Séminaire Choquet.
Initiation à l'analyse

Stephen Gelbart

Introduction to the theory of group representations

Séminaire Choquet. Initiation à l'analyse, tome 11-12 (1971-1973), exp. n° 3, p. 1-7
<http://www.numdam.org/item?id=SC_1971-1973__11-12__A4_0>

Séminaire CHOQUET
(Initiation à l'analyse)
11-12e années, 1971-1973, n° 3, 7 p.
15 juin 1972

INTRODUCTION TO THE THEORY OF GROUP REPRESENTATIONS

by Stephen GELBART

It is not our purpose to give a complete survey of the theory of group representations. Rather we hope to quickly exhibit the subject's flavor and to collect some elementary facts which are needed for the conpanion article [2]. Since [2], principally concerning the group $SL(2, \underset{\sim}{R})$, our presentation is clearly biased toward that group. The reader interested in a more comprehensive account of representation theory should consult the text of VILENKIN [5] and its exhaustive bibliography.

1. Definitions.

Let G denote a locally compact group which is separable and unimodular. Let H denote a complex separable Hilbert space, and $U(H)$ the group of unitary operators on H equipped with the topology of strong operator convergence.

By a unitary representation of G on H, we shall understand a continuous homomorphism from G to $U(H)$, that is, a mapping $g \to \pi(g)$ such that

(a) $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$, $\forall g_1, g_2 \in G$; and

(b) for each $x \in H$, the map $g \to \pi(g)x$ is continuous from G to H.

Such a representation is called reducible when there exists a closed proper subspace of H invariant for all $\pi(g)$, or equivalently, when there exists a nontrivial projection operator P in H which commutes with every $\pi(g)$. (In this case, π is the direct sum of the subrepresentations $\pi_1 = P \circ \pi$ and $\pi_2 = (I-P)\circ\pi$). When π is expressible as the direct sum of countably many irreducible subrepresentations we say that π is completely reducible.

EXAMPLE 1. - Let Γ denote a closed unimodular subgroup of G, and H the Hilbert space $L^2(\Gamma\backslash G, \mu)$ where μ is G-invariant measure on $\Gamma\backslash G$. Then the right shift representation of G on H is given by

$$R(g)\, f(x) = f(xg), \tag{1}$$

where $g \in G$, $f \in H$, and $x \in \Gamma\backslash G$. As we shall see, this representation is rarely completely reducible.

Two representations (π_1, H_1) and (π_2, H_2) are said to be equivalent whenever there exists an isometry A from H_1 onto H_2 such that

$$\pi_2(g)A = A\pi_1(g), \quad \forall g \in G.$$

Note that two finite-dimensional representations will be equivalent if and only if the matrices describing them coincide for an appropriate choice of bases. Thus it is natural to deal not with the set of irreducible unitary representations but rather with the set of equivalence classes of such representations. We shall denote this set by $\hat{G}$.

2. The fundamental problems of representation theory.

These are :

PROBLEM I : Describe $\hat{G}$ completely ; and

PROBLEM II : Decompose a given unitary representation into irreducibles ; in particular, decompose the right shift representation of Example 1.

(Here we must allow for continuous direct sum decompositions as well.)

For compact and abelian groups these problems are essentially solved. However, for non-compact non-abelian groups (even non-compact semi-simple Lie groups) the situation is more complicated. Therefore, after surveying the basic results for compact and abelian groups, we shall focus attention on a special example, namely $SL(2, \mathbb{R})$. For this group the first problem is solved, but the second (in all its generality) certainly is not (See [2]).

3. A basic principle of representation theory.

If $\mathfrak{L}$ is a linear space of functions defined on $\Gamma\backslash G$ then G operates on $\mathfrak{L}$ by the familiar formula

$$R(g)\, f(x) = f(xg) \ , \qquad f \in \mathfrak{L} \ .$$

Suppose that A is an operator defined in $\mathfrak{L}$, and that A permutes with every $R(g)$. Then obviously every eigenspace of A is an invariant subspace for $R(g)$; in particular, each eigenvalue λ of A defines a subrepresentation π_λ of $R(g)$. This simple observation proves to be surprisingly useful. A special case of of it is :

Schur's lemma. - Suppose A is a bounded operator which commutes with an irreducible representation π ; that is, A is defined in the representation space of π , and

$$A\pi(g) = \pi(g)A \ , \qquad \forall\, g \in G \ .$$

Then A is a multiple of the identity operator I .

Proof. - Without loss of generality, we may assume that A is self-adjoint. Indeed

$$A = \frac{A + A^*}{2} + i\left(\frac{A - A^*}{2i}\right) \ .$$

Then, by the spectral theorem,

$$A = \int \lambda \, dP(\lambda) \ ,$$

where each of the projections $P(\lambda)$ commutes with π . But π being irreducible, we must have $P(\lambda) = 0$ or I .

Q. E. D.

4. Abelian groups.

From Schur's lemma it follows that the irreducible unitary representations of an abelian group G are one dimentional. Moreover, such representations are of the form $\chi(g)I$, where I denotes the identity operator on $\underset{\sim}{C}$, and χ is a continuous homomorphism from G to the circle group T . Thus $\hat{G}$ coincides with the familiar character group $\hat{G}$.

