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THE DECOMPOSITION OF $L^2(\Gamma\backslash G)$

by Stephen GELBART

Let G denote the real unimodular group $SL(2, R)$ and Γ its discrete subgroup of integral matrices. By $R(g)$, we denote the right shift representation of G on the Hilbert space $L^2(\Gamma\backslash G)$. Thus

$$R(g)\, f(x) = f(xg) .$$

We note that $L^2(\Gamma\backslash G)$ is formed with respect to a G-invariant measure μ defined on the quotient $\Gamma\backslash G$.

This exposition concerns the following :

PROBLEM. - Decompose the representation $R(g)$ into irreducibles ; equivalently, find a measure $d\nu(\lambda)$ on $\hat{G}$ such that

$$L^2(\Gamma\backslash G) = \int \oplus H_\lambda \, d\nu(\lambda)$$

(1) and

$$R(g) = \int \oplus \pi_\lambda \, d\nu(\lambda)$$

in the sense of direct integrals of Hilbert spaces and unitary operators.

By the theory of von NEUMANN and F. MAUNTNER, we know *a priori* that such a decomposition is possible (See, for example, [1]). Moreover, all the irreducible unitary representations π_λ of G are known (See [2] for a convenient summary). Thus it remains only to describe $d\nu(\lambda)$. We shall refer to the support of this measure as the spectrum of $R(g)$.

We note that since $\mu(\Gamma\backslash G)$ is finite the identity representation of G occurs once in $R(g)$. The space of constant functions in $L^2(\Gamma\backslash G)$ is one-dimensionsal. In particular, the discrete spectrum of $R(g)$ is non-empty.

We shall see that the continuous spectrum of $R(g)$ is also non-empty and is completely known. The remainder of the discrete spectrum, on the other hand, has not yet been completely described.

Part 1 : Spectral decomposition of the Laplacian.

1. Automorphic forms.

We recall the principle which asserts that a unitary representation is reduced by the eigenspaces of an operator commuting with it.

In the context of the representation $R(g)$, we are thus led to introduce the

Casimir operator Δ , described in [2]. This operator is essentially self-adjoint in $L^2(\Gamma\backslash G)$ and commutes with $R(g)$. Its eigenfunctions of immediate interest to us are the following.

Definition. - An automorphic form of type (k , λ , Γ) on G is a C^∞ function $f : \Gamma\backslash G \to \mathbb{C}$ such that

(i) $\Delta f = \lambda f$;

(2) (ii) $f(gr(\theta)) = \exp(- ik\theta) f(g)$ for $r(\theta) = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$; and

(iii) f satisfies a mild growth condition.

We denote the space of such functions by $A_{\lambda,k}(\Gamma)$. The question immediately arises as to whether such functions exist for given λ and k .

Remarks. - The essential feature of an automorphic form is that it is an eigenfunction of Δ . Condition (iii), which varies from example to example, is also crucial. It is included to insure that (possibly) non square-integrable eigenfunctions of Δ still have to do with the decomposition of $L^2(\Gamma\backslash G)$. Condition (ii), on the other hand, is unimportant, and is really no restriction at all. (The restriction of $R(g)$ to $K = SO(2)$ is completely reducible.) It is included merely for the sake of convenience.

2. Coordinates for G .

Before describing some examples of automorphic forms, it will be convenient to collect several facts related to the parameterization of G .

(i) Let $\mathfrak{H}$ denote the upper half-plane $\{z = x + iy ; \; y > 0\}$. Then G acts transitively on $\mathfrak{H}$ by the fractional linear transformations

$$g \circ z = \frac{az + b}{cz + d} , \qquad g = \begin{bmatrix}a & b\\ c & d\end{bmatrix} .$$

The stability subgroup of G at i is

$$K = \{\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\}$$

and therefore $\mathfrak{H} \cong G/K$.

