Astérisque

MILES REID

# La correspondance de McKay

*Astérisque*, tome 276 (2002), Séminaire Bourbaki, exp. n° 867, p. 53-72

<http://www.numdam.org/item?id=SB_1999-2000__42__53_0>

© Société mathématique de France, 2002, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (http://smf4.emath.fr/Publications/Asterisque/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

*Article numérisé dans le cadre du programme Numérisation de documents anciens mathématiques*

http://www.numdam.org/

Séminaire BOURBAKI Novembre 1999
52e année, 1999-2000, nº 867, p. 53 à 72

**LA CORRESPONDANCE DE McKAY**

by **Miles REID**

## 1. COMMENT C'EST

### 1.1. Model case: the binary dihedral group $\mathrm{BD}_{4n}$

For $G \subset \mathrm{SL}(2,\mathbb{C})$ a finite group, the quotient variety $X = \mathbb{C}^2/G$ is called a *Klein quotient singularity.* I draw the quotient map $\pi\colon \mathbb{C}^2 \to X$ and the minimal resolution of singularities $Y \to X$ together in the diagram:

$$\begin{array}{ccc} & & \mathbb{C}^2 \\ & & \downarrow\pi \\ Y & \xrightarrow{\varphi} & X \end{array}$$

This situation has been well studied, since Klein around 1870 and Coxeter and Du Val in the 1930s: the subgroup $G$ is classified as cyclic, binary dihedral or a binary group corresponding to one of the Platonic solids; the quotient singularity is a hypersurface $X \subset \mathbb{C}^3$ with defining equation one of a list of simple functions. The resolution $Y$ is a surface with $K_Y = \varphi^* K_X$, and the exceptional locus $\varphi^{-1}(0) \subset Y$ of the resolution consists of a bunch of $-2$-curves $E_i$ (that is, $E_i \cong \mathbb{P}^1_{\mathbb{C}}$ and $E_i$ has self-intersection $E_i^2 = -2$), and the intersection $E_iE_j$ is given by one of the Dynkin diagrams $A_n$, $D_n$, $E_6$, $E_7$, $E_8$. To avoid writing out lists, let us simply discuss the binary dihedral group

$$G = \mathrm{BD}_{4n} = \langle \alpha, \beta\rangle\,, \quad \text{where} \quad \alpha = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1}\end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$$

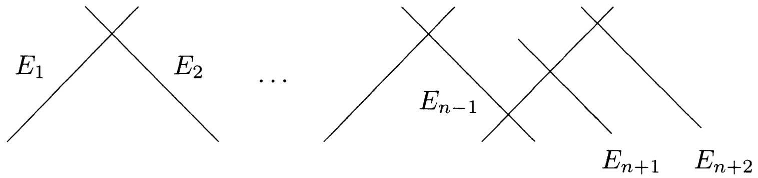
where $\varepsilon = \exp\frac{2\pi i}{2n}$. If $u, v$ are coordinates on $\mathbb{C}^2$, the $G$-invariant polynomials are

$$\mathbb{C}[x,y,z]/(z^2 - yx^2 + 4y^{n+1}), \quad \text{where} \quad x = u^{2n} + v^{2n}, y = u^2v^2, z = uv(u^{2n} - v^{2n});$$

thus the quotient variety is the singularity $X : (z^2 = yx^2 - 4y^{n+1}) \subset \mathbb{C}^3$ of type $D_{n+2}$, and the quotient morphism $\pi\colon (u,v) \mapsto (x,y,z)$. The resolution of singularities

$Y \to X$ has exceptional locus consisting of $-2$-curves $E_1, \dots, E_{n+2}$ forming the $D_{n+2}$ configuration:

(1.1)

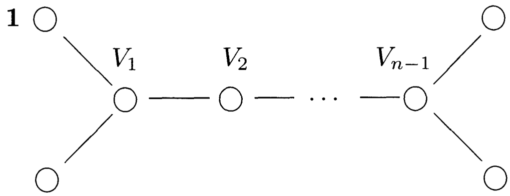



The classical McKay correspondence begins in the late 1970s with the observation that the same graph arises in connection with the representation theory of $G$. For a group $G$ and a given representation $Q$, the *McKay graph* (or McKay quiver) has a node for each irreducible representation, and an edge $V \to V'$ whenever $V'$ is a direct summand of $V \otimes Q$. In our case, $\mathrm{BD}_{4n}$ has the 2-dimensional representations

$$V_i \cong \mathbb{C}^2, \quad \text{with action} \quad \alpha = \begin{pmatrix} \varepsilon^i & 0 \\ 0 & \varepsilon^{-i} \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ (-1)^i & 0 \end{pmatrix} \quad \text{for } i = 0, \dots, n.$$

