nu_x \in M_x$ λ-a.e., and $\mu \geqslant \int \nu_x \, d\lambda(x)$, and $0 \ll \mu - \int \nu_x \, d\lambda(x)$.

Proof. - Note that $\{x ; M_x = \emptyset\}$ is a G_δ λ-null subset of X since $\lambda(- \hat{1}(y)) \geqslant \mu(- 1(y)) > - \infty$. We shall use the notations of the proof of § 20.
Suppose v is the projection of $\mathcal{M}^+(X) \times \mathcal{M}^+(Y)$ on $\mathcal{M}^+(X)$. We have $v(\pi) = \lambda$ as conical measures on $\mathcal{M}^+(X)$. Let $A_0 = \{(0 , \beta) ; \beta \in \mathcal{M}_1^+(Y)\}$, and π_0 the part of π carried by A_0.
Let $\pi' = \pi - \pi_0$.
Suppose $\pi' = \pi_1' + \dots + \pi_n' + \dots$ is a decomposition of π' such that, for each n, π_n' lives on $A_n = \{(\alpha , \beta) ; \alpha \in \mathcal{M}^+(X) , \beta \in \mathcal{M}^+(Y) \text{ and } \alpha(1) \geqslant 1/n\}$.
Let π_n'' the Radon measure on $[1 , n]\bar{K}$ such that, for each $f \in \mathcal{C}([1 , n]\bar{K})$, we have

$$\pi_n''(f) = \int \alpha(1) \, f(\alpha/\alpha(1) , \beta/\alpha(1)) \, d\pi_n'(\alpha , \beta) .$$

π_n' and π_n'' induce the same conical measure on $\mathcal{M}^+(X) \times \mathcal{M}^+(Y)$. Then $v(\pi_n'')$ is carried by $\{\varepsilon_x , x \in X\}$; the image of λ by the map $x \longmapsto \varepsilon_x$ of X into $\mathcal{M}^+(X)$ is $\sum_n v(\pi_n'')$. If $\varepsilon_x \longmapsto \nu_x^n$ is a disintegration ([2], p. 58) of π_n'' with respect to v, then, for each n, we have $v(\pi_n'')$-a.e., that ν_x^n lives on the set $\{(\varepsilon_x , \nu) ; \nu \in M_x \text{ and } \nu(1) \leqslant n\}$, let $\nu_x^n \in M_x$ such that

$$r(\varepsilon_{\varepsilon_w} \otimes \pi_x^n) = (\varepsilon_x , \nu_x^n) .$$

For each $x \in X$, we identify x and ε_x. Let $\mu_0 \in \mathcal{M}^+(Y)$ be such that $(0 , \mu_0) = r(\pi_0)$. We have $\lambda = \sum_n v(\pi_n'')$ and $\mu - \mu_0 = \sum_n \nu_x^n \, dv(\pi_n'')$.

If we let $\nu_x = \sum_n \nu_x^n \,(d\nu(\pi''_n)/d\lambda)$, then, we have $\nu_x \in M_x$, λ-a.e. and $\mu - \mu_0 = \int \nu_x \, d\lambda(x)$. As π_0 is carried by A_0 , we have $0 \ll \mu_0$.

22. Remark. - Strictly speaking, in [8] (chap. 11), Strassen theorem is T51 which admits T52 as a consequence, but T51 can be also derived from T52. We sketch a proof, with the notations of [8]. Suppose E'_1 is the unit ball of E' equiped with the weak-*-topology. For each $\omega \in \Omega$, let P_ω be the set

$$\{y \; ; \; y \in E' \; , \; y \leq p_\omega\} \; .$$

We suppose $P_\omega \subset E'_1$. Let $M_\omega = \{\nu \; ; \; \nu \in \mathfrak{M}_1^+(E'_1) \; , \; r(\nu) \in P_\omega\}$.
Now, suppose (x_n) is a sequence of E everywhere norm-dense in the unit ball E_1 of E. Let φ be the map $\Omega \longmapsto [-1 \, , \, 1]^N$ such that $(\varphi(\omega))_n = p_\omega(x_n)$. We let $X = \overline{\varphi(\Omega)}$ and $\Lambda = \varphi(\lambda)$, which is a regular Borel measure on X ([9] prop. II7.2).
For each $t = (t_n)$ in X , because of [8] (p. 300 footnote), there exists a sub-linear form p_t on E such that $p_t(x_n) = t_n$, for each n , and $p_t(E_1) \in [-1, 1]$.
Then the definition of P_t and M_t (given for $t = p_\omega$) are meaningfull, and the set $\{(t \, , \, \nu) \; ; \; t \in X \; , \; \nu \in M_t\}$ is a compact subset of $X \times \mathfrak{M}_1^+(E'_1)$.
Now it is sufficient to apply T52 to X and measure Λ , with $Y = E'_1$ using the map $X \longmapsto \mathcal{P}(\mathfrak{M}_1^+(Y))$ defined by $t \longmapsto M_t$, and taking for μ an extension of x' to $\mathcal{C}(Y)$ such that, for each $f \in \mathcal{C}(Y)$, $\mu(f) \leq \Lambda(\hat{f})$, then T51 follows since in X , $\varphi(\Omega)$ is of Λ-outer measure equal to $\Lambda(1)$.

II (B). Theory of balayage.