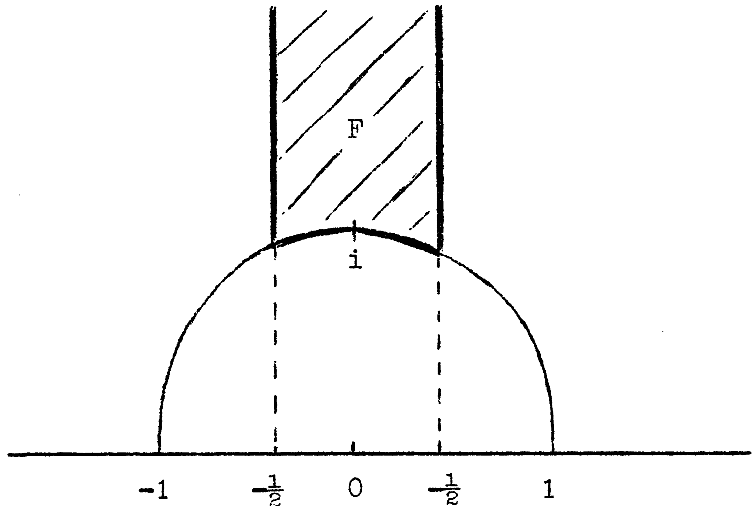
(ii) To g in G , we associate the coordinates $(z = x + iy , \theta)$ if

$$g = \begin{bmatrix}y^{\frac12} & xy^{-\frac12}\\ 0 & y^{-\frac12}\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} , \quad (y > 0 , \quad 0 < \theta \leqslant 2\pi) .$$

Thus

$$z = \frac{g_{11}\, i + g_{12}}{g_{21}\, i + g_{22}} , \text{ if } g = \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix} .$$

(iii) The quotient $\Gamma\backslash G$ is parameterized by pairs (z , θ) where z belongs to the fundamental domain F of Γ in $\mathfrak{H}$:



In particular, if $f(g)$ is right K-invariant and left Γ-invariant, it defines a function $\varphi(z)$ on $\mathfrak{H}$ such that

$$\int_{\Gamma\backslash G} |f(g)|^2 \, d\mu(g) = \iint_F |\varphi(z)|^2 \frac{dx\, dy}{y^2} .$$

(iv) For the coordinates just described the operator Δ assumes the form

$$\Delta = - y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y \frac{\partial^2}{\partial x\, \partial \theta} . \tag{3}$$

3. Examples of automorphic forms.

Suppose H is a closed subspace of $L^2(\Gamma\backslash G)$ equivalent to an irreducible representation π of G. Then the restriction of Δ to H coincides with the Laplacian for π, and is described by the scalar $\frac{k}{2}(\frac{k}{2} - 1)$ or $(1 + t^2)/4$ (See [2], § 7). These considerations motivate the following examples.

EXAMPLE 1 : Holomorphic cusp forms.

(i) $\lambda = \frac{k}{2}(\frac{k}{2} - 1)$;

(4) (ii) k is an even positive integer ; and

(iii) our growth condition is that $f(g)$ be bounded (in particular,

$$\int_{\Gamma\backslash G} |f(g)|^2 \, dg < \infty ,$$

since $\Gamma\backslash G$ has finite measure).

We consider the formula

$$\varphi(z) = f(g)\, j(g , i)^k ,$$

where g is such that $g \circ i = z$, and

$$j(g , z) = g_{21}\, z + g_{22} .$$

Then φ defines a single-valued function in $\mathfrak{H}$ satisfying the properties :

(i) $\varphi(z)$ is holomorphic in $\mathfrak{H}$;

(5) (ii) $\varphi(\frac{az+b}{cz+d}) = (cz+d)^k \varphi(z)$, $\forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$; and

(iii) $\varphi(z)(\mathrm{Im}\ z)^{k/2}$ is bounded (in particular,

$$\iint_F |\varphi(z)|^2 y^k \frac{dx\,dy}{y^2}$$

is finite).

Establishing this claim is completely straight forward since

$$j(g_1 g_2 , z) = j(g_2 , z)\, j(g_1 , g_2 z)$$

(i. e. j is a factor of automorphy for G) and $j(g , i) = y^{-\frac{1}{2}}$ for $g = (x,y,\theta)$. The result is that $A_{(k/2)((k/2)-1),k}(\Gamma)$ is isomorphic to the classical space $S_k(\Gamma)$ of cusp forms of weight k for Γ .

Holomorphic cusp forms are known to exist and the dimension of the space they span is well known (See, for example, [6]).