This is irreducible for $0 < i < n$, and splits into 2 eigenlines when $i = 0$ or $n$. The inclusion $G \subset \mathrm{SL}(2, \mathbb{C})$ provides the *given* representation $Q = V_1$. It is a simple exercise [Homework] to write down the action of $G$ on a basis $\{e_i \otimes e'_j\}$ of $Q \otimes V_i$ to get $V_i \otimes Q = V_{i-1} \oplus V_{i+1}$ for $0 < i < n$, so that the McKay graph of $\mathrm{BD}_{4n}$ is the extended Dynkin diagram $\widetilde{D}_{n+2}$:

(1.2) $\widetilde{D}_{n+2}$




Here **1** is the trivial 1-dimensional representation.

This example, and the other $\mathrm{SL}(2, \mathbb{C})$ cases observed by McKay, suggest that there is a one-to-one correspondence between the components of the exceptional locus of $Y \to X$ in (1.1) and the nontrivial irreducible representations of $G \subset \mathrm{SL}(2, \mathbb{C})$ in (1.2). This talk explains this coincidence in several different ways, and discusses higher dimensional generalisations.

## 1.2. General assumption

I use the following diagram throughout:

$$(1.3) \qquad \begin{array}{ccc} & & M \\ & & \downarrow \pi \\ Y & \xrightarrow{\varphi} & X \quad = M/G \end{array}$$

Here $M$ is a quasiprojective algebraic manifold with $K_M = 0$ and $G$ a finite automorphism group of $M$ that acts trivially on a global basis $s_M \in H^0(K_M)$. The object of study is the quotient variety $X = M/G$ and its resolutions $Y \to X$, sometimes assumed to have $K_Y = 0$. An important motivating case is a finite subgroup $G \subset \mathrm{SL}(3, \mathbb{C})$ acting on $M = \mathbb{C}^3$.

### 1.3. Definition–Reassurance

The quotient varieties $X = M/G$ occuring here are singular. The theory of minimal models of higher dimensional algebraic varieties (Mori theory) has a whole battery of definitions that deal systematically with singular varieties; here I only need one small item: the orbifolds $X$ here have trivial canonical class $K_X = 0$ (or trivial Serre–Grothendieck dualising sheaf $\omega_X = \mathcal{O}_X$). In concrete terms, this means the following: $X$ is a complex $n$-fold (algebraic or analytic variety), nonsingular in codimension 1, and its nonsingular locus NonSing $X$ has an everywhere nondegenerate holomorphic $n$-form $s_X$ (deduced from $s_M$). So $s_X$ is a complex volume element at every nonsingular point of $X$, or in other words, it is a global basis of $\Omega^n_{\mathrm{NonSing}\, X}$. A resolution of singularities $\varphi\colon Y \to X$ is *crepant* if $K_Y = \varphi^* K_X$ or $\omega_Y = \varphi^* \omega_X$, which simply means that $Y$ is a nonsingular $n$-fold with $K_Y = 0$ or $\omega_Y = \Omega^n_Y = \mathcal{O}_Y \cdot s_Y$, where $s_Y = \varphi^* s_X$. More generally, an arbitrary proper birational map $\varphi\colon V \to X$ has a *discrepancy divisor* $\Delta_\varphi = \sum a_i E_i$ defined by $K_V = \varphi^* K_X + \sum a_i E_i$ with $a_i \geq 0$; a divisor $E_i$ is *crepant* if $a_i = 0$. The discrepancy $\Delta_\varphi$ is the divisor of zeros on $V$ of the basic $n$-form $s_X$ on $X$, generalising the divisor of zeros of the Jacobian determinant; in Mori theory, it measures how far $V$ is from minimal.

### 1.4. Summary and slogan

I start with a preview of different approaches to the McKay correspondence, which are treated in more detail in later sections. Each of these approaches gives a result in the case of a finite subgroup $G \subset \mathrm{SL}(3, \mathbb{C})$ acting on $M = \mathbb{C}^3$.

