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TRAVAUX DE ZINK

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1. INTRODUCTION AND PRELIMINARY DEFINITIONS AND RESULTS

Fix a prime number p. All rings considered will be $\mathbf{Z}_{(p)}$-algebras. If R is a ring we will consider p-divisible groups over R and in particular those which are formal groups. If $\frac{1}{p} \in R$, then p-divisible groups are étale and consequently given by continuous representations $\rho : \pi_1(\mathrm{Spec}(R)) \to \mathrm{GL}_h(\mathbf{Z}_p)$. Hence we shall assume p is either nilpotent in R or R is separated and complete for a topology having a neighborhood basis of 0 consisting of ideals and that p is topologically nilpotent.

With these conventions, the aim of the various Dieudonné theories is to classify the category of p-divisible groups over R via functors to categories living in the realm of (semi)linear algebra. One should think of them as analogous to the functor $G \mapsto \mathrm{Lie}(G)$ which establishes an equivalence of categories between formal groups and Lie algebras when R is a $\mathbf{Q}$-algebra. We will not give an overview of the various Dieudonné theories, but rather concentrate on the most recent, Zink's theory of displays. Nevertheless it will be necessary for us to relate Zink's theory to Cartier's theory and to the crystalline theory. We refer the reader to [Ta], [Ser], [Gr1], [Gr2], [Dem], [Fon1] for p-divisible groups, to [Car1], [Car2], [Haz], [Laz], [Z1], [Z2] for Cartier theory, to [Gr1], [Gr2], [MM], [M], [BBM], [BM1], [BM2], [dJ2], [dJM] for crystalline Dieudonné theory.

If R is a perfect field of characteristic p, these theories are, for p-divisible groups (formal in the case of Cartier's theory), all equivalent. Indeed it was one of Zink's motivations in developing his theory to relate the Cartier theory to the crystalline theory. But, in establishing properties of his theory, he uses both the Cartier and the crystalline theories. Hence there is a symbiotic relationship between the three theories.

We refer to [Bour] for the standard facts about the Witt vector ring, $W(R)$. We write $w_n : W(R) \to R$ for the ghost component maps, $f : W(R) \to W(R)$ for the Frobenius ring endomorphism and $v : W(R) \to W(R)$ for the additive Verschiebung endomorphism. Let $I_R = \mathrm{Ker}(w_0) = \mathrm{im}(v)$. If $a \in R$, $[a]$ denotes its Teichmüller representative.

Lemma 1.1. — *If R is separated and complete in the p-adic topology, then $W(R)$ is separated and complete in both its p-adic and I_R-adic topologies. If p is nilpotent in R, these topologies coincide and it is finer than the v-adic topology.*

Definition 1.2. — *A display $\mathscr{P}$ over R is a quadruple (P, Q, F, F_1) where P is a finitely generated projective $W(R)$-module, Q a submodule, $F : P \to P$, $F_1 : Q \to P$ are f-semilinear such that*

(i) $I_R P \subset Q$.

(ii) $0 \to Q/I_R P \to P/I_R P \to P/Q \to 0$ is a split sequence of R-modules.

(iii) P is generated by $\mathrm{im}(F_1)$.

(iv) $F_1(v(\xi)x) = \xi F(x)$ for $\xi \in W(R)$, $x \in P$.

If $u : M \to N$ is a f-semilinear map of $W(R)$-modules, we set $M^{(1)} = W(R) \otimes_{f,W(R)} M$ for the extension of scalars using f and denote by $u^\sharp : M^{(1)} \to N$ the associated linear map.

With the obvious notion of morphisms, displays form an additive category and, if we define a morphism of displays $u : \mathscr{P}' \to \mathscr{P}$ to be an admissible monomorphism (resp. epimorphism) provided $u : P' \to P$ is injective (resp. surjective) and $u^{-1}(Q) = Q'$ (resp. $u(Q') = Q$), we equip $\mathrm{Displays}_R$ with the structure of an exact category.

Definition 1.3. — *A normal decomposition for a display $\mathscr{P}$ over R is a direct sum decomposition $P = L \oplus T$ such that $Q = L \oplus I_R T$.*

If R is a p-adic ring, in particular if p is nilpotent in R, normal decompositions always exist. This is a consequence of the fact that finitely generated projective modules can always be lifted for surjections $A \to B$ whose kernel is a nilideal or such that A is separated and complete for the topology given by powers of the kernel.

Examples.– (i) The display corresponding to the formal multiplicative group $\mathscr{G} = (W(R), I_R, f, v^{-1})$.

(ii) If $R = k$, a perfect field of characteristic p, $M \mapsto \mathscr{P}_M = (M, V(M), F, V^{-1})$ establishes an equivalence of categories between Dieudonné modules over k and displays over k.

From now on we assume p is nilpotent in R, unless we explicitly state the contrary.

If $u : R \to R'$ is a ring homomorphism and $\mathscr{P}$ is a display over R, the base changed display $u_*(\mathscr{P})$ is the display over R', $\mathscr{P}' = (P', Q', F', F_1')$, where $P' = W(R') \otimes_{W(R)} P$, $Q' = \mathrm{Ker}(P' \to R' \otimes_R P/Q)$, $F' = f \otimes F$ and F_1' is determined by

$$F_1'\big(v(\xi) \otimes x\big) = \xi \otimes F(x)\,,\ \xi \in W(R')\,,\ x \in P$$

and

$$F_1'\big(\xi \otimes y\big) = f(\xi) \otimes F_1(y)\,,\ \xi \in W(R')\,,\ y \in Q.$$

Using a normal decomposition, it is easy to show that F_1' exists and $\mathscr{P}'$ is a display.

Definition 1.4. — *Let $\mathscr{P}, \mathscr{P}'$ be displays over R. A bilinear form of displays $(\ ,\ ) : \mathscr{P} \times \mathscr{P}' \to \mathscr{G}$ is a bilinear map $P \times P' \to W(R)$ such that $v(F_1 y, F_1' y') = (y, y')$ for $y \in Q$, $y' \in Q'$.*

If $\mathscr{P}$ is a display over R, its dual display $\mathscr{P}^t = (P^\vee, \widehat{Q}, F, F_1)$ where $P^\vee = \mathrm{Hom}_{W(R)}(P, W(R))$, $\widehat{Q} = \{z \in P^\vee | z(Q) \subset I_R\}$ and F and F_1 are determined by

$$\begin{array}{lcll} (F_1 x, Fz) & = & f(x,z) & \text{for } x \in Q\,,\ z \in P^\vee \\ (Fx, Fz) & = & pf(x,z) & \text{for } x \in P\,,\ z \in P^\vee \\ (Fx, F_1 z) & = & f(x,z) & \text{for } x \in P\,,\ z \in \widehat{Q} \\ v(F_1 x, F_1 z) & = & (x,z) & \text{for } x \in Q\,,\ z \in \widehat{Q}. \end{array}$$

We have a canonical isomorphism

$$\mathrm{Bil}(\mathscr{P}, \mathscr{P}'; \mathscr{G}) \simeq \mathrm{Hom}(\mathscr{P}', \mathscr{P}^t).$$

Proposition 1.5. — *There is a unique linear map $V^\sharp : P \to P^{(1)}$ determined by $V^\sharp(\xi F x) = p\xi \otimes x$, $V^\sharp(\xi F_1 y) = \xi \otimes y$, for $\xi \in W(R)$, $x \in P$, $y \in Q$.*

This is established by taking a normal decomposition $P = L \oplus T$, showing that $F_1^\sharp \oplus F^\sharp : L^{(1)} \oplus T^{(1)} \to P$ is bijective and defining $V^\sharp$ to be the composite

$$(\mathrm{id} \oplus p \cdot \mathrm{id}) \circ (F_1^\sharp \oplus F^\sharp)^{-1} : P \to L^{(1)} \oplus T^{(1)} = P^{(1)}.$$

One has $F^\sharp \circ V^\sharp = p \cdot \mathrm{id}_P$, $V^\sharp \circ F^\sharp = p \cdot \mathrm{id}_{P^{(1)}}$. If $P^{(i)}$ is the scalar extension of P using f^i, then $V^\sharp$ gives rise to $V_i^\sharp : P^{(i)} \to P^{(i+1)}$.

Definition 1.6. — *$\mathscr{P}$ satisfies the nilpotence condition or $\mathscr{P}$ is a* nilpotent display *provided there is an N such that $V_N^\sharp \circ V_{N-1}^\sharp \circ \cdots \circ V^\sharp$ is zero modulo $I_R + p\,W(R)$.*

Remark 1.7. — In [Z5], displays were called $3n$-displays ($3n$ for "not necessarily nilpotent") and nilpotent displays were called displays. We follow Zink's more recent terminology (*cf.* his Paris 13 lectures of February, 2006) here. Also in [Z5], F_1 was denoted by V^{-1}. Zink and Langer have initiated a theory of higher displays, [LZ2], in which $P = P_0$, $Q = P_1$ and there are higher P_i and $F_i : P_i \to P$. For this reason we write, following Zink, F_1 instead of his original V^{-1}.

Remark 1.8. — Locally on $\operatorname{Spec}(R)$, if $L\oplus T$ is a normal decomposition we will have L and T free modules and if T has basis $\{e_1,\dots,e_d\}$ and L has basis $\{e_{d+1},\dots,e_h\}$, the map $F_1^\sharp\oplus F^\sharp$ will be expressed in terms of these bases by a matrix $(\alpha_{ij})\in \mathrm{GL}_h(W(R))$. Conversely any such invertible matrix will determine a display. If the matrix (α_{ij}) has inverse $(\beta_{k\ell})$, and B is the $(h-d)\times(h-d)$ matrix with entries in $R/p\,R$ given by $B=(w_0(\beta_{k\ell}))\mathrm{mod}\,p)_{k,\ell=d+1,\dots,h}$, then $\mathscr{P}$ is nilpotent if and only if there is an N such that

$$B^{(p^N)}\dots B=0,$$

where $B^{(p^i)}$ is the matrix obtained by applying the i-th iterate of Frobenius to B.