An important example for arithmetic is the additive group of the rational number field $\underset{\sim}{Q}$ whose character group is $\underset{\sim}{A}/\underset{\sim}{Q}$ where A denotes the ring of adeles of $\underset{\sim}{Q}$. Simpler examples are the additive real line $\underset{\sim}{R}$ and its compact subgroup $T = \underset{\sim}{R}/\underset{\sim}{Z}$ whose character groups are well known.

As regards problem II with Γ the trivial subgroup we note that for abelian groups a solution is provided immediately by Plancherel's theorem (see, for example, WEIL [6]).

5. Compact groups.

The basic facts are as follows (for proofs, see [3]) :

(a) Every irreducible unitary representation is finite dimensional ;

(b) Every unitary representation of G is completely reducible (this fact is certainly special to compact groups ; witness the representation discussed in [2] or the representation $R(g)$ for $G = \underset{\sim}{R}$ and $\Gamma = \{0\}$) ;

(c) In the right regular representation $R(g)$ (take Γ to be trivial) every irreducible unitary representation of G occurs with multiplicity equal to its dimension ; equivalently, the matrix coefficients of the irreducible unitary representations of G comprise a complete orthonormal set in $L^2(G)$ (this is the celebrated Peter-Weyl theorem).

To describe the flavor of an explicit solution to problems I and II in this setting we consider the following :

EXAMPLE 2. - Suppose $G = SO(3)$.

(a) Description of $\hat{G}$. - Fix k a non-negative integer. Let $\mathcal{H}_k$ denote the $(2k+1)$-dimensional space of solid spherical harmonics in $\underset{\sim}{R}^3$, i. e. harmonic polynomials in $\underset{\sim}{R}^3$ homogeneous of degree k . Let $\pi_k(g)$ denote the representation of G on $\mathcal{H}_k$ given by $\pi_k(g)\, p(x) = p(xg)$. Then these π_k exhaust $\hat{G}$.

(b) A decomposition problem. - Let S^2 denote the unit sphere in $\underset{\sim}{R}^3$, and H the Hilbert space $L^2(S^2 , \mu)$ where μ is the (essentially unique) rotationally invariant measure on S^2. Since $SO(3)$ acts transitively on S^2 (by right matrix multiplication) and since the stability group for $(1, 0, 0)$ is isomorphic to $SO(2)$, we may identify S^2 with $SO(2)\backslash SO(3)$ and consider the right shift representation $r(g)$ of $SO(3)$ on H. Our problem is to decompose this representation.

To illustrate the basic principle of section 3, we introduce the angular Laplacian Δ defined for smooth functions in $L^2(S^2)$ by

$$\Delta = \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\frac{\partial}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}$$

where (φ, θ) denote the familiar spherical coordinates.

This operator is self-adjoint in $L^2(S^2)$ and permutes with each $r(g)$.

Now it is a simple matter to compute that

$$\Delta p = -k(k+1)p \tag{2}$$

when p is the restriction to S^2 of a solid harmonic of order k. Thus, by our "basic principle", the decomposition of $r(g)$ (namely the statement that each irreducible representation of G occurs exactly once in $r(g)$) is entirely equivalent to the spectral decomposition of the differential operator Δ (namely the statement that (2) gives the complete set of eigenfunctions for Δ).

6. The representations of $SL(2, \mathbb{R})$.

The group $G = SL(2, R)$ consists of all 2×2 real matrices with determinant 1. It is the simplest example of a non-compact semi-simple Lie group of matrices. Its Lie algebra, by definition the set

$$\mathfrak{G} = \{X \;;\; \exp(tX) \in G, \;\forall\; \text{real } t\},$$

consists of all 2×2 real matrices of trace zero equipped with the bracket operation

$$[X, Y] = XY - YX.$$

(Any Lie group of matrices is called semi-simple when its Lie algebra is semi-simple, i. e. the direct sum of simple ideals ; these algebras and groups have an especially rich structure theory that facilitates the study of their representations : see, for example, [4]).

It can be shown that the non-trivial irreducible unitary representations of semi-simple groups must be infinite-dimensional. According to BARGMANN [1], every such representation for $SL(2, \mathbb{R})$ belongs to one of the following series of representations :

(a) The principal series of representations π^{ε}_{it}, where t is a real number and $\varepsilon = 0$ or 1.

The representation space is $L^2(\mathbb{R})$ and

$$\pi^{\varepsilon}_{it}(g)\, f(x) = [\operatorname{sign}(bx+d)]^2\, |bx+d|^{-it-1}\, f\left(\frac{ax+c}{bx+d}\right)$$

if $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Roughly speaking, these representations are indexed by the characters $|a|^{it}[\mathrm{sign}(a)]^{\varepsilon}$ of the abelian subgroup

$$A = \{\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} ; \quad a \neq 0\} .$$

(b) The discrete series of representations $\pi_k(g)$, indexed by integers $k \neq 0, 1, -1$.