(1) Gonzalez-Sprinberg and Verdier sheaves: the first direct link from the representation theory of $G$ to the geometry of the resolution $Y \to X$ was the work of Gonzalez-Sprinberg and Verdier [GSpV]: for a Kleinian subgroup $G \subset \mathrm{SL}(2, \mathbb{C})$, they constructed *sheaves $\mathcal{F}_\rho$ on $Y$, indexed by the irreducible representations of $G$, whose first Chern classes base the cohomology of $Y$.*

(2) String theory: the first hint of a McKay correspondence in higher dimensions comes from work of the string theorists Dixon, Harvey, Vafa and Witten [DHVW] around 1985: if $G \subset \mathrm{SL}(3, \mathbb{C})$ and $Y \to X = \mathbb{C}^3/G$ is a crepant resolution of the quotient $\mathbb{C}^3/G$, *the Euler number of $Y$ equals the number of conjugacy classes of $G$ (or the number of its irreducible representations).*

(3) Explicit methods: the finite subgroups $G \subset \mathrm{SL}(3, \mathbb{C})$ are classified, and work in the early 1990s of Roan, Ito, Markushevich and others proved case-by-case *the*

*existence of crepant resolutions, and the validity of the formula of* [DHVW] for the Betti numbers of $Y$.

(4) Valuation theory: for a finite subgroup $G \subset \mathrm{SL}(n, \mathbb{C})$, the paper [IR] shows that $G$ has a grading by *age*, analogous to the weight grading in Hodge theory, and proves that *the conjugacy classes of* junior *elements $g \in G$ (elements of age 1) correspond one-to-one with the crepant divisors of a resolution* (more precisely, their discrete valuations). This result holds for any $G \subset \mathrm{SL}(n, \mathbb{C})$ and is intrinsic, classification-free; but for $n \geq 4$ it only gives a small part of a McKay correspondence (so far).

(5) Nakamura's $G$-Hilbert scheme: a resolution of singularities $Y \to X$, even if it is a Mori minimal model theory, is not at all unique. Moreover, if $X = M/G$, the construction of a resolution $Y$ need not have much to do with $G$. In 1995, Nakamura made the revolutionary suggestion that in many interesting cases, *the $G$-Hilbert scheme is a preferred resolution $Y$ of $X$* (see [IN2], [N], [R]). When this holds, $Y$ is a "very good" moduli space over $M$, and the general yoga of moduli suggests that there should be a "tautological" treatment of the geometry of $Y$ (comparable to the cohomology of Grassmann varieties).

(6) Fourier–Mukai transform: the derived category $\mathrm{D}(V)$ of coherent sheaves on a variety $V$ (considered up to isomorphism of triangulated categories) can be used as a geometric characteristic of $V$, in place of K theory or cohomology. The Fourier–Mukai transform is a general method for constructing isomorphisms of derived categories (see [Mu], [O], [BO1], [Br], [BrM]). Bridgeland and others [BKR] have recently used this technique to prove that, if $Y = G\text{-Hilb}\, M$ is a crepant resolution, then $\mathrm{D}^G(M) = \mathrm{D}(Y)$. This implies the corresponding result in K theory.

(7) Motivic integration: the motivic integration of Batyrev, Denef and Loeser, and Kontsevich is a rigorous and comparatively simple mathematical trick that mimics some aspects of the path integrals of QFT. Very roughly, if $\varphi\colon Y \to X$ is a resolution of singularities, possibly far from minimal, with discrepancy divisor $K_Y - \varphi^* K_X = \sum a_i E_i$, the calculation amounts to defining the *stringy homology* of $X$ by picking only $\frac{1}{a_i+1}$th of the homology of $E_i$. Quite remarkably, *this is well defined, agrees with the predictions of* [DHVW] *mentioned in (2) above, and provides an exact form of the homological McKay correspondence for finite subgroups $G \subset \mathrm{SL}(n, \mathbb{C})$.*

(8) Explicit methods (bis): for a finite group $G \subset \mathrm{SL}(3, \mathbb{C})$, the results of (6) (maybe also (7)) imply that Gonzalez-Sprinberg–Verdier sheaves $\mathcal{F}_\rho$ base the K theory of the resolution $Y \to X$, so that their Chern classes or Chern characters base the cohomology. Reworking this in explicit terms presents a treasure chest of delightful computational problems – already the Abelian cases lead to lovely pictures (compare [R], [CR], [C2]).