If e_i is a basis for a free module over the Cartier ring, then the relations

$$\begin{aligned} Fe_i &= \sum \alpha_{ji}e_j\,, & i&=1,\dots d\,;\\ e_i &= V\Big(\sum \alpha_{ji}e_j\Big)\,, & i&=d+1,\dots h\end{aligned}$$

define a reduced Cartier module. Relations of this form were called by Norman [N] "displayed structural equations" of a reduced Cartier module. This is the origin of Zink's use of the term display.

Remark 1.9. — Let $S\overset{u}{\twoheadrightarrow}R$ be a surjection whose kernel is a nilideal. Let $\mathscr{P}$ be a display over R. Then there is a display $\mathscr{P}'$ over S and an isomorphism $u_*(\mathscr{P}')\xrightarrow{\sim}\mathscr{P}$.

This is proven using the fact that finitely generated projective $W(R)$-modules can be lifted to finitely generated projective $W(S)$-modules and using normal decompositions. Nakayama's lemma then shows that lifting modules are determined up to isomorphism (non-unique!).

If $\mathscr{P}/R$ is a nilpotent display and $\mathscr{P}'$ is a lifting to S, then $\mathscr{P}'$ is nilpotent too. This is clear as $\operatorname{Ker}(S\to R)$ is a nilideal.

We ask about the ambiguity in the *lifting* $\mathscr{P}'$ of $\mathscr{P}$. If $\mathscr{P}'=(P',Q',F',F_1')$, $J=\operatorname{Ker}(S\overset{u}{\twoheadrightarrow}R)$ and $\alpha: P'\to W(J)\otimes_{W(S)}P'$, we define a display $\mathscr{P}'_\alpha$ over S lifting $\mathscr{P}$ by $\mathscr{P}'_\alpha=(P',Q',F'_\alpha,F'_{1\alpha})$, where $F'_\alpha(x)=F'x-\alpha(F'x)$, for $x\in P'$, $F'_{1\alpha}(y)=F'_1y-\alpha(F'_1y)$, for $y\in Q'$. Then $\mathscr{P}'_\alpha$ is a display and Zink shows any lifting of $\mathscr{P}$ is isomorphic to a $\mathscr{P}'_\alpha$.

Remark 1.10. — Assume $p\cdot 1_R=0$. Let $\mathscr{P}$ be a display over R, $\mathscr{P}^{(p)}$ be the display over R given by $(\mathrm{Frob})_*\mathscr{P}$. Then $V^\sharp$ commutes with F and F_1 and hence defines a morphism of display $\mathscr{F}_{\mathscr{P}}:\mathscr{P}\to\mathscr{P}^{(p)}$. Similarly $F^\sharp$ defines a morphism of displays $\mathscr{V}_{\mathscr{P}}:\mathscr{P}^{(p)}\to\mathscr{P}$. Of course both composites are multiplications by p.

If $R\to R'$ is a ring homomorphism, there is an obvious notion of a descent datum for $\mathscr{P}'$ a R'-display and, if $\mathscr{P}$ is a R-display, $\mathscr{P}_{R'}$ has a canonical descent datum, can.

Zink proves:

Proposition 1.11. — *If $R \to R'$ is faithfully flat and p is nilpotent in R, then $\mathscr{P} \mapsto (\mathscr{P}_{R'}, \mathrm{can})$ is an equivalence of categories between* Displays$/R$ *and the category of R'-displays equipped with descent data. The same is true for nilpotent displays.*

2. THE CRYSTALS ASSOCIATED TO DISPLAYS

We refer to [Ber] for a detailed discussion of crystals, crystalline cohomology,... and recall the bare minimum here. An ideal $J \subset A$ has divided powers if we are given maps $\gamma_n : J \to J$, $n \geq 1$, satisfying axioms imposed by thinking of $\gamma_n(x)$ as $\frac{x^n}{n!}$. The ideal $(p) \subset \mathbf{Z}_{(p)}$ has unique divided powers since $\frac{p^n}{n!} \in (p)$. It follows that for any ring A, $p \cdot A$ has divided powers. If $J \subset A$ is an ideal with divided powers we require that its divided powers agree with those on $J \cap pA$. This is called the compatibility condition. If R is a $\mathbf{Z}_{(p)}$-algebra, then $I_R \subset W(R)$ has canonical divided powers which are compatible with those on $p \cdot W(R)$. These are determined by $\gamma_n\big(v(x)\big) = \frac{p^{n-1}}{n!} v(x^n)$, [Gr2]. The ideals $v^m(W(R))$ are sub-divided power ideals. We refer to [Ber] for the definition of nilpotent divided powers and to [M], [Z3] for a weaker notion.

We continue to assume p is nilpotent in R. If A is an R-algebra, a divided power thickening of A is a surjection $A' \overset{\pi}{\twoheadrightarrow} A$ such that p is nilpotent in A' and $\mathrm{Ker}(\pi)$ is equipped with divided powers (satisfying the compatibility condition). A morphism of divided power thickenings is a commutative diagramm

$$(*) \qquad \begin{array}{ccc} A' & \xrightarrow{\pi} & A \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\phi} \\ B' & \xrightarrow{\widetilde{\pi}} & B \end{array}$$

such that $\mathrm{Ker}(\pi)$, $\mathrm{Ker}(\widetilde{\pi})$ have divided powers, $\psi(\gamma_n(x)) = \gamma_n(\psi(x))$, $n \geq 1$ for $x \in \mathrm{Ker}\ \pi$.

A *crystal* in *modules* M on R is the giving for every divided power thickening $A' \overset{\pi}{\twoheadrightarrow} A$ of a A'-module, $M_{(A' \xrightarrow{\pi} A)}$, and for every morphism of divided power thickenings of an isomorphism

$$T_{(\psi,\phi)} : B' \underset{A'}{\otimes} \big(M_{(A' \xrightarrow{\pi} A)}\big) \xrightarrow{\sim} M_{(B' \xrightarrow{\widetilde{\pi}} B)},$$

these isomorphisms being required to satisfy the obvious transitivity condition.

Similarly we define a Witt-crystal on R as the giving for any divided power thickening of an R-algebra $\big(A' \xrightarrow{\pi} A\big)$ of a $W(A')$-module $K_{(A' \xrightarrow{\pi} A)}$ together with, for any diagram $(*)$, an isomorphism

$$T'_{(\psi,\phi)} : W(B') \underset{W(A')}{\otimes} K_{(A' \xrightarrow{\pi} A)} \xrightarrow{\sim} K_{(B' \xrightarrow{\tilde{\pi}} B')}.$$

We want now to explain Zink's functors

$$\begin{aligned} \text{Nilpotent Displays}/R &\longrightarrow \text{Crystals}/R \\ \mathscr{P} &\longmapsto \mathscr{D}_{\mathscr{P}} \end{aligned}$$

$$\begin{aligned} \text{Nilpotent Displays}/R &\longrightarrow \text{Witt crystals} \\ \mathscr{P} &\longmapsto \mathscr{K}_{\mathscr{P}}. \end{aligned}$$

Note first, and for later use as well, that if C is a $\mathbf{Z}_{(p)}$-algebra commutative and associative, but not necessarily with an identity, $W(C)$ is defined.

Assume C has divided powers; then we can divide the n-th ghost component and write $w'_n =$ "$\frac{w_n}{p^n}$" via

$$w'_n(\gamma) = \frac{\sum_{i=0}^n p^i x_i^{p^{n-i}}}{p^n} = \sum_{i=0}^n (p^{n-i} - 1)!\, \gamma_{p^{n-i}}(x_i).$$

In particular if $J = \operatorname{Ker}(\pi)$, $\pi : A' \twoheadrightarrow A$ has divided powers, then the map $W(J) \xrightarrow{w'} J^{\mathbf{N}}$, $x \longmapsto \big(w'_0(x_0), w'_1(x_0, x_1), \ldots, w'_n(x_0, \ldots, x_n), \ldots\big)$ is an isomorphism of $W(A')$-modules where the target is made into a $W(A')$-module via

$$\xi \cdot \underline{y} = \big(w_0(\xi)y_0, w_1(\xi)y_1, \ldots, w_n(\xi)y_n, \ldots\big)\,, \quad \underline{y} \in J^{\mathbf{N}}.$$

We also refer to the $y_i = w'_i(x)$ as the logarithmic coordinates and write this isomorphism as $\log : W(J) \xrightarrow{\sim} J^{\mathbf{N}}$. Then $\log^{-1}(J, 0, 0, \ldots) \subset W(J)$ is an ideal in $W(A')$ which we abusively denote by J.