It will be convenient to reinterpret the growth conditions 4(iii) and 5(iii) as follows. From property 4(ii) (with $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$), it follows that we may expand $\varphi(z)$ in a Fourier series

$$\varphi(z) = \sum_{n=-\infty}^{\infty} a_n \exp 2\pi i n z$$

where

$$a_n \exp(-2\pi n y) = \int_0^1 \varphi(x+iy) \exp(-2\pi i n x)\, dx \,.$$

The boundedness of $\varphi(z)y^{k/2}$ implies then that

$$|a_n| \leqslant M \exp(2\pi n y)\, y^{-k/2} , \quad \forall\ y ,$$

hence that $a_n = 0$ for $n \leqslant 0$ (let $y \to +\infty$). In particular,

$$a_0 = 0 = \int_0^1 \varphi(x+iy)\, dx = y^{-k/2} \int_0^1 f(\begin{bmatrix} y^{\frac{1}{2}} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{bmatrix} r(\theta))\, dx$$

for all $y > 0$ and $r(\theta)$, so the condition that $f(g)$ be bounded is equivalent to the "cuspidal condition"

(6) $$\int_0^1 f(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g)\, dx = 0 , \quad \forall\ g \in G$$

(or to the condition that the zeroth Fourier coefficient of $\varphi(z)$ vanish).

EXAMPLE 2 : Non-holomorphic cusp forms.

$$\lambda = \frac{1+t^2}{4} ,$$

$$k = 0 ,$$

and the growth condition is the "cuspidal condition" of example 1 (i. e. $f(g)$ satisfies (6)).

We denote this space of functions by $M_t(\Gamma)$. Since $k = 0$, it follows immediately from the formula for Δ that $M_t(\Gamma)$ is identifiable with the space of functions $f(z)$ in $\mathfrak{H}$ such that

(i) $-y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\, f(x , y) = (\frac{1+t^2}{4})\, f(x , y)$ (i. e. f is an eigenfunction of the Laplace-Beltrami operator in the Poincaré upper half-space) ;

(ii) $f(\frac{az+b}{cz+d}) = f(z)$, $\forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$; and

(iii) $\lim_{y\to+\infty} f(x\ ,\ y) = 0$ uniformly in x (in particular,

$$\int_F |f(z)|^2 \frac{dx\,dy}{y^2} < \infty ,$$

since $\int_F \frac{dx\,dy}{y^2} < \infty$).

Thus $M_t(\Gamma)$ consists of cuspidal "wave forms" in the sense of MAASS [8]. Unfortunately, it is not yet known for which values of t such functions exist.

EXAMPLE 3 : Eisenstein series.

$$\lambda = \frac{1+t^2}{2} ,$$

$$k = 0 ,$$

and the growth condition is that $f(g) = f(z) = O(y^c)$ as $y \to +\infty$, for some positive constant c .

It is possible to construct automorphic forms satisfying this growth condition as follows. Let $f_\mu(z) = y^\mu$. Then $f_\mu(z)$ is an eigenfunction for Δ (with eigenvalue $-\mu(\mu-1)$) and is already invariant (on the left) by $N = \{\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} ;\ x \in \underset{\sim}{R}\}$. To obtain a function invariant for Γ , we must sum the γ-translates of f_μ for γ in Γ . Thus we define (the Eisenstein series)

$$E(z\ ,\ \mu) = \sum_{N\cap\Gamma\backslash\Gamma} f_\mu(\gamma \circ z)$$

and compute that

$$E(z\ ,\ \mu) = \sum_{(c,d)=1} \frac{y^\mu}{|cz+d|^{2\mu}} .$$

This serie converges, for $\mathrm{Re}(s) > 1$, has a meromorphic continuation to the whole plane, and satisfies a simple functional equation of Riemann type (See, for example, SELBERG [10]). In particular, for

$$\mu = \frac{1}{2} + \frac{it}{2} ,$$

$E(\mu\ ,\ z)$ is a well defined element of $A_{((1+t^2)/4),0}(\Gamma)$.

4. Preliminary decomposition of $R(g)$.

Roughly speaking, the three examples just described already suffice to decompose $L^2(\Gamma\backslash G)$.