I believe that many other approaches to the McKay correspondence remain to be discovered, and many interrelations between the different approaches; this problem area is recommended to afficionados of noncommutative geometry, perverse sheaves, Gromov–Witten invariants, elliptic cohomology, Chow groups, etc. Here is an attempt to describe the subject in a single statement:

PRINCIPLE 1.1. — *Let $M$ be an algebraic manifold, $G$ a group of automorphisms of $M$, and $Y \to X$ a resolution of singularities of $X$. Then the answer to any well posed question about the geometry of $Y$ is the $G$-equivariant geometry of $M$.*

I give two illustrations

I. If $G \subset \mathrm{SL}(n, \mathbb{C})$ acts on $\mathbb{C}^n$ and the quotient $X = \mathbb{C}^n/G$ has a crepant resolution $Y \to X$, the homology or K theory of $Y$ is expected (or known) to be independent of $Y$. In this case, the principle says that the homology or K theory of $Y$ is the representation theory of $G$ (equal to the $G$-equivariant geometry of $\mathbb{C}^n$ because $\mathbb{C}^n$ is contractible).

II. Let $M$ be a Calabi–Yau $n$-fold and $G$ a group of automorphisms of $M$ that acts trivially on $\Omega^n_M$. The stringy homology of $X = M/G$ (see Sections 3 and 4) is well defined by [DL1]. The principle says that it must agree with the $G$-equivariant homology of $M$. (I expand on what this means in Section 4.)

Viewed as an orbifold or stack, $X = M/G$ contains $M$ and the $G$ action, and you can of course derive tautological question-and-answer pairs from this (this is often popular as a source of questions after the talk). The content of my slogan is that *the equivariant geometry of $M$ already knows about the crepant resolution $Y \to X$.* Minimal models exist for surfaces by classical work, and for 3-folds by Mori theory (or by explicit methods). Minimal models of orbifolds by finite subgroups $G \subset \mathrm{SL}(3, \mathbb{C})$ provide infinitely many examples of local models of Calabi–Yau 3-folds; calculating their Betti numbers or K theory in a priori terms is in no sense a tautology. If you prefer to think of the singular $X$ as the fundamental object, and not resolve it (a perfectly sensible alternative), the content is that *$X$ has invariants that can be described from the orbifold $M/G$, but are birationally invariant* under appropriate conventions about resolutions.

## 2. AGE AND DISCREPANCY

Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite group; any element $g \in G$ has finite order $r$, say. For any such $r$, I choose at the outset a primitive $r$th root of 1, say $\exp \frac{2\pi i}{r}$. A choice of eigenbasis diagonalises the action of $g \in G$ on $M = \mathbb{C}^n$, giving

$$g = \mathrm{diag}(\varepsilon^{a_1}, \dots, \varepsilon^{a_n}) \quad \text{with } 0 \le a_i < r. \tag{2.1}$$

I write $g = \frac{1}{r}(a_1, \dots, a_n)$, possibly depending on the choices made. Now $\sum a_i \equiv 0 \mod r$ because $g \in \mathrm{SL}(n, \mathbb{C})$. Following [IR], define the *age* of $g$ by $\operatorname{age} g = \frac{1}{r} \sum a_i$. As we will see, this is an analog of *weight* in Hodge theory; Denef and Loeser [DL2] refer to it by the long-winded but not inappropriate term *valuation theoretic weight.*

Clearly, $\operatorname{age} g$ is an integer in the range $[0,\ldots,n-1]$, and only the identity has age 0. The elements of age 1 are *junior*.

Junior elements of $G$ give rise to crepant divisors of a resolution $V \to M/G$ by the following toric mechanism (for more details, and a picture, see [IR], 2.6–7). For $g \in G$ (not the identity), consider the $a_i$ of (2.1), and suppose $(a_1,\ldots,a_n) \in \mathbb{Z}^n$ is primitive. The coordinate subspace corresponding to the $x_i$ with $a_i = 0$ is the fixed locus $\operatorname{Fix} g$; it splits off as a direct product, and I assume that all $a_i > 0$ to short-cut some simple arguments. A useful example to bear in mind is when all the $a_i = 1$ (compare Example 4.1).

I view the integers $(a_1,\ldots,a_n)$ as *weights*. They define the grading $\operatorname{wt} x_i = a_i$ on the coordinate ring $\mathbb{C}[x_1,\ldots,x_n]$, or equivalently, the action $x_i \mapsto \lambda^{a_i} x_i$ of $\mathbb{C}^*$ on $M = \mathbb{C}^n$ that defines the weighted projective space

$$\mathbb{P}(a_1,\ldots,a_n) = (\mathbb{C}^n \setminus 0)/\mathbb{C}^*.$$

We obtain the weighted blowup $B_g \to M$ as the closed graph of the quotient map $M \dashrightarrow \mathbb{P}(a_1,\ldots,a_n)$; it has the exceptional divisor $B_g \supset E_g = \mathbb{P}(a_1,\ldots,a_n)$. Obviously $g$ acts on $B_g$, and fixes $E_g$ pointwise (because $g$ acts on $M$ as $\varepsilon \in \mathbb{C}^*$). Therefore $B_g \to B_g/\langle g\rangle$ is totally ramified along $E_g$.