This embedding $J \subset W(A')$ depends of course on the divided power structure on J. We have $J \oplus I_{A'} \subset W(A')$ and $f(J) = (0)$. By using a normal decomposition, one easily obtains the basic lemma:

Lemma 2.1. — *Assuming $\mathscr{P}'$ is a display over A', p is nilpotent in A' and $J \subset A'$ is an ideal with divided powers, F'_1 has a unique extension to $W(J) \cdot P + Q$ such that $F'_1(J \cdot P) = (0)$.*

Corollary 2.2. — *Assume p is nilpotent in A', $J \subset A'$ is a nilpotent ideal and $\mathscr{P}'_1$, $\mathscr{P}'_2$ are nilpotent displays over A'. Let $\pi : A' \to A'/J$, $\mathscr{P}_i = \pi_*(\mathscr{P}'_i)$. Then*

$$\operatorname{Hom}(\mathscr{P}'_1, \mathscr{P}'_2) \longrightarrow \operatorname{Hom}(\mathscr{P}_1, \mathscr{P}_2)$$

is injective and its cokernel is p-torsion. Further, if the nilradical of A' is nilpotent, $\operatorname{Hom}(\mathscr{P}'_1, \mathscr{P}'_2)$ *is p-torsion free.*

This result is the analogue of rigidity for p-divisible groups.

We now introduce the concept of a $\mathscr{P}$-triple where $\mathscr{P}$ is a nilpotent display over R. Let $R' \overset{\pi}{\twoheadrightarrow} R$ be a surjection whose kernel J is equipped with divided powers and where p is nilpotent in R'.

Definition 2.3. — *A $\mathscr{P}$-triple over R' is a triple (P', F', F_1') such that P' is a finitely generated projective $W(R')$-module which lifts P, and if Q' = the inverse image of Q, $F' : P' \to P'$, $F_1' : Q' \to P'$ are f-semilinear and satisfy*

(i) *F' (resp. F_1') lifts F (resp. F_1).*
(ii) *$F_1'(v(\xi) \cdot x) = \xi F'(x)$, $\xi \in W(R')$, $x \in P'$.*
(iii) *$F_1'(J \cdot P') = (0)$.*

If $\mathscr{P}_i$, $i = 1, 2$ are nilpotent displays and $\mathscr{T}_i$ are $\mathscr{P}_i$-triples and $\alpha : \mathscr{P}_1 \to \mathscr{P}_2$, there is an obvious notion of an α-morphism $\widetilde{\alpha} : \mathscr{T}_1 \to \mathscr{T}_2$.

There is also the notion of base change for $\mathscr{P}$-triples. If

$$\begin{tikzcd} R' \arrow[r, "u'"] \arrow[d] & S' \arrow[d] \\ R \arrow[r, "u"] & S \end{tikzcd}$$

is a morphism of divided power thickenings and $\mathscr{T}$ is a $\mathscr{P}$-triple, $u'_*(\mathscr{T})$ is a $u_*(\mathscr{P})$-triple with $P'_{S'} = W(S') \otimes_{W(R)} P'$ and $F'_{S'}$, $F'_{1S'}$ determined in the obvious way.

Theorem 2.4. — *Let $\alpha : \mathscr{P}_1 \to \mathscr{P}_2$ be a morphism of nilpotent displays over R, $R' \to R$ a divided power thickening and $\mathscr{T}_i$, $\mathscr{P}_i$-triples over R'. Then there exists a unique α-morphism $\widetilde{\alpha} : \mathscr{T}_1 \to \mathscr{T}_2$.*

Using this theorem, we construct functors:

$$\mathscr{D} : \text{nilpotent displays}/R \longrightarrow \text{crystals}/R$$
$$\mathscr{K} : \text{nilpotent displays}/R \longrightarrow \text{Witt crystals}/R$$

If $\mathscr{P}$ is a nilpotent display over R and $\mathscr{T} = (P', F, F_1)$ is a $\mathscr{P}$-triple over the divided power thickening $R' \to R$, then $\mathscr{D}_{\mathscr{P}}(R') = P'/I_{R'}\, P'$ and $\mathscr{K}_{\mathscr{P}}(R') = P'$.

Let $\mathscr{P} = (P, Q, F, F_1)$ be a nilpotent display over R. We consider its "Hodge filtration" $Q/I_R\, P \subset P/I_R\, P$. If $R' \to R$ is a divided power thickening and $\mathscr{T} = (P', F, F_1)$ is a $\mathscr{P}$-triple over R', we call a lift of the Hodge filtration the giving of a direct summand $L \subset P'/I_{R'}\, P'$ such that $L \otimes_{R'} R = Q/I_R\, P$. If Q'_L is the inverse image of L in P', then $\mathscr{P}' = (P', Q'_L, F, F_1)$ is a nilpotent display over R' which lifts $\mathscr{P}$.

Consider a divided power thickening $R' \to R$. Let $\mathscr{C}$ be the category whose objects are pairs $(\mathscr{P}, L)$ where $\mathscr{P}$ is a nilpotent display over R and L is a direct summand of $\mathscr{D}_{\mathscr{P}}(R')$ lifting the Hodge filtration. A morphism $\phi : (\mathscr{P}_1, L_1) \to (\mathscr{P}_2, L_2)$ in $\mathscr{C}$ is a morphism of nilpotent displays $\phi : \mathscr{P}_1 \to \mathscr{P}_2$ such that $\mathscr{D}(\phi)(R')L_1 \subset L_2$.

The following result is the analogue of [M, V (1.6)].

Proposition 2.5. — *The functor nilpotent display/R' $\to \mathscr{C}$ given by $\mathscr{P}' = (P', Q', F, F_1) \mapsto \big(\mathscr{P}'_R, Q'/I_{R'}\, P'\big)$ is an equivalence of categories.*

If R is a ring of characteristic p, then the morphisms

$$\mathscr{F}_{\mathscr{P}} : \mathscr{P} \longrightarrow \mathscr{P}^{(p)} \;,\; \mathscr{V}_{\mathscr{P}} : \mathscr{P}^{(p)} \longrightarrow \mathscr{P}$$

define associated morphisms of the crystals. In particular $\mathscr{D}_{\mathscr{P}}$ is endowed with the structure of a Dieudonné crystal in the sense of [Gr1], [Gr2], [BBM].

Let $R' \xrightarrow{\pi} R$ be a divided power thickening where p is nilpotent in R'. If $J = \mathrm{Ker}(\pi)$, then $J \subset W(R')$ has divided powers given by transport of structure via log. $I_{R'}$ has divided powers and as $J + I_{R'} = J \oplus I_{R'}$, we obtain divided powers on this ideal. Hence $W(R') \xrightarrow{\pi \circ w_0} R$ is a (topological) divided power thickening, inducing divided power thickenings $W_n(R') \to R$.

Using the Cartier map $W(R) \xrightarrow{\delta} W(W(R))$, characterized by $w_n \circ \delta = f^n$, $n \in \mathbf{N}$, Zink proves:

Proposition 2.6. — *Let $\mathscr{P}$ be a nilpotent display. There is a canonical isomorphism*

$$\mathscr{K}_{\mathscr{P}}(R') \simeq \varprojlim \mathscr{D}_{\mathscr{P}}(W_n(R')).$$

If R has characteristic p, this isomorphism is compatible with Frobenius and Verschiebung.

Proposition 2.7. — *Let R have characteristic p. Assume there is a topological divided power thickening $S \to R$ which is a flat $\mathbf{Z}_p$-algebra. Then the functor $\mathscr{P} \mapsto \big(\mathscr{D}_{\mathscr{P}}, Q/I_R\, P\big)$ from nilpotent displays to filtered Dieudonné crystals is fully-faithful in the following weak sense. Given a morphism ψ between the filtered Dieudonné crystals, there is a morphism of displays ϕ inducing ψ for every divided power thickening $R' \to R$ which receives a map from (some) S to R'.*

Remark 2.8. — The same result is true for the filtered Witt-crystals, the filtration being given by $\widehat{Q}$. Also one need only assume ψ is compatible with Frobenius. Finally, if $p \neq 2$, the same result holds for the nilpotent crystalline site.

3. THE FUNCTOR BT

Recall that an R-algebra N without identity is nilpotent provided $N^n = (0)$ for some positive integer n. There are many equivalent ways to define formal groups. For us we will regard a formal group G over R as a functor from nilpotent R-algebras to abelian groups satisfying:

(i) if $0 \to N_1 \to N_2 \to N_3 \to 0$ is exact (as a sequence of R-modules)

$$0 \longrightarrow G(N_1) \longrightarrow G(N_2) \longrightarrow G(N_3) \longrightarrow 0$$

is exact;

(ii) if any R-module M is viewed as a nilpotent R-algebra ($M^2 = (0)$), then $\oplus_{i\in I} G(M_i) \xrightarrow{\sim} G\big(\oplus_{i\in I} M_i\big)$;

(iii) $G(R)$ is a finitely generated projective module, R being viewed as having square zero. This is the tangent space, denoted $\mathrm{Lie}(G)$.

We will define a functor $BT : \text{Displays}/R \longrightarrow \text{Formal Groups}/R$.

If N is a nilpotent R-algebra, let $\widehat{W}(N) \subset W(N)$ be the $W(R)$-subalgebra consisting of all Witt vectors almost all whose components are zero. Given a display $\mathscr{P} = (P, Q, F, F_1)$ over R, we set $G^0_{\mathscr{P}}(N) = \widehat{W}(N) \otimes_{W(R)} P$, $G^{-1}_{\mathscr{P}}(N) \subset G^0_{\mathscr{P}}(N)$ is the subgroup generated by $\{v(\xi) \otimes x,\ \xi \otimes y \mid \xi \in \widehat{W}(N),\ x \in P,\ y \in Q\}$. The functors $G^{-1}_{\mathscr{P}}(N)$ and $G^0_{\mathscr{P}}(N)$ are exact in the sense of condition (i) in our definition of a formal group.