SELBERG [10] proved that if $f(z)$ is a smooth function which is simultaneously of mild growth and orthogonal to all cuspidal Γ-invariant eigenfunctions of Δ (in the sense that

$$\iint_F f(z)\,\phi(z)\,\frac{dx\,dy}{y^2} = 0$$

for all such ϕ) then

$$f(z) = \int_{\mu=\frac{1}{2}+\frac{it}{2}} \hat{f}(\mu)\, E(\mu\ ,\ z)\ dt\ ,$$

where

$$\hat{f}(\mu) = \iint_F f(z)\ E(\mu\ ,\ z)\ \frac{dx\ dy}{y^2}\ .$$

Thus Eisenstein series provide the complete continuous spectrum of Δ restricted to $L^2(\Gamma\backslash G/K)$. To obtain an analogous result for $L^2(\Gamma\backslash G)$ one need only construct "slightly more general" Eisenstein series ("slightly more general" meaning that these functions should transform according to some character of K , but not necessarily the trivial character ; see condition 2(ii)). Since Eisenstein series are obviously parameterized by the principal series representations π^0_{it} (see [2]), the following result should not seem surprising.

THEOREM 1.

(a) $L^2(\Gamma\backslash G) = \{\underset{\sim}{C}\} \oplus \int_0^\infty H_{it}\ dt \oplus {}^0L^2(\Gamma\backslash G)$, where $\{\underset{\sim}{C}\}$ denotes the 1-dimensional space of constant functions on $\Gamma\backslash G$, $\int_0^\infty H_{it}\ dt$ is the direct integral of the representation spaces of the principal series π^0_{it} (with respect to Lebesgue measure dt), and ${}^0L^2(\Gamma\backslash G)$ denotes the space of cuspidal functions in $L^2(\Gamma\backslash G)$ (functions satisfying the cuspidal condition (6)).

(b) The restriction of $R(g)$ to ${}^0L^2(\Gamma\backslash G)$ is the direct sum of irreducible unitary representations of G each occuring with finite multiplicity ; therefore

$$R(g) = I \oplus \int_0^\infty \pi^0_{it}\ dt \oplus \sum_{\pi\in\hat{G}} m(\pi)\pi$$

with $m(\pi)$ finite (usually zero !).

(c) Multiplicity formula for the discrete spectrum :

$$m(\pi) = \text{dimension}\ \ (A_\pi(\Gamma))\ ,$$

where $A_\pi(\Gamma)$ denotes $A^{cusp}_{(k/2)((k/2)-1)}(\Gamma)$ for $\pi = \pi_k$ and denotes $A^{cusp}_{((1+t^2)/4),0}(\Gamma)$ for $\pi = \pi^0_{it}$. (We call $A_\pi(\Gamma)$ the space of automorphic cusp forms belonging to the representation π .)

Remarks concerning the proof of THEOREM 1.

(a) Part (a) is essentially due to SELBERG [10]. (SELBERG stated his results independently of the language of group representations.) A more rigorous and modern proof may be found in GODEMENT [5].

(b) Part (b) is due to GEL'FAND and PJATECKIJ-ŠAPIRO, and first appeared in [3].

(c) Part (c) is an immediate consequence of the fact that in the representation space of a principal series representation π^0_{it} (resp. discrete series representation π_k) there is precisely one vector v such that $\pi(r(\theta))v = v$ (resp. $\pi(r(\theta))v = \exp(-\ ik\theta)\ v)$ for all $r(\theta) \in K$.

Theorem 1 provides an almost complete solution to our initial problem. The solution is not complete because we do not yet know (for example) which representations

π^0_{it} occur discretely in $R(g)$. Clearly new methods and approaches must be found.

Part 2 : An alternate approach : Hecke theory

5. Classical Hecke theory.

An alternate approach to our problem is motivated by the pioneering work of HECKE relating automorphic forms to Dirichlet series with functional equation.