Theorem 2.1 ([IR], 2.6–7). — *Suppose that $V \to X$ is any resolution of singularities of the quotient $X = M/G$. Then $V$ contains a divisor $F_g$ rationally dominated by $E_g$ under the rational map $B_g \to M \dashrightarrow V$. The discrepancy of $F_g$ is given by $a_{F_g} = \operatorname{age} g - 1$, and in particular*

$$F_g \text{ is crepant} \iff g \text{ is junior.}$$

*Every crepant divisor of any resolution $V$ occurs thus.*

*Discussion of proof.* — Write $X_g = M/\langle g\rangle$ for the partial quotient. Then $B_g/\langle g\rangle \to X_g$ is a partial resolution, with the single exceptional divisor $E_g$. An easy toric calculation gives the discrepancy of $E_g \subset B_g$ or $E_g \subset B_g/\langle g\rangle$ (compare [YPG], 4.8): the standard basis of $\Omega^n_M$ is $s_M = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$. For $K_{B_g}$, choose a Laurent monomial $y_1 = x^{\mathbf{m}}$ of weight 1 (recall that the $a_i$ were coprime). Then $y_1$ is the defining equation of $E_g \subset B_g$ at a general point of $E_g$ (away from all the coordinate hyperplanes), and $y_1^r$ that of $E_g \subset B_g/\langle g\rangle$. Choosing Laurent monomials $y_2,\ldots,y_n$ forming a basis of the lattice of monomials of weight 0, we get that

$$s_{B_g} = \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n \in \Omega^n_{B_g}$$

is the required basis. The discrepancy is the exponent $a$ in $s_M = (\text{unit}) \cdot y_1^a s_{B_g}$, and is determined by weighty considerations: $s_M$ has weight $\sum a_i$ and $s_{B_g}$ weight 1, so $a = \sum a_i - 1$. On the quotient $B_g/\langle g\rangle$ we only have $y_1^r$, so we get the stated discrepancy $\frac{1}{r}\big(\sum a_i - 1 - (r-1)\big) = \operatorname{age} g - 1$.

The quotient morphism $M \to X$ is a Galois cover with group $G$; a cyclic subgroup $\langle g \rangle$ corresponds to an intermediate cover $M \to M/\langle g \rangle = X_g \to X$. The reduction to a cyclic group is in terms of ramification theory; see [IR], 2.6–7. Roughly, over the general point of any exceptional divisor $F$ of $V \to X$, the Galois extension of function fields $k(X) \subset k(M)$ forms a tower, starting with a cyclic ramified cover. For a crepant exceptional divisor, the cyclic ramification can be chased back up to a conjugacy class of junior elements $g \in G$.

*Remark 2.2.* — This argument works in all dimensions, but it only identifies the *divisors* of a crepant resolution $Y$, and thus only gives a basis of $H^2(Y, \mathbb{Q})$ or $H_{2n-2}(Y, \mathbb{Z})$ corresponding in McKay style to junior conjugacy classes of $G$. In 3 dimensions, we used Poincaré duality to bootstrap ourselves up to a basis of $H^*(Y, \mathbb{Q})$ in [IR]. Historically, this was the first intrinsic proof of the conjectured formula of [DHVW] for the Betti numbers of a crepant resolution.

As Brylinski [B] remarks (following Mumford), if $V \to X$ is any resolution, the group $G$ can be viewed as the fundamental group of $V$ minus the branch locus, so that an exceptional divisor $F$ of a resolution $V$ corresponds directly to a conjugacy class of $G$ as a little anticlockwise loop around $F$; for crepant divisors, this is of course the same relation as in [IR]. But I don't know how to use this idea to get a well defined relation between, say, codimension 2 cycles of $Y$ and age 2 conjugacy classes of $G$.

## 3. L'INNOMMABLE

This section is mainly for sociological and historical interest, but some harmless hilarity may derive from my garrulous display of incompetence and ignorance in physics.