Lemma 3.1. — (i) *The map $F_1 : Q \to P$ extends to a map $G^{-1}_{\mathscr{P}}(N) \xrightarrow{F_1} G^0_{\mathscr{P}}(N)$ determined by $F_1\big(v(\xi) \otimes x\big) = \xi \otimes F(x)$, $F_1\big(\xi \otimes y\big) = f(\xi) \otimes F_1(y)$.*

(ii) *If N is equipped with nilpotent divided powers, the F_1 of* (i) *extends to a nilpotent endomorphism of $G^0_{\mathscr{P}}(N)$.*

(iii) *If i is the inclusion of $G^{-1}_{\mathscr{P}}(N)$ in $G^0_{\mathscr{P}}(N)$, $F_1 - i : G^{-1}_{\mathscr{P}}(N) \to G^0_{\mathscr{P}}(N)$ is injective.*

Definition 3.2. — *Let $BT_{\mathscr{P}}$ be the functor on nilpotent R-algebras $BT_{\mathscr{P}}(N) =$* $\mathrm{coker}(F_1 - i)$.

Theorem 3.3. — (i) *The functor BT takes values in the category of formal groups and commutes with base change.*

(ii) *If N has nilpotent divided powers, there is a canonical isomorphism*

$$\exp : N \underset{R}{\otimes} \frac{P}{Q} \xrightarrow{\sim} BT_{\mathscr{P}}(N).$$

(iii) *If $\mathscr{P}$ is a nilpotent display, $BT_{\mathscr{P}}$ is a p-divisible formal group.*

(iv) *If R has characteristic p, BT transforms $\mathscr{F}_{\mathscr{P}}$ and $\mathscr{V}_{\mathscr{P}}$ to Frobenius and Verschiebung.*

Recall that the Cartier ring C_R is by definition $\mathrm{End}(\widehat{W})^0$ and that its elements may be uniquely expressed as $c = \sum_{n,m\geq 0} V^n[a_{n,m}]\,F^m$ where, for fixed n, $a_{nm} = 0$ for all but finitely many m. For $x \in \widehat{\widehat{W}}(N)$, $c \in C_R$ as above, we have $x\cdot c = \sum_{n,m\geq 0} v^m\big([a_{n,m}]\,f^n(x)\big)$. The Cartier module associated to a formal group G is the left C_R-module $\mathrm{Hom}(\widehat{W}, G) = M(G)$.

Proposition 3.4. — *Let $\mathscr{P}$ be a display over R, $C_R \otimes_{W(R)} P \xrightarrow{\sim} \mathrm{Hom}\big(\widehat{W}, G^0_{\mathscr{P}}\big)$ via $c\otimes z \longmapsto (x \mapsto xc\otimes z)$. The map $G^0_{\mathscr{P}} \to BT_{\mathscr{P}}$ induces a surjection $C_R \otimes_{W(R)} P \longrightarrow M\big(BT_{\mathscr{P}}\big)$ whose kernel is the C_R-submodule generated by*

$$\{F\otimes x - 1\otimes Fx\,,\ V\otimes F_1 y - 1\otimes y \,|\, x\in P\,,\ y\in Q\}.$$

Recall that if G/R is a p-divisible formal group, the crystalline Dieudonné theory associates to G a crystal as follows. For any divided power thickening $R' \to R$ with nilpotent divided powers and any p-divisible group G' lifting G, we consider the universal extension by a vector group

$$0 \longrightarrow \omega_{G^*} \longrightarrow E(G') \longrightarrow G' \longrightarrow 0.$$

By definition $\mathbf{D}(G)_{R'}$ is $\mathrm{Lie}(E(G'))$ which is, up to canonical isomorphism, independent of G'. Because G is a formal group, this definition extends to the site consisting of divided power thickening $R' \xrightarrow{\pi} R$ where $\mathrm{Ker}(\pi)$ is a nilpotent ideal [MM], [Z5].

The following is one of Zink's main theorems.

Theorem 3.5. — (i) *The functors $\mathscr{P} \mapsto \mathscr{D}_{\mathscr{P}}$ and $\mathscr{P} \mapsto \mathbf{D}(BT_{\mathscr{P}})$ from the category of nilpotent displays to the category of crystals (where the divided power ideals of thickenings are nilpotent) are canonically isomorphic. The canonical isomorphism transforms Hodge filtration to Hodge filtration.*

(ii) *If $\mathscr{P}$ is a nilpotent display over R, $R' \overset{\pi}{\twoheadrightarrow} R$ has nilpotent kernel and G'/R' is a p-divisible group lifting $BT_{\mathscr{P}}$, then there is a nilpotent display $\mathscr{P}'$ lifting $\mathscr{P}$ and an isomorphism $BT_{\mathscr{P}'} \to G'$ lifting the identity.*

With the notation of (ii),

(iii) *if $\alpha : \mathscr{P} \to \widetilde{\mathscr{P}}$ is a morphism of nilpotent displays over R and $\widetilde{G}'$ is a lift to R' of $BT_{\widetilde{\mathscr{P}}}$ with corresponding display $\widetilde{\mathscr{P}}'$, then α lifts to $\alpha' : \mathscr{P}' \to \widetilde{\mathscr{P}}'$ if and only if $BT(\alpha)$ lifts to an homomorphism $G' \to \widetilde{G}'$.*

(iv) *The functor BT from nilpotent displays to p-divisible formal groups is faithful.*

Parts (ii), (iii) of Theorem 3.5 follow from Proposition 2.5 and [M] in the case where $R' \twoheadrightarrow R$ has kernel with nilpotent divided powers. As usual this extends inductively to the case where this kernel is a nilpotent ideal. For (iv), let $\alpha : \mathscr{P} \to \mathscr{P}'$ induce $a : BT_{\mathscr{P}} \to BT_{\mathscr{P}'}$. If $a = 0$, $\mathbf{D}(a) = 0$. But evaluating $\mathbf{D}(a)$ on the thickenings $W_n(R) \to R$ and passing to the projective limit, we find, by (i) and Proposition 2.6, that $\alpha = 0$.

4. PROPERTIES OF THE FUNCTOR BT FOR NILPOTENT DISPLAYS

All displays in this paragraph are nilpotent. Before stating the next result recall that a noetherian ring R is said to be a Nagata ring if, for every prime ideal $\mathfrak{p}$ of R, the integral closure of $R/\mathfrak{p}$ in a finite extension of $\mathrm{Frac}(R/\mathfrak{p})$ is a finitely generated $R/\mathfrak{p}$-module. Examples of such rings include excellent rings. The concept of Nagata ring is discussed in [Bour, IX, § 4] and, of course, in E.G.A. IV, § 7.7, where these rings are called (noetherian and) universally Japanese(1).

Proposition 4.1. — (i) *Let p be nilpotent R and assume the nilradical of R is nilpotent. Then BT : nilpotent displays over $R \to p$-divisible formal groups is fully faithful.*

(ii) *Assume R has characteristic p and admits a topological divided power thickening $S \overset{\pi}{\twoheadrightarrow} R$,* i.e. *$J = Ker(\pi)$ has divided powers and S is separated and complete for the linear topology defined by a sequence of sub-divided power ideals $J = J_1 \supset J_2 \supset \dots$ such that* $\mathrm{Ker}(S/J_n \to R)$ *is a nilpotent ideal and S is flat as a* $\mathbf{Z}_p$*-algebra. Then BT is fully-faithful.*

Theorem 4.2. — *Assume R is a Nagata ring which is separated and complete in the p-adic topology. Then BT is an equivalence of categories between nilpotent displays over R and p-divisible formal groups over R.*

Proof.— The theorem is proved in successive steps. As each category is the projective limit in the sense of $\varprojlim$ of the categories relative to R/p^n, we may assume p is nilpotent in R. Then, by Proposition 4.1, we know BT is fully-faithful. Hence we need show that it is essentially surjective.

(i) Assume $R = k$ a field. Let $K = k^{p^{-\infty}}$. By classical Dieudonné theory the result is true over K. If G/k is a p-divisible formal group, let $\mathscr{P}_K$ be a nilpotent display over K such that $BT(\mathscr{P}_K) = G_K$. If C is a Cohen ring for k we choose a map $C \to W(K)$ lifting the inclusion of k in K. Let $A = W(K) \otimes_C W(K)$, a p-torsion free ring such that $A/p\,A = K \otimes_k K$. If $S = \widehat{A}$, the p-adic completion, then Proposition 4.1 (ii) tells us that BT is fully-faithful over $K \otimes_k K$. Then $\mathscr{P}_K$ is equipped by Proposition 4.1 (ii), with descent data and, by Proposition 1.11, there is a nilpotent display $\mathscr{P}$ over k which descends $\mathscr{P}_K$. Then $BT(\mathscr{P})$ is isomorphic to G.

(ii) R is an artin local ring with residue field k. Let G/R be a p-divisible formal group. Let $\mathscr{P}_k$ be a nilpotent display such that $BT(\mathscr{P}_k) = G_k$. By theorem 3.5 (ii), there is a nilpotent display $\mathscr{P}$ lifting $\mathscr{P}_k$ and an isomorphism $BT(\mathscr{P}) \to G$.

(1)Gabber has told me that if V is a complete height 1 valuation ring with algebraically closed fraction field, then $V\{X_1, \dots, X_n\}[Y_1, \dots, Y_m]$ is universally Japanese, but of course not noetherian.