23. Notations. - Suppose X is a compact (HAUSDORFF) space, Γ a convex subcone of $\mathcal{C}(X)$ which is an inf-lattice (i. e. if $f \, , \, g \in \Gamma$, then $g\ell b(f \, , \, g) \in \Gamma$), and Γ^0 is the polar of Γ in $\mathfrak{M}(X)$.
Using the previous notations, we take $Y = X$,

$$M_x = (\varepsilon_x - \Gamma^0) \cap \mathfrak{M}^+(X) = \{\mu \; ; \; \mu \in \mathfrak{M}^+(X) \text{ and } \mu_{|\Gamma} \leq \varepsilon_{x|\Gamma}\} \; .$$

Note that we do not suppose as in [8] (p. 294-297) that Γ contains a strictly positive function.

24. Definition (of f_Γ and r_λ). - For each $f \in \mathcal{C}(X)$, we let

$$f_\Gamma = g\ell b(g \; ; \; g \in \Gamma \text{ and } g \geq f)$$

and for each $\lambda \in \mathfrak{M}^+(X)$, we let $r_\lambda(f) = \lambda(f_\Gamma)$. r_λ is a sublinear functional on $\mathcal{C}(X)$, with values in $R \cup (+\infty)$, and we have $p_\lambda \leq q_\lambda \leq r_\lambda$.

25. PROPOSITION (Extension of a balayage formula of MOKOBODZKI [8] chap. 11 T45).
For each $f \in \mathcal{C}(X)$ with $f < 0$, we have $f_\Gamma = \hat{f}$.
Moreover the following properties are equivalent :

1° There is no element > 0 in Γ , 2° $\hat{1} \equiv +\infty$ <u>everywhere on</u> X ,

3° $\hat{1}$ <u>is equal to infinity in at least one point of</u> X ,

4° $\hat{1}$ <u>is unbounded on</u> X .

<u>Proof</u>. - Let us prove that $f_\Gamma = \hat{f}$ for each $f < 0$ of $\mathcal{C}(X)$. If $\lambda \in \mathcal{M}^+(X)$, because of the theorem of Hahn-Banach recalled in § 1, for each $k \in]-r_\lambda(-f) , r_\lambda(f)[$, there exists $\mu_k \in \mathcal{M}^+(X)$ with $\mu_k(f) = k$ and $\mu_k \leqslant r_\lambda$. It suffices now to take $\lambda = \varepsilon_x$ and $k = f_\Gamma(x) = r_{\varepsilon_x}(f)$. Now 1° $\Longrightarrow$ 2° can be proved in the same way, and we see that 4° $\Longrightarrow$ 1°.

26. PROPOSITION.

(a) <u>Suppose</u> f <u>is an</u> u. s. c. <u>function</u> < 0 <u>on</u> X . <u>We have</u> $f_\Gamma = \hat{f}$ (<u>the definition of</u> $\hat{f}$ <u>is as in</u> § 16 <u>and that of</u> f_Γ <u>as in</u> § 24).

(b) <u>If</u> (f_i) <u>is a family of</u> u. s. c. <u>functions</u> < 0 <u>on</u> X , <u>directed downward, having a limit</u> f , <u>we have</u> $(f_i) \longrightarrow f_\Gamma$.

<u>Proof</u>.

(a) can be proved as in [7] (prop. 5.6) because it is enough to work, for each $x \in X$, on a compact subset of M_x .

(b) can be proved as in [7] (prop. 5.6).

Proposition 25 enables us to give a balayage proof of the following result of CHOQUET-DENY [4].

27. PROPOSITION. - <u>Suppose</u> Γ <u>is a closed convex subcone of</u> $\mathcal{C}^-(X)$ <u>which is an inf-lattice and contains</u> -1 . <u>If we let</u>

$$\hat{\Gamma} = \{f \; ; \; f \in \mathcal{C}^-(X) \text{ with } m(f) \leqslant f(x) , \; \forall x \in X , \; \forall m \in \mathcal{M}^+(X) \text{ with } m|_\Gamma \leqslant \varepsilon_x|_\Gamma\} ,$$

<u>then we have</u> $\Gamma = \hat{\Gamma}$.

<u>Proof</u>. - $\hat{\Gamma}$ is a closed convex subcone of $\mathcal{C}^-(X)$ which is an inf-lattice and $\Gamma \subset \hat{\Gamma}$. For each $f \in \mathcal{C}(X)$ such that $f < 0$, we have, because of 25, $f_\Gamma(x) = \mathrm{lub}_{\nu \in M_x}(\nu(f))$, and we see that $f_{\hat{\Gamma}}(x) = \mathrm{lub}_{\nu \in M_x}(\nu(f))$, hence $f_\Gamma = f_{\hat{\Gamma}}$. Therefore, by Dini lemma, we have $f = f_\Gamma$ if, and only if, $f \in \Gamma$ and $f = f_{\hat{\Gamma}}$ if, and only if, $f \in \hat{\Gamma}$, hence $\Gamma \cap \{f < 0\} = \hat{\Gamma} \cap \{f < 0\}$. Then $\Gamma = \hat{\Gamma}$, since Γ and $\hat{\Gamma}$ are the closure of $\Gamma \cap \{f < 0\}$ and $\hat{\Gamma} \cap \{f < 0\}$.

28. <u>Remark</u>. - Suppose Γ is separating. Then we can apply to Γ the theorem 48 of [8] (chap. 11) about the Silov compacts. It is enough to apply [8] (chap. 11, th. 48) to the cone $\Gamma_1 = \{f \; ; \; f = g + a$ with $g \in \Gamma$ and $a \geqslant 0\}$ which is an inf-lattice.

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