A theoretical prediction of string theory: Fermionic strings propagate in 10-dimensional space-time. Indeed, a universe of any other dimension would have particles moving faster than the speed of light. Since this prediction, on the face of it, contradicts the empirically observed 4-dimensions of space-time, string theorists want 6 of the dimensions to be filled up with tiny Calabi–Yau 3-folds. (This means (i) a 6-dimensional Riemannian manifold with SU(3) holonomy, or (ii) a complex manifold $V$ with a Ricci flat Kähler metric and $H^1(V, \mathbb{R}) = 0$, or (iii) an algebraic manifold $V$ with $K_V = 0$ and $H^1(V, \mathcal{O}_V) = 0$. It seems that the holonomy or Kähler conditions on $V$, together with some finite volume, are required by the physics, whereas making $V$ nonsingular, compact, and a constant fibre over macroscopic space-time are just convenient choices when you try to guess a model.)

The two papers [DHVW] were concerned with trying to calculate string theory on examples of Calabi–Yau varieties obtained by dividing a 3-dimensional complex torus $M$ by a finite group $G$ preserving a basic holomorphic 3-form, so that the stabiliser subgroup at any point is a subgroup of $\mathrm{SL}(3, \mathbb{C})$. A closed string on the quotient may lift either to a closed string on the cover, or to a path that goes from $x$ to $g \cdot x$. The latter are called *twisted sectors*. The physicists need to take care of these in order to relate $\int_X$ to $G$-equivariant $\int_M$, and they are the key to the form of the McKay correspondence in Theorem 4.4, (4).

Taking limits is a tradition in physics, where the old is frequently the limit of the new: Newtonian mechanics is the limit of special relativity as $c \to \infty$, classical mechanics the limit of quantum physics as $\hbar \to 0$, groups and their Hopf algebras the limit of quantum groups as $q \to 1$. In string theory, if the scale (or radius of curvature) of the tiny Calabi–Yau tends to zero, the theory should approximate ordinary Lorentz 4-dimensional space-time, whereas letting it tend to macroscopic proportions would approximate flat Lorentz 10-dimensional space-time. In this context, the twisted sector near a point $x \in M^H$ plays the role of strings that are topologically nontrivial, but are allowed to remain of finite length (and so contribute to path integrals) as the scale becomes large. To calculate something called the 1-*loop partition function*, DHVW considered mapping the elliptic curve $S^1 \times S^1$ (with parameters $\sigma$ and $\tau$ along the copies of $S^1$) into $X$, or the $\sigma, \tau$ square into $M$ with equivariant boundary conditions depending on $g, h$. Thinking about twisted sectors and limits led DHVW (I confess that their logic eludes me somewhat) to the formula

$$e_{\text{string}}(X) = e(M,G) := \frac{1}{|G|} \sum_{\substack{g,h \in G \\ \text{commuting}}} e(M^{\langle g,h \rangle}). \tag{3.1}$$

Here $e(M,G)$ on the left-hand side is the $G$-*equivariant Euler number of* $M$; on the right-hand side, the sum runs over all commuting pairs of elements of $G$, $\langle g,h \rangle$ is the Abelian group they generate, $M^{\langle g,h \rangle}$ its fixed locus in $M$, and $e$ is the usual Euler number. The formula is a replacement for the Euler number of the singular orbifold $X$. The papers [DHVW] contain more-or-less explicitly the conjecture that this number is the Euler number of a minimal resolution of singularities.

It is not hard (see [HH], [Roan] and [Homework]) to rearrange the sums in (3.1) to give

$$e_{\text{string}}(X) = e(M,G) = \sum_{[H] \subset G} e(X^H) \times \operatorname{card}\{[h] \in H\}, \tag{3.2}$$

where (i) the first sum runs over conjugacy classes of subgroups $H \subset G$; (ii) the stratum $X^H$ is the set of $x \in X$ such that $\operatorname{Stab} y$ is conjugate to $H$ for any point $y \in M$ over $x$; (iii) the second factor is the number of conjugacy classes in $H$. This means that $X^H \subset X$ contributes to $e(M,G)$ with multiplicity the representation theory of $H$.

*Remark 3.1.* — The physicists want to do path integrals, that is, they want to integrate some "Action Man functional" over the space of all paths or loops $\gamma \colon [0,1] \to Y$. This impossibly large integral is one of the major schisms between math and fizz.

The physicists learn a number of computations in finite terms that approximate their path integrals, and when sufficiently skilled and imaginative, can use these to derive marvellous consequences; whereas the mathematicians give up on making sense of the space of paths, and not infrequently derive satisfaction or a misplaced sense of superiority from pointing out that the physicists' calculations can equally well be used (or abused!) to prove $0 = 1$. Maybe it's time some of us also evolved some skill and imagination. The motivic integration treated in the next section builds a miniature model of the physicists' path integral, by restricting first to germs of holomorphic paths $\gamma\colon U \to Y$, where $0 \in U \subset \mathbb{C}$ is a neighbourhood of 0, then to formal power series $\gamma\colon \operatorname{Spec}\mathbb{C}[\![z]\!] \to Y$.