(iii) R is a complete local ring. By the result just used in case (ii) we may assume R is reduced. Let $G_n = G \otimes R/\mathfrak{m}^n$. Then $G_n = BT\big(\mathscr{P}_n\big)$ and let $\mathscr{P}$ be the display corresponding to the system $\big(\mathscr{P}_n\big)$. We must show that $\mathscr{P}$ is a nilpotent display. If $H = BT(\mathscr{P})$ we look at $M(H)$ described in Proposition 3.4, the Cartier module. We embed R in a finite product of algebraically closed fields and hence reduce to the case where $R = k$ is an algebraically closed field. Then $\mathscr{P} = \mathscr{P}_{\mathrm{nil}} \oplus \mathscr{P}_{\text{ét}}$, where $\mathscr{P}_{\mathrm{nil}}$ is a nilpotent display and $P_{\text{ét}}$ has a basis $\{e_1, \dots, e_h\}$, $Q = P_{\text{ét}}$ and $F_1(e_i) = e_i$ for $i = 1, \dots, h$ and $M(BT(\mathscr{P}_{\text{ét}}))$ has a presentation $\oplus_{i=1}^h C_k \cdot \frac{e_i}{Ve_i - e_i}$. But $V - 1$ is a unit in C_k so $M\big(BT(\mathscr{P}_{\text{ét}})\big) = (0)$ and $M\big(BT(\mathscr{P}_{\mathrm{nil}})\big) = M(BT(\mathscr{P}))$. But $\mathrm{height}(G) = \mathrm{rank}\big(P_{\mathrm{nil}}\big)$ and $\mathrm{height}(G) = \mathrm{rank}(P)$. Then $\mathscr{P}_{\mathrm{nil}} = \mathscr{P}$, finishing the case when R is a complete local ring.

(iv) R is a Nagata local ring. Let $\widehat{R}$ be its completion. We may assume R is reduced. Then $\widehat{R}$ and $\widehat{R} \otimes_R \widehat{R}$ are both reduced [Bour, IX, § 4, théorème 3]. Given G/R, let $\widehat{\mathscr{P}}$ be a nilpotent display over $\widehat{R}$ such that $BT(\widehat{\mathscr{P}}) = G_{\widehat{R}}$. Using Proposition 4.1 (i), Theorem 3.5 (iv), we see $\widehat{\mathscr{P}}$ is equipped with descent data and this gives the result just as in the case when R is a field.

(v) The general case. We may assume that the h = height of G is constant on $\mathrm{Spec}(R)$. Let $R' = \Pi\, R_{\mathfrak{m}}$, the product running over all maximal ideals of R. Then to give a p-divisible formal group G of height h over R' is the same as giving for each $\mathfrak{m}$ a p-divisible formal group $G_{\mathfrak{m}}$ of height h over each $R_{\mathfrak{m}}$. Similarly to give a display $\mathscr{P}$ over R', with $\mathrm{rank}(P) = h$, is the same as giving displays $\mathscr{P}_{\mathfrak{m}}$ over each $R_{\mathfrak{m}}$, these each having $\mathrm{rank}(P_{\mathfrak{m}}) = h$. Both statements follow because idempotents in $M_n(R')$ are given by families of idempotents in the $M_n(R_{\mathfrak{m}})$. If all the $\mathscr{P}_{\mathfrak{m}}$ are nilpotent displays, then $\mathscr{P}$ is a nilpotent display also because the exponent of nilpotency for $V_{\mathfrak{m}}^{\sharp}$ is bounded above by h. As each $R_{\mathfrak{m}}$ is a Nagata local ring we conclude from Part (iv) that there is a nilpotent display $\mathscr{P}'$ over R' such that $BT(\mathscr{P}') = G_{R'}$. We will be able to apply descent to finish the proof, provided we can show $R' \otimes_R R'$ is reduced (so to be able to invoke Proposition 4.1 (i), Theorem 3.5 (iv) again). For any ring A, let $Q(A)$ be its full ring of quotients. As R' is faithfully flat over R, $R' \otimes_R R' \hookrightarrow Q(R') \otimes_{Q(R)} Q(R')$. As R is noetherian and reduced, $Q(R) = \prod_{i=1}^n Q(R/\mathfrak{p}_i)$ if $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ is the set of minimal prime ideals of R. The ring $Q(R') \otimes_{Q(R)} Q(R')$ then decomposes as the $\prod_{i=1}^n \big(Q(R') \otimes_{Q(R)} Q(R')\big) \otimes_{Q(R)} Q(R/\mathfrak{p}_i)$. As $Q(R') \hookrightarrow \prod Q(R_{\mathfrak{m}})$, we see that $R' \otimes_R R'$ embeds into the product of the rings

$$\prod_{\mathfrak{m}} Q\Big(\frac{R_{\mathfrak{m}}}{\mathfrak{p}_i}\Big) \underset{Q(R/\mathfrak{p}_i)}{\otimes} \prod_{\mathfrak{m}} Q\Big(\frac{R_{\mathfrak{m}}}{\mathfrak{p}_i}\Big), \quad i = 1, \dots n.$$

This reduces us to showing that if K is a field and I any index set $K^I \otimes_K K^I$ is reduced. This is standard as products and tensor products of separable algebras are separable.

Remark 4.3. — In [Z5], Zink stated Theorem 4.2 only for p nilpotent in R and R an excellent local ring or R/pR of finite type over a field (which implies R is excellent). But his proof works, as we have seen, in general. Indeed the proof shows that all we need to assume is that if R is noetherian and reduced, for each localization $R_{\mathfrak{m}}$, $\widehat{R}_{\mathfrak{m}}$ and $\widehat{R}_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} \widehat{R}_{\mathfrak{m}}$ are both reduced.

If R is a complete noetherian local ring which is normal, with fraction field K of characteristic zero and residue field of characteristic p, then for a nilpotent display $\mathscr{P}$ over R, the Tate module $T_p(BT_{\mathscr{P}})$ can be described explicitly in terms of $\mathscr{P}$. Let $\overline{K}$ be an algebraic closure of K, $\overline{R}$ the integral closure of R in $\overline{K}$, $\overline{\mathfrak{m}}$ its maximal ideal. For $K \subset E \subset \overline{K}$ with $[E : K]$ finite, set $\widehat{W}(\mathfrak{m}_E) = \varprojlim \widehat{W}(\mathfrak{m}_E/\mathfrak{m}_E^n)$ and $\widehat{W}(\overline{\mathfrak{m}}) = \varinjlim \widehat{W}(\mathfrak{m}_E)$. Let $\overline{W}(\overline{\mathfrak{m}})$ be the p-adic completion of $\widehat{W}(\overline{\mathfrak{m}})$. Let $\widetilde{\mathfrak{m}}$ be the p-adic completion of $\overline{\mathfrak{m}}$. For $\mathscr{P}$ a nilpotent display over R, let $G = BT_{\mathscr{P}}$ and define $\overline{G}^0_{\mathscr{P}}$ to be $\overline{W}(\overline{\mathfrak{m}}) \otimes_{W(R)} P$,

$$\overline{G}^{-1}_{\mathscr{P}} = \operatorname{Ker}\Big(\overline{W}(\overline{\mathfrak{m}}) \underset{W(R)}{\otimes} P \longrightarrow \widetilde{\mathfrak{m}} \underset{R}{\otimes} P/Q\Big).$$

As G is p-divisible we may write $G = \varinjlim G_n$ where G_n is the kernel of p^n on G. The Tate module $T_p(G)$ is by definition $\operatorname{Hom}\big(\mathbf{Q}_p/\mathbf{Z}_p, \varinjlim G_n(\overline{K})\big)$.

Proposition 4.4. — *There is an exact sequence of* $\operatorname{Gal}(\overline{K}/K)$*-modules*

$$0 \longrightarrow T_p(G) \longrightarrow \overline{G}^{-1}_{\mathscr{P}} \xrightarrow{F_1 - i} \overline{G}^0_{\mathscr{P}} \longrightarrow 0.$$

5. DUALITY

We briefly sketch now the duality theory for nilpotent displays and corresponding p-divisible formal groups. From Zink's perspective it is based upon a canonical homomorphism

$$\operatorname{Bil}(\mathscr{P}, \mathscr{P}'; \mathscr{G}) \longrightarrow \operatorname{Biext}\big(BT_{\mathscr{P}}, BT_{\mathscr{P}'}; \widehat{\mathbf{G}}_m\big).$$

We do not review here the formalism of biextensions referring the reader to Mumford's original paper [Mu], Grothendieck's geometric and homological versions of the theory [Gr3] and Zink's reformulation of these in his context [Z5]. Let us only say that we have, if A, B, C are abelian groups (in a topos) and we are given exact sequences

$$0 \longrightarrow K_1 \longrightarrow K_0 \longrightarrow B \longrightarrow 0$$
$$0 \longrightarrow L_1 \longrightarrow L_0 \longrightarrow C \longrightarrow 0,$$

an exact sequence

$$\begin{aligned}\mathrm{Hom}(K_0\otimes L_0, A) \longrightarrow \mathrm{Hom}(K_0\otimes L_1, A) \underset{\mathrm{Hom}(K_1\otimes L_1,A)}{\times} \mathrm{Hom}(K_1\otimes L_0, A)\\ \longrightarrow \mathrm{Biext}(B,C;A) \longrightarrow \mathrm{Biext}(K_0,L_0;A).\end{aligned}$$

We apply this formalism to the exact sequences

$$0 \longrightarrow G^{-1}_{\mathscr{P}} \xrightarrow{F_1 - i} G^0_{\mathscr{P}} \longrightarrow BT_{\mathscr{P}} \longrightarrow 0$$

$$0 \longrightarrow G^{-1}_{\mathscr{P}'} \xrightarrow{F_1 - i} G^0_{\mathscr{P}'} \longrightarrow BT_{\mathscr{P}'} \longrightarrow 0$$

where $\mathscr{P}$, $\mathscr{P}'$ are displays. Mumford proved that $\mathrm{Biext}\big(G^0_{\mathscr{P}}, G^0_{\mathscr{P}'}, \widehat{\mathbf{G}}_m\big) = (0)$ so every biextension of $BT_{\mathscr{P}}$, $BT_{\mathscr{P}'}$, by $\widehat{\mathbf{G}}_m$ is defined by a pair of morphism

$$\alpha_1 : G^{-1}_{\mathscr{P}} \otimes G^0_{\mathscr{P}'} \longrightarrow \widehat{\mathbf{G}}_m$$

$$\alpha_2 : G^0_{\mathscr{P}} \otimes G^{-1}_{\mathscr{P}'} \longrightarrow \widehat{\mathbf{G}}_m$$

agreeing on $G^{-1}_{\mathscr{P}} \otimes G^{-1}_{\mathscr{P}'}$.