Following HECKE, we associate to each cusp form

$$f(z) = \sum_{n=1}^{\infty} a_n \exp 2\pi i n z \tag{7}$$

in $S_k(\Gamma)$ the Dirichlet series

$$D(f , s) = \sum_{n=1}^{\infty} a_n n^{-s} .$$

According to HECKE, this serie converges for $Re(s) > \frac{k}{2} + 1$, extends to an entire function in $\underset{\sim}{C}$, and satisfies the functional equation where

$$L(f , k - s) = i^k L(f , s) ,$$

$$L(f , s) = (2\pi)^{-s} \Gamma(s) D(f , s) .$$

Conversely, every Dirichlet series possessing these properties arises in this fashion. The proof of this result rests on the fact that

$$L(f , s) = \int_0^{\infty} f(iy) y^{s-1} dy , \tag{8}$$

that is, $L(f , s)$ is the Mellin transform of $f(iy)$. HECKE also proved that $\sum a_n n^{-s}$ has an Euler product expansion

$$\prod(1 - a_p p^{-s} - p^{k-1-2s})^{-1} \tag{9}$$

if, and only if, f is an eigenfunction of a certain family of self-adjoint operators T_p defined on $S_k(\Gamma)$ (the so-called Hecke operators).

The results of HECKE were extended first by MAASS [8] to his space of non-holomorphic wave forms $M_t(\Gamma)$ and then considerably improved and generalized by WEIL in [11] so as to deal with Dirichlet series defined over arbitrary number fields (not just $\underset{\sim}{Q}$). As a result the construction of cuspidal automorphic forms is equivalent to the problem of constructing " L-functions" which are defined by Euler product expansions and which possess an analytic continuation and prescribed functional equation.

By the last section, a basis for the space of cuspidal automorphic forms of type π is in one-to-one correspondence with irreducible summands of $L^2(\Gamma\backslash G)$ equivalent to π . Therefore it is clearly indicated that one should reinterpret Hecke theory from the point of view of group representations.

This is precisely the aim of JACQUET-LANGLANDS' work.

6. Jacquet-Langlands theory.

To describe this theory, it will be necessary to view automorphic forms not as functions on $SL(2, \mathbb{R})$ but rather as functions on the adele group of the general linear group $GL(2)$. Therefore we must introduce some more notation.

For each prime p let $\mathbb{Q}_p$ denote the completion of $\mathbb{Q}$ with respect to a p-adic valuation $|\ |_p$. Let $\mathcal{O}_p$ denote the ring of integers of $\mathbb{Q}_p$, that is, the set $\{x \in \mathbb{Q}_p ;\ |x|_p \leq 1\}$. Then the groups $G_p = GL(2, \mathbb{Q}_p)$ are locally compact and the subgroups $K_p = GL(2, \mathcal{O}_p)$ are compact.

By $\mathbb{Q}_\infty$ we shall understand $\mathbb{R}$, and by $\mathbb{A}$ the ring of adeles of $\mathbb{Q}$. Thus

$$\mathbb{A} = \prod_{p \leq \infty} (\mathbb{Q}_p : \mathcal{O}_p)$$

where the direct product is restricted with respect to the subrings $\mathcal{O}_p$. Similarly,

$$G_{\mathbb{A}} = GL(2, \mathbb{A}) = \prod_{p \leq \infty} (G_p : K_p) .$$

Moreover, if $G_\infty^+ = \{g \in GL(2, \mathbb{R}) ;\ \det(g) > 0\}$ then

(10)
$$G_{\mathbb{A}} = G_{\mathbb{Q}}\, G_\infty^+\, K_0$$

where $G_{\mathbb{Q}} = GL(2, \mathbb{Q})$ and $K_0 = \prod_{p<\infty} K_p \subset G_{\mathbb{A}}$.

<u>Given</u> f <u>in</u> $S_k(\Gamma)$, <u>we may use</u> (10) <u>to define</u>

(11)
$$\tilde{f}(g) = f(g_\infty \circ i)\, j(g_\infty , i)^{-k}$$

if $g = \gamma g_\infty k_0$, $\gamma \in G_{\mathbb{Q}}$, $g_\infty \in G_\infty^+$ and $k_0 \in K_0$. This function $\tilde{f}$ is well-defined on $G_{\mathbb{A}}$ (since $G_{\mathbb{Q}} \cap G_\infty^+ K_0 = \Gamma$ and f is an automorphic form for Γ) and satisfies the "cuspidal" condition

(12)
$$\int_{\mathbb{Q}\backslash\mathbb{A}} \tilde{f}\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) dx = 0 , \quad \forall\ g \in G_{\mathbb{A}}$$

since $f(z)$ is itself a cusp form.