## 4. MOTIVIC INTEGRATION

The material in this section is due to Batyrev [Ba1], [Ba2], Denef and Loeser [DL1], [DL2] and Kontsevich [K]. I recommend Craw [C1] as a readable first introduction to these ideas.

Rather than trying to restrict to crepant resolutions, take an arbitrary normal crossing resolution $\varphi\colon Y \to X$, marked by the discrepancy divisor $D = \Delta_\varphi = \sum_{i\in I} a_i D_i$ (here $I$ is the indexing set of the components $D_i$). The normal crossing divisor $D$ defines a stratification of $Y$, with

$$\textit{closed strata } D_J = \bigcap_{j\in J} D_j, \quad \text{and} \quad \textit{open strata } D_J^\circ = D_J \setminus \bigcup_{J'\supsetneq J} D_{J'}$$

for $J \subset I$ (including, of course, $Y = D_\varnothing$ and $Y \setminus D = D_\varnothing^\circ$).

Motivic integration is discussed and defined below, but it is convenient to start from the answer: the *stringy motive* of $(Y, D)$, or of $X$ itself, turns out to be

$$h_{\text{string}}(X) = h(Y, D) = \sum_{J\subset I} [D_J^\circ] \cdot \prod_{j\in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_j+1} - 1}. \tag{4.1}$$

Here $\mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}] = [\mathbb{C}]$ is the Tate motive, and the formula takes place in a certain ring of motives with formal power series in $\mathbb{L}^{-1}$ adjoined. We will worry about the coefficient ring later, but in lucky cases it will happen that the cyclotomic polynomials in the denominators cancel out, leaving an integral motive (see Example 4.1 and [Homework] for examples). It follows from Theorem 4.4, (2) and (3) that $h(Y, D)$ is independent of the choice of the normal crossing resolution $Y$, so depends only on $X$. In the case when $D = aE$ has a single component with discrepancy $a$, it boils down to

$$[Y - E] + \frac{[E]}{1 + \mathbb{L} + \mathbb{L}^2 + \cdots + \mathbb{L}^a} = [Y - E] + \frac{[E]}{[\mathbb{P}^a]}. \tag{4.2}$$

*Example 4.1.* — Let $n = ab$, and consider the $n$-fold quotient singularity $X$ of type $\frac{1}{b}(1,\dots,1)$, that is, the quotient $\mathbb{C}^n/(\mathbb{Z}/b)$, with the diagonal action of $\varepsilon = \exp\frac{2\pi i}{b}$. It is the cone over the $b$th Veronese embedding of $\mathbb{P}^{n-1}$, so that its resolution $Y \to X$ has exceptional divisor $E = \mathbb{P}^{n-1}$ with $\mathcal{O}_E(E) = \mathcal{O}_{\mathbb{P}^{n-1}}(-b)$. The discrepancy is $a-1$, to fit the adjunction formula, with $K_Y = (a-1)E$, and $K_E = \mathcal{O}_E(aE) = \mathcal{O}_{\mathbb{P}^{n-1}}(-n)$.

Now whereas $Y$ is homotopy equivalent to $\mathbb{P}^{n-1}$, so has $n$ homology classes, one in each dimension $0, 2, \dots, 2(n-1)$, the effect of dividing by $[\mathbb{P}^{a-1}]$ in (4.3) is to throw away most of these, leaving only the $b$ stringy homology classes in dimension $0, 2a, 4a, 2a(b-1)$. This is exactly what we need for the McKay correspondence: the $b$ elements of $\mathbb{Z}/b$ have age $0, a, 2a, \dots, a(b-1)$ and correspond to the stringy classes in dimension $2ia$.

*Example 4.2.* — Consider the blowup $\sigma\colon Y_1 \to Y$ of a subvariety $C \subset Y$ that intersects all the strata of $D$ transversally, and set $D_1 = \sigma^* D + (c-1)E$, where $E = \sigma^{-1}C$ is the exceptional divisor of the blowup and $c = \operatorname{codim} C$. The coefficient is the discrepancy of $E$, so that $K_{Y_1} - D_1 = \sigma^*(K_Y - D)$. It is an exercise to see that

$$h(Y, D) = h(Y_1, D_1).$$

(This is rather trivial if $C \cap D = \varnothing$ in view of Grothendieck's *formule clef* for the motive of a blowup; see [Homework] for more hints.) This is good evidence for the birational invariance of $h(Y, D)$.