Let $\alpha \in \mathrm{Bil}(\mathscr{P}, \mathscr{P}', \mathscr{G})$. We will associate to α a pair of such homomorphisms as follows. First, for N a nilpotent R-algebra we have

$$\alpha_N : G^0_{\mathscr{P}}(N) \otimes G^0_{\mathscr{P}'}(N) \longrightarrow \widehat{W}(N)$$

defined by $(\xi \otimes x) \otimes (\xi' \otimes x') \mapsto \alpha(x \otimes x')\xi\xi'$.

Next recall that the Artin-Hasse exponential defines an exact sequence

$$0 \longrightarrow \widehat{W}(N) \xrightarrow{v-\mathrm{id}} \widehat{W}(N) \xrightarrow{AH} \widehat{\mathbf{G}}_m \longrightarrow 0.$$

Here for $x \in \widehat{W}(N)$,

$$AH(x) = \Big[\exp\Big(\sum_{n=0}^{\infty} w_n(x)\frac{T^{p^n}}{p^n}\Big)\Big]_{T=1}.$$

Then with these notations we define

$$\begin{aligned}\alpha_1(y,x') &= AH\big(\alpha_N(F_1 y\otimes x')\big), && y\in G^{-1}_{\mathscr{P}}(N)\ ,\ x'\in G^0_{\mathscr{P}'}(N)\\ \alpha_2(x,y') &= \big[AH\big(\alpha_N(x\otimes y')\big)\big]^{-1}, && x\in G^0_{\mathscr{P}}(N)\,,\ y'\in G^{-1}_{\mathscr{P}'}(N).\end{aligned}$$

The verification that α_1, α_2 agree on $G^{-1}_{\mathscr{P}} \otimes G^{-1}_{\mathscr{P}'}$ is easy. One can show that we obtain the same biextension if $F_1 y$ is replaced by y in the equation defining α_1 and, simultaneously, y' is replaced by $F_1 y'$ in the equation defining α_2.

The category of nilpotent R-algebras is isomorphic to that of augmented R-algebras such that the augmentation ideal is nilpotent, Aug_R. We endow $(\mathrm{Aug}_R)^\circ$ with the fpqc topology and consider formal groups as abelian sheaves on this site. For any abelian sheaf F, we define a subsheaf F^+ by setting $F^+(X) = \mathrm{Ker}\big(F(X) \xrightarrow{F(\varepsilon)} F(\mathrm{Spec}(R))\big)$, where $\varepsilon : \mathrm{Spec}(R) \to X$ is the canonical section.

The homological formalism provides a canonical homomorphism $\mathrm{Biext}(B,C;A) \to \mathrm{Hom}(B, \underline{\mathrm{Ext}}^1(C,A))$. We apply this taking a nilpotent display $\mathscr{P}$, its dual display

$\mathscr{P}^t$ and the tautological $\alpha \in \mathrm{Bil}(\mathscr{P}^t, \mathscr{P}, \mathscr{G})$. This furnishes us with a homomorphism $BT_{\mathscr{P}^t} \xrightarrow{\chi} \underline{\mathrm{Ext}}^1(BT_{\mathscr{P}}, \widehat{\mathbf{G}}_m)$.

PROPOSITION 5.1. — *The map χ defines an isomorphism of formal groups*

$$BT_{\mathscr{P}^t} \longrightarrow \underline{\mathrm{Ext}}^1\big(BT_{\mathscr{P}}, \widehat{\mathbf{G}}_m\big)^+.$$

Remark. — In the case where $\mathscr{P}^t$ is also a nilpotent display, this result was given in [MM], with a completely different proof using [Ill3] and [Ill4].

PROPOSITION 5.2. — *Assume $\mathscr{P}$, $\mathscr{P}'$ are nilpotent displays and $(\mathscr{P}')^t$ is also a nilpotent display. Then the map* $\mathrm{Bil}(\mathscr{P}, \mathscr{P}', \mathscr{G}) \to \mathrm{Biext}\big(BT_{\mathscr{P}}, BT_{\mathscr{P}'}; \widehat{\mathbf{G}}_m\big)$ *is an isomorphism.*

6. DIEUDONNÉ DISPLAYS AND p-DIVISIBLE GROUPS

Assume R is a complete noetherian local ring with perfect residue field k of characteristic p. If $p = 2$, assume in addition that $2R = (0)$. We will modify the definition of a display so as to obtain an equivalence of categories

$$BT : \text{Dieudonné displays}/R \xrightarrow{\approx} p\text{-divisible groups}/R.$$

If R is artinian we will consider a subring $\widehat{W}(R) \subset W(R)$, stable under f and v, functorial in R and having $\widehat{W}(k) = W(k)$. Let $\mathfrak{m}$ be the maximal ideal of R and let $x \in W(k)$. Let $y_n \in W(k)$ satisfy $f^n(y_n) = x$. Let $\widetilde{y}_n \in W(R)$ be any lift of y_n. Then for $n \gg 0$, $f^n(\widetilde{y}_n)$ is independent of the choice and the map $\delta : W(k) \to W(R)$ defined by $\delta(x) = f^n(\widetilde{y}_n)$, n large, is a ring homomorphism which is a section of $W(R) \xrightarrow{\pi} W(k)$.

We consider the sequence

$$0 \longrightarrow W(\mathfrak{m}) \longrightarrow W(R) \xrightarrow{\pi} W(k) \longrightarrow 0.$$

Let $\widehat{W}(\mathfrak{m}) = \big\{x_0, \dots, x_i, \dots) \in W(\mathfrak{m}) \,|\, \text{almost all } x_i = 0\big\}$. Then $\widehat{W}(\mathfrak{m})$ is stable under f and v and we define $\widehat{W}(R) = \{x \in W(R) \,|\, x - \delta\pi x \in \widehat{W}(\mathfrak{m})\}$. $\widehat{W}(\mathfrak{m})$ is an ideal in $W(R)$ and consequently $\widehat{W}(R)$ is a ring. This ring is stable under f and v. It is easy to see that $\widehat{W}(R)$ is a (non-noetherian) local ring which is separated and complete in the topology defined by its maximal ideal $\widehat{W}(\mathfrak{m}) + \delta(p \cdot W(k))$.

Now we define a Dieudonné display $\mathscr{P}$ over R in exactly the same manner as we defined a display in Definition 1.2, replacing $W(R)$ by $\widehat{W}(R)$. Just as before there is a map $V^\sharp : P \to P^{(1)}$ and we have

$$F^\sharp \circ V^\sharp = p \cdot \mathrm{id}_P \ , \ V^\sharp \circ F^\sharp = p \cdot \mathrm{id}_{P^{(1)}}.$$

We consider divided power thickenings $S \xrightarrow{\pi} R$ such that for each $a \in \mathrm{Ker}(\pi) = J$, we have $\gamma_n(a) = 0$ for $n \gg 0$ (depending on a). We have an exact sequence

$$0 \longrightarrow \widehat{W}(J) \longrightarrow \widehat{W}(S) \longrightarrow \widehat{W}(R) \longrightarrow 0.$$

The condition imposed on the divided powers implies that log induces an isomorphism

$$\log \widehat{W}(J) \xrightarrow{\sim} J^{(\mathbf{N})}.$$

With this nilpotency assumption on the divided powers, Lemma 2.1 and Theorem 2.4 extend to Dieudonné displays. Just as before this enables us to define the Witt and Dieudonné crystals associated to a Dieudonné display.

We have a "forgetful" functor Dieudonné displays $\xrightarrow{\mathscr{F}}$ displays, $P \mapsto W(R) \otimes_{\widehat{W}(R)} P$.

We say that a Dieudonné display $\mathscr{P}$ is nilpotent if $\mathscr{F}(\mathscr{P})$ is nilpotent. The functor $\mathscr{F}$ establishes an equivalence of categories between nilpotent Dieudonné displays and nilpotent displays. This follows easily from the fact that, for a divided power thickening $S \xrightarrow{\pi} R$, the functor

Dieudonné displays$/S \longrightarrow$ Dieudonné displays$/R +$ lifts of the Hodge filtration

is an equivalence of categories.

Proposition 6.1. — *The category of nilpotent Dieudonné displays is equivalent via a functor BT to the category of p-divisible formal group/R. If A is a finite R-algebra* (*so a product of artinian local rings*) *and $\mathscr{P}$ is such a nilpotent display with base change to A, $\mathscr{P}_A = (P', Q', F, F_1)$, then we have an exact sequence*

$$0 \longrightarrow Q' \xrightarrow{F_1 - i} P' \longrightarrow BT_{\mathscr{P}}(A) \longrightarrow 0.$$

Proposition 6.2. — *Let P be a finitely generated projective $\widehat{W}(R)$-module and $\phi : P^{(1)} \to P$* (resp. *$\phi : P \to P^{(1)}$*). *Then there is a direct summand P^{tm} of P* (resp. *a projective quotient $P^{\text{ét}}$ of P*) *such that ϕ induces on P^{tm}* (resp. *on $P^{\text{ét}}$*) *an isomorphism. Further, if M is any $\widehat{W}(R)$-module equipped with an isomorphism $\psi : M^{(1)} \to M$* (resp. an isomorphism *$\psi : M \to M^{(1)}$*) *and $\alpha : M \to P$* (resp. *$\alpha : P \to M$*) *and $\phi \circ \alpha^{(1)} = \alpha \circ \psi$* (resp. *$\alpha^{(1)} \circ \phi = \psi \circ \alpha$*), *then α factors uniquely through P^{tm}* (resp. *$P^{\text{ét}}$*).

Proposition 6.3. — *Let $\mathscr{P} = (P, Q, F, F_1)$ be a Dieudonné display.*

a) *$P = Q \Longleftrightarrow V^{\sharp}$ is an isomorphism.*

b) *$I_R\, P = Q \Longleftrightarrow F^{\sharp}$ is an isomorphism.*

We say $\mathscr{P}$ is étale if a) *holds and $\mathscr{P}$ is of multiplicative type if* b) *holds.*

Proposition 6.4. — *Let $\mathscr{P}/R$ be a Dieudonné display. There is a map $\mathscr{P} \to \mathscr{P}^{\text{ét}}$ to an étale Dieudonné display which is universal with respect to morphisms from $\mathscr{P}$ to étale Dieudonné displays. The map $P \to P^{\text{ét}}$ is surjective and if P^{nil} is its kernel, then $\mathscr{P}^{\text{nil}} = (P^{\text{nil}}, P^{\text{nil}} \cap Q, F, F_1)$ is a nilpotent Dieudonné display.*

This proposition has a dual proposition which we do not state explicitly.

Our artin ring R is canonically a $W(k)$-algebra. Let $\overline{k}$ be an algebraic closure of k, $\Gamma = \mathrm{Gal}(\overline{k}/k)$ and $\overline{R} = R \otimes_{W(k)} W(\overline{k})$, equipped with its continuous action of Γ. If H is a finitely generated free $\mathbf{Z}_p$-module endowed with a continuous, for its p-adic topology, action of Γ we set

$$P(H) = \left(\widehat{W}(\overline{R}) \otimes_{\mathbf{Z}_p} H\right)^{\Gamma}.$$

The natural map $\widehat{W}(\overline{R}) \otimes_{\widehat{W}(R)} P(H) \to \widehat{W}(\overline{R}) \otimes_{\mathbf{Z}_p} H$ is an isomorphism. Let $\mathscr{P}(H)$ be the étale Dieudonné display $(P(H), Q(H), F, F_1)$ where $P(H) = Q(H)$ and F_1 is induced by $f \otimes_{\mathbf{Z}_p} \mathrm{id}_H$ on $\widehat{W}(\overline{R}) \otimes_{\mathbf{Z}_p} H$, $F = p \cdot F_1$.

Conversely, if $\mathscr{P}$ is an étale Dieudonné display over R define $H(\mathscr{P})$ to be the kernel of the $\mathbf{Z}_p$-linear homomorphism.

$$F_1 - \mathrm{id} : \widehat{W}(\overline{R}) \otimes_{\widehat{W}(R)} P \longrightarrow \widehat{W}(\overline{R}) \otimes_{\widehat{W}(R)} P.$$

Proposition 6.5. — *These functors establish equivalences of categories between étale Dieudonné displays over R and continuous Γ-modules, finitely generated and free as $\mathbf{Z}_p$-modules.*

Proposition 6.6. — *Let $\mathscr{P}$ be a nilpotent Dieudonné display over R, $\overline{\mathscr{P}}$ the corresponding nilpotent Dieudonné display over $\overline{R}$ and $C_{\overline{R}}$ the cokernel of $F_1 - i : \overline{Q} \to \overline{P}$. Then there is an isomorphism* $\mathrm{Hom}_{\Gamma}(H, C_{\overline{R}}) \xrightarrow{\sim} \mathrm{Ext}(\mathscr{P}(H), \mathscr{P})$.

This is proved using Galois cohomology to establish that $H^1(\Gamma, \mathrm{Hom}_{\mathbf{Z}_p}(H, \overline{Q})) = (0)$.

Let G be a p-divisible group over R; we wish to associate to it a Dieudonné display. As R is an artin local ring we have an exact sequence

$$0 \longrightarrow \widehat{G} \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow 0$$

where $G^{\text{ét}}$ is étale and $\widehat{G}$ is a p-divisible formal group.

Write $G^{\text{ét}} = BT(H) = \varinjlim p^{-n}H/H$ and $\widehat{G} = BT_{\widehat{\mathscr{P}}}$, where $\widehat{\mathscr{P}}$ is a nilpotent Dieudonné display.

Proposition 6.7. — *There is a canonical isomorphism*

$$\mathrm{Hom}_{\Gamma}(H, \widehat{G}(\overline{R})) \xrightarrow{\sim} \mathrm{Ext}(BT(H), \widehat{G}).$$

Theorem 6.8. — *There is an equivalence of categories*

$$\text{Dieudonné displays}/R \xrightarrow{\approx} p\text{-divisible groups}/R.$$

This follows from Propositions 6.1, 6.6, 6.7 and the fact that there are no non-trivial homomorphisms in either direction between étale and nilpotent Dieudonné displays (resp. étale and p-divisible formal groups over R).

Remark 6.9. — The following questions raised by Zink in [Z3] remain open.

(i) If G/R is a p-divisible group with associated Dieudonné displays $\mathscr{P}$, is the Dieudonné crystal associated to $\mathscr{P}$ (canonically) isomorphic to that associated to G by crystalline Dieudonné theory? The answer is of course yes by Theorem 3.5 if G is a p-divisible formal group.

(ii) Lau has shown that the equivalence between p-divisible groups and Dieudonné displays is compatible with duality, [L2].

(iii) The classification when R is a complete noetherian local ring is obtained by defining $\widehat{W}(R) = \varprojlim \widehat{W}(R/\mathfrak{m}^n)$ and Dieudonné displays in the obvious manner since p-divisible groups$/R \xrightarrow{\approx} \underleftarrow{\mathrm{LIM}}$ p-divisible groups $R/\mathfrak{m}^n$.

7. WINDOWS AND DIEUDONNÉ DISPLAYS

If K is a local field of characteristic zero with perfect residue field of characteristic p, then one knows how to classify p-divisible groups over the ring of integer $\mathscr{O}_K$. If the absolute ramification index $e \leq p-2$, the maximal ideal $\mathfrak{m}_K$ has nilpotent divided powers and [Gr1] explains how to do this using the filtered Dieudonné module. A more direct approach to this classification is given in [Fon1]. Breuil has extended this result if $p \neq 2$ to the case where e is arbitrary,[Br], see also the appendix of [K]. Zink has generalized Breuil's result to give a classification of p-divisible groups over a local finite flat $W(k)$-algebra S. When $S = \mathscr{O}_K$, Zink further shows that the category Breuil uses, strongly divisible modules, is naturally equivalent to the category of Dieudonné displays over $\mathscr{O}_K$. We indicate now Zink's generalization.

Let R be a local ring with perfect residue field of characteristic $p \geq 3$. Assume there is a n such that $x^n = 0$ for all $x \in \mathfrak{m}$. If $S \xrightarrow{\pi} R$ is a divided power thickening and S satisfies the same hypotheses we have the log : $W(J) \xrightarrow{\sim} J^{\mathbf{N}}$, $J = \mathrm{Ker}(\pi)$. Let $\widetilde{W}(J) = \log^{-1}(J^{(\mathbf{N})})$. Then $\widehat{W}(J) \subset \widetilde{W}(J)$ and they are equal if the $\gamma_m(x)$ of each $x \in J$ are zero, for $m \gg 0$, depending on x. Denote by $\widetilde{W}(S)$ the subring of $W(S)$ generated by $\widehat{W}(S)$ (*cf.* § 6) and $\widetilde{W}(J)$. If A is a p-adic local ring equipped with an homomorphism $A \xrightarrow{\pi} R$ and $\mathrm{Ker}(\pi)$ has divided powers compatible with those on pA, we define $\widetilde{W}(A) = \varprojlim \widetilde{W}(A/p^n)$, the limit over those n such that $p^n = 0$ in R. Similarly we define $\widehat{W}(A)$.