By definition, $\tilde{f}$ is actually a function on the quotient of $G_{\mathbb{A}}$ by its discrete subgroup $G_{\mathbb{Q}}$. Therefore, as in part 1, it is natural to consider the space ${}^0L^2(G_{\mathbb{Q}}\backslash G_{\mathbb{A}})$ consisting of <u>all</u> cuspidal functions in $L^2(G_{\mathbb{Q}}\backslash G_{\mathbb{A}})$ and to introduce the right shift representation $R_{\mathbb{A}}(g)$ of $G_{\mathbb{A}}$ in this space.

In direct analogy with Hecke's theory, JACQUET and LANGLANDS characterize the representations of $G_{\mathbb{A}}$ which occur in $R_{\mathbb{A}}(g)$ in terms of associated L-functions. More precisely, let π denote an arbitrary irreducible unitary representation of $G_{\mathbb{A}}$. Then π is expressible in the form

$$\pi = \bigotimes \pi_p$$

where each π_p is an irreducible unitary representation of G_p (See, for example, [4], chapter 3). To each π_p is associated a canonical Euler factor $L(\pi_p , s)$ which for finite p is of the form $[(1 - A_p^{-s})(1 - B_p^{-s})]^{-1}$ (compare with (9)) and for $p = \infty$ is a product of gamma functions (possibly one). Then to π itself is associated the L-function

$$L(\pi , s) = \prod_{p\leqslant\infty} L(\pi_p , s) .$$

THEOREM 2. - π <u>occurs in $^0L^2(G_{\mathbb{Q}}\backslash G_{\mathbb{A}})$ if, and only if, $L(\pi , s)$ has a specified analytic behaviour and functional equation of Hecke type</u>.

7. Epilogue.

To relate this theory to part 1 (and the classical Hecke theory), let us suppose that $f \in S_k(\Gamma)$ is such that $T_p f = a_p f$ for all Hecke operators T_p. Then $\tilde{f}$ (defined by (11)) belongs to an irreducible subspace H_{π_f} of $^0L^2(G_{\mathbb{Q}}\backslash G_{\mathbb{A}})$ such that $\pi_f = \bigotimes_{p\leqslant\infty} \pi_p$ with $\pi_\infty = \pi^k$ (each discrete series representation $\tilde{\pi}_k$ of $SL(2 , \mathbb{R})$ canonically determines the representation π^k of $GL(2 , \mathbb{R})$) and π_p for each finite p is determined by the eigenvalues a_p of T_p. (Thus to f is naturally associated the L-function $L(\pi_f , s)$ which, of course, agrees with the L-function $L(f , s)$ described by (8)).

Conversely, if $\pi = \otimes \pi_p$ occurs in $^0L^2(G_{\mathbb{Q}}\backslash G_{\mathbb{A}})$, and each π_p has a vector fixed by K_p, then π_∞ occurs in $L^2(\Gamma\backslash G)$ and H_π contains a one-dimensional space of automorphic forms of type π_∞ (whose elements satisfy $T_p f = a_p f$ with the a_p determined by π_p).

Roughly speaking, the problem of constructing automorphic forms of type π_∞ for Γ (hence of finding the spectrum of $^0L^2(\Gamma\backslash G)$) is reduced to constructing "nice" L-functions, determining which representations π they belong to, and unraveling the consequences for $SL(2 , \mathbb{R})$. Unfortunately, this recipe remains somewhat theoretical since such L-functions are themselves difficult to find. Nevertheless (to close on a less sour note) let us remark that the study of a certain L-function (introduced first by MAASS) leads to the conclusion that the principle series representation π^0_{it}, where

$$t = \frac{\pi}{\log(\sqrt{2} - 1)}$$

occurs discretely in $L^2(\Gamma\backslash G)$, not necessarily for Γ the full modular group of part 1 but for Γ some subgroup of finite index in $SL(2 , \mathbb{Z})$!

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