I now describe briefly the mechanics of motivic integration, following [C1]. Start from the Grothendieck ring $K_0(\mathcal{V})$ of classes of varieties under the equivalence relation $[V] = [V \setminus W] + [W]$. Addition and multiplication are quite harmless. The Tate motive is $\mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}] = [\mathbb{C}]$. We formally adjoin $\mathbb{L}^{-1}$ to $K_0(\mathcal{V})$, and make a fairly mild $(\mathbb{L}^{-1})$-adic completion to give the value ring $R = \widehat{K}_0(\mathcal{V})[\mathbb{L}^{-1}]$. This value ring is the really clever thing about the whole construction. (Exercise: $(\mathbb{L}^a - 1)^{-1}$ can be written as a formal power series in $\mathbb{L}^{-1}$, so all the terms on the right-hand side of (4.2) are in $R$.)

Motivic integration takes place over the infinite jet space $J_\infty Y$, which coincides with the set $Y(\mathbb{C}[\![z]\!])$ of points of $Y$ with values in the formal power series ring $\mathbb{C}[\![z]\!]$. An element $\gamma \in Y(\mathbb{C}[\![z]\!])$ is a point $y = \gamma(0) \in Y$ together with a formal arc $\gamma\colon \operatorname{Spec}\mathbb{C}[\![z]\!] \to Y$ starting at $y$; if convergent, $\gamma$ is the Taylor series of a holomorphic germ $\widetilde{\gamma}\colon (\mathbb{C}, 0) \to Y$. The infinite jet space $J_\infty Y$ is the profinite limit $\varprojlim_k J_k Y$ of the finite jet spaces $J_k Y$; recall that $J_0 Y = Y$, $J_1 Y$ is the total space of the tangent bundle $T_Y$, and $J_{k+1} \to J_k$ is a $\mathbb{C}^n$-fibre bundle.

The projection maps $\pi_k\colon J_\infty Y \to J_k$ of the profinite limit allow us to define a *cylinder set* in $J_\infty Y$ to be $\pi_k^{-1}(B_k)$ for a constructible set $B_k \subset J_k$. The measure on

$J_\infty Y$ is initially defined on these, by setting[(1)]

$$\mu(\pi_k^{-1}(B_k)) := [B_k] \cdot \mathbb{L}^{-nk} \in R. \tag{4.3}$$

It is straightforward to see that this is independent of $k$, and is a "finitely additive measure".

As our measurable functions, consider an effective divisor $D$ on $Y$, and define a function $F_D\colon J_\infty Y \to \mathbb{Z}_{\geq} 0$ by $F_D(\gamma) = D \cdot \gamma$ (intersection number). In other words, suppose $\gamma(0) = P \in Y$ and let $g_D$ be the local defining equation of $D$ at $P$; then $F_D(\gamma)$ is the order of $\gamma^*(g_D) \in \mathbb{C}[\![z]\!]$. Since the first $s$ coefficients of $\gamma^*(g_D)$ clearly only depend on $\pi_s(\gamma) \in J_s$, it is obvious that $F_D^{-1}(s)$ is a cylinder set.

The grand definition is now: for $Y$ a nonsingular variety and $D$ a normal crossing divisor, the motivic integral is

$$h(Y, D) = \int_{J_\infty Y} \mathbb{L}^{-F_D} := \sum_{s \in \mathbb{Z}_{\geq 0}} \mu\big(F_D^{-1}(s)\big) \cdot \mathbb{L}^{-s} \in R. \tag{4.4}$$

*Remark 4.3.* — I omit some tricky details on convergence required to get a genuine measure (involving the $[\mathbb{L}^{-1}]$-adic completion). To tell the truth, I don't know if they are at all essential. A basic point for applications is that the measure of $F_D^{-1}(s)$ tends to 0 as $s \to \infty$; this is plausible enough (because arcs $\gamma$ with $\gamma \cdot D \geq s$ have codimension $\geq s$ in $J_\infty Y$), and is an intuitive reason behind birational invariance: the arcs in a Zariski closed subset of $Y$ have measure zero.

THEOREM 4.4. — *$h(Y, D)$ of (4.4) has the following properties:*

(1) *If $D = 0$ then $h(Y, D) = [Y]$.*

(2) *$h(