Definition 7.1. — (i) *A frame for R is a flat $\mathbf{Z}_p$-algebra A which is a p-adic ring equipped with a surjection $\pi : A \to R$ whose kernel has divided powers and an endomorphism $\sigma : A \to A$ which lifts Frobenius.*

(ii) *A Dieudonné frame for R is a frame for R which, in addition, satisfies A/p^nA, is for all n a local ring whose maximal ideal satisfies the nilpotence condition imposed on R and the Cartier map $A \xrightarrow{\delta} W(A)$ factors through $\widetilde{W}(A)$.*

(iii) *A Dieudonné window for a frame* (A, σ) (resp. a Dieudonné frame) *is a finitely generated projective A-module M a submodule M_1 which contains JM, $J = \mathrm{Ker}(\pi)$, a σ-linear map $\Phi : M \to M$ such that M/M_1 is a projective R-module, $\Phi\, M_1 \subset pM$ and M is generated by $\Phi M \cup \frac{1}{p}\,\Phi\, M_1$.*

The hypothesis that $A \xrightarrow{\delta} W(A)$ factors through $\widetilde{W}(A)$ implies that the composite

$$A \xrightarrow{\delta} W(A) \xrightarrow{W(\pi)} W(R)$$

factors through $\widehat{W}(R)$. Hence we may associate to a Dieudonné window a Dieudonné display over R as follows

$$P = \widehat{W}(R) \otimes_A M$$
$$Q = \ker\Big(\widehat{W}(R) \otimes_A M \longrightarrow M/M_1\Big)$$

$$\begin{array}{lll} F(\xi \otimes x) & = f\xi \otimes \Phi x\,, & \xi \in \widehat{W}(R)\,,\ x \in M \\ F_1(\xi \otimes y) & = f\xi \otimes \frac{1}{p}\Phi y\,, & \xi \in \widehat{W}(R)\,,\ y \in M_1 \\ F_1(v\xi \otimes x) & = \xi \otimes \Phi x\,, & \xi \in \widehat{W}(R)\,,\ x \in M. \end{array}$$

Theorem 7.2. — *Let R satisfy our hypotheses and (A, σ) be a Dieudonné frame for R. Then the functor* Dieudonné A-windows $\to$ Dieudonné displays/R *is an equivalence of categories.*

This theorem is established by constructing a quasi-inverse functor. To do this one associates to a Dieudonné display a crystal generalizing the discussion of § 6 in that we no longer require any nilpotence condition on divided power thickenings. Lemma 2.1 and Theorem 2.4 extend to this situation. Note if the divided powers on $J = \ker\big(S \xrightarrow{\pi} R\big)$ are not nilpotent it is essential that one works with $\widetilde{W}(S)$ and not $\widehat{W}(S)$ for proving Lemma 2.1 in this context. Then, if $\mathscr{P}$ is a Dieudonné display over R and $\widetilde{\mathscr{P}}$ is a triple over A which lifts $\mathscr{P}$ in the sense that $\widetilde{\mathscr{P}} = (\widetilde{P}, F, F_1)$ where $\widetilde{P}$ is a finitely generated free $\widetilde{W}(A)$-module, the window associated to $\mathscr{P}$ is

$$M = A \underset{\widetilde{W}(A)}{\otimes} \widetilde{P} \quad , \quad M_1 = \mathrm{Ker}\Big(M \longrightarrow R \underset{\widehat{W}(R)}{\otimes} P \longrightarrow P/Q\Big).$$

The crystal associated to $\mathscr{P}/R$ depends only on $\overline{\mathscr{P}} = \mathscr{P}_{R/p}$. For this Dieudonné display we have

$$\mathscr{V}_{\overline{\mathscr{P}}} : \overline{\mathscr{P}}^{(p)} \longrightarrow \overline{\mathscr{P}}$$

and this induces $\Phi : M \to M$, σ-similinear.

Let S be a finite flat local $W(k)$-algebra. Consider a presentation

$$0 \longrightarrow I \longrightarrow W(k)[T_1, \ldots, T_d] \longrightarrow S \longrightarrow 0,$$

where each T_i has image in the maximal ideal of S.

Let A_0 = the divided power envelope of I modulo p-torsion (*i.e.* if $K = \mathrm{Frac}(W(k))$, the subring of $K[T_1, \dots, T_d]$ generated over $W(k)$ by $\{\frac{x^n}{x!}, x \in I\}$. We have a surjection $A_0 \twoheadrightarrow S$. Let $\sigma : W(k)[T_1, \dots, T_d] \to W(k)[T_1, \dots, T_d]$ be f semi-linear with $\sigma(T_i) = T_i^p$. Then σ leaves $I + (p)$ stable. If $J_0 = \mathrm{Ker}(A_0 \to S)$, that is the divided power ideal, $J_0 + p\,A_0$ has divided powers and σ extends to A_0, σ lifts Frobenius. Let A be the p-adic completion of A_0. If $R = S/p^n\,S$ for some fixed n, then (A, σ) is a Dieudonné frame for R since one shows that $A \xrightarrow{\delta} W(A)$ factors through $\widetilde{W}(A)$. By Theorem 6.8 and 7.2, Dieudonné windows for (A, σ) classify p-divisible groups over R.

Passing to the limit over n we obtain:

THEOREM 7.3. — *The category of Dieudonné windows for the frame (A, σ) is equivalent to the category of p-divisible groups over S.*

8. FURTHER RESULTS

Zink has applied techniques from his theory of displays to give new and simpler proofs of a purity result for p-divisible groups. This result [dJO] states the following:

PROPOSITION 8.1 (de Jong–Oort). — *Let R be a noetherian local ring of dimension ≥ 2, U the complement of the closed point in* $\mathrm{Spec}(R)$. *If G/R is a p-divisible group which has constant Newton polygon over U, then G has constant Newton polygon over* $\mathrm{Spec}(R)$.

Zink's proof of this result uses, in part, ideas of Vasiu, who proved a more general result of [V].

Langer and Zink have developed a theory of the de Rham–Witt complex valid for X an arbitrary scheme over a $\mathbf{Z}_{(p)}$-algebra R. This theory is related to, but not the same as the "absolute theory" of [HM]. It generalizes the classical theory of Bloch–Illusie, [Ill1]. If R is a ring with p nilpotent in R and X is an R-scheme there are complexes $W_n\,\Omega^\bullet_{X/R}$ of $W_n(\mathscr{O}_X)/W_n(R)$ differential graded algebras such that

$$W_n(\mathscr{O}_X) = W_n\,\Omega^0_{X/R}\,,\ d : W_n(\mathscr{O}_X) \to W_n\,\Omega^1_{X/R}$$

satisfies $d(\gamma_n(v\xi)) = \gamma_{n-1}(v\xi)d(v\xi)$. There are algebra homomorphisms $F : W_{n+1}\,\Omega^\bullet_{X/R} \to W_n\,\Omega^\bullet_{X/R}$ and additive maps $V : W_n\,\Omega^\bullet_{X/R} \to W_{n+1}\,\Omega^\bullet_{X/R}$. If X/R is smooth, then $\mathbb{H}^*(X, W_n\,\Omega^\bullet_{X/R})$ is canonically isomorphic to $H^*_{\mathrm{crys}}(X/W_n(R))$. Passing to the inverse limit one defines $W\Omega^\bullet_{X/R}$. It has operators F, V, d satisfying the standard relations $FdV = d$, $Vd = dVp$, $dF = pFd$.

If X/R is proper and smooth, we have

$$\mathbb{H}^*(X, W\Omega^\bullet_{X/R}) \simeq H^*_{\mathrm{crys}}(X/W(R)).$$

As an application we give the Zink–Langer construction of the display associated to an abelian scheme. Let X/R be an abelian scheme of relative dimension g. Let $P = H^1_{\text{crys}}\big(X/W(R)\big)$, a projective $W(R)$-module of rank $2g$. Consider the subcomplex of $W\Omega^\bullet_{X/R}$ obtained by replacing $W\big(\mathscr{O}_X\big)$ by $I_X = v(\big(W(\mathscr{O}_X)\big)$. Denote this complex by $IW\Omega^\bullet_{X/R}$.

We have a commutative diagram

$$(*)\qquad \begin{tikzcd} I_X \arrow[r, "d"] \arrow[d, "V^{-1}"'] & W\Omega^1_{X/R} \arrow[r, "d"] \arrow[d, "F"] & W\Omega^2_{X/R} \arrow[r] \arrow[d, "pF"] & \cdots \\ W(\mathscr{O}_X) \arrow[r, "d"] & W\Omega^1_{X/R} \arrow[r, "d"] & W\Omega^2_{X/R} \arrow[r] & \cdots \end{tikzcd}$$

Let Q be the $W(R)$-module obtained as the $\mathbb{H}^1\big(X, IW\Omega^\bullet_{X/R}\big)$.

We have an exact sequence

$$0 \longrightarrow IW\Omega^\bullet_{X/R} \longrightarrow W\Omega^\bullet_{X/R} \longrightarrow \mathscr{O}_X \longrightarrow 0$$

where $\mathscr{O}_X$ is viewed as a complex concentrated in degree zero.

Taking $\mathbb{H}^1$ of this sequence we find

$$0 \longrightarrow Q \longrightarrow P \longrightarrow \operatorname{Lie}(X) \longrightarrow 0.$$

Proposition 8.2. — *Let $F_1 : Q \to P$ be the f-linear map induced by the diagram $(*)$. Then $\mathscr{P} = (P, Q, F, F_1)$ is a display over R.*

We end with a question and two comments:

1) Can the de Rham–Witt complex be used to directly construct the nilpotent display associated to a p-divisible formal group over a Nagata ring R. More generally we ask whether using crystalline techniques will allow us to associate a Dieudonné display to a p-divisible group over a complete noetherian local ring whose residue field is perfect of characteristic p.

2) Breuil has, extending work of Fontaine and Conrad, classified finite flat commutative p-group schemes over $\mathscr{O}_K$ (notation as in § 7) for $p \neq 2$. Crystalline Dieudonné theory is defined for such group schemes over any base where p is nilpotent and has good faithfullness properties if the base is of characteristic p and is a reasonably nice scheme, [BM2], [dJ1], [dJM]. Thus it seems reasonable to hope that a good theory of displays which will classify finite flat p-group schemes can be developed.

3) In [Z5], page 132, Zink says he "would expect" that BT defines an equivalence of categories between nilpotent displays and p-divisible formal groups over any noetherian R. Recently Lau, [L1], has proven this conjecture of Zink without the noetherian hypothesis on R, requiring only that R is separated and complete for the p-adic topology. His proof uses the Grothendieck-Illusie theory, [Ill2], of deformations of truncated Barsotti-Tate groups.

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