For $k > 1$, the representation space consists of all the holomorphic functions in the upper half-plane such that

$$\iint |f(x + iy)|^2 \; |y|^k \frac{dx\,dy}{y^2} < \infty ,$$

and for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\pi_k(g)\; f(z) = (bz + d)^{-k} \; f(\frac{az + c}{bz + d}) .$$

The operators π_k for $k < 1$ are defined similarly in the lower half-plane.

Roughly speaking, the discrete series representations are indexed by characters of the compact subgroup $K = SO(2)$.

(c) The complementary series of representations, indexed by points of the interval $(-1 , 1)$.

We shall not describe these representations explicitly, except to say that they are indexed by certain non-unitary characters of the subgroup A .

Summing up, the set $\hat{G}$ for $SL(2 , \mathbb{R})$ consists of continuous as well as discrete components. It must be emphasized that although a great many representations of an arbitrary semi-simple group have been constructed, a complete description of $\hat{G}$ is not yet available (except for scattered special examples).

7. Infinitesimal representations.

Let π denote a unitary representation of G . Then π defines a representation of the Lie algebra of G (called infinitesimal representation of π) as follows.

For $X \in \mathfrak{G}$, set

$$\pi(X)v = \lim_{t \to 0} \frac{\pi(\exp t X)\; v - v}{t}$$

for v in H , the representation space of π . The set of v in H for which this limit exists is dense and comprises the space $C^{\infty}(\pi)$ of so-called C^{∞}-vectors for π .

It is not difficult to show that the map $X \longrightarrow \pi(X)$ defines a representation of $\mathfrak{G}$ in $C^{\infty}(\pi)$ (in particular, this map preserves the bracket operation in $\mathfrak{G}$). The study of these "infinitesimal operators" plays a crucial role in the general theory of Lie group representations. For example, Bargmann's description of $\hat{G}$ for $SL(2 , \mathbb{R})$ proceeds from a classification of the irreducible (infinitesimal) representations of $\mathfrak{G}$. Of particular interest is the Casimir operator of π which we shall presently describe.

Fix π any representation of G, and consider the basis $\{\ell_0, \ell_1, \ell_2\}$ for $\mathfrak{G}$ where

$$\ell_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \ell_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ and } \ell_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

To this π and these ℓ_j $(j = 0, 1, 2)$ correspond the infinitesimal operator $H_j = \pi(\ell_j)$. These latter operators are unbounded self-adjoint operator in the representation space of π with dense common domain $C^\infty(\pi)$. The Casimir (or Laplacian) associated to π is, by definition, the operator

$$\Delta = -\frac{1}{4}(H_0^2 - H_1^2 - H_2^2). \tag{3}$$

This operator, in addition to being essentially self-adjoint in H, commutes with the operators $\pi(g)$. In particular, one can prove that when π is irreducible its associated Casimir operator is a scalar multiple of the identity. Specifically, if π is the discrete series representation π_k, then

$$\Delta = \frac{k}{2}\left(\frac{k}{2} - 1\right), \tag{4}$$

and if π is the principal series representation π_{it}^{ε}, then

$$\Delta = \frac{1 + t^2}{4} \qquad \text{(See [1]).} \tag{5}$$

It should not seem surprising that in problems concerning the decomposition of representations of $SL(2, \underset{\sim}{R})$ this operator plays a role analogous to that of the angular Laplacian of example 2 (See [2]).

EXAMPLE 3. - Let G denote $SL(2, \underset{\sim}{R})$ and $R(g)$ the right regular representation of G in $L^2(G, \mu)$ (where μ is Haar measure on G). By examining the eigenfunction expansion of the Casimir operator of $R(g)$, BARGMANN [1] was able to show that only the principal and discrete series representations of G occur in $R(g)$. In other words, the representations of the complementary series do not appear in the Fourier expansion of square integrable functions on G. (This fact is in direct contrast with the situation for compact G.)

BIBLIOGRAPHY

[1] BARGMANN (V.). - Irreducible unitary representations of the Lorentz group, Annals of Math., t. 48, 1947, p. 568-640.

[2] GELBART (S.). - The decomposition of $L^2(\Gamma\backslash G)$, Séminaire Choquet : Initiation à l'analyse, 11-12e années, 1970-72, n° 4, 10 p.

[3] GÉRARDIN (P.). - Décomposition des fonctions de carré sommable sur un espace homogène (Cas d'un groupe compact), Séminaire Choquet : Initiation à l'analyse, 7e année, 1967/68, n° A-10, 12 p.

[4] HELGASON (S.). - Differential geometry and symmetric spaces. - New York, Academic Press, 1962 (Pure and applied Mathematics, Academic Press, 12).

[5] VILENKIN (N.). - Fonctions spéciales et théorie de la représentation des groupes. - Paris, Dunod, 1969 (Monographies universitaires de Mathématiques, 33).

[6] WEIL (A.). - L'intégration dans les groupes topologiques et ses applications. 2e édition. - Paris, Hermann, 1940 (Act. scient. et ind., 849 ; Publ. Inst. math. univ. Strasbourg, 4).

Stephen GELBART
Princeton University
Dept of Mathematics
PRINCETON, N. J. 08540 (Etats-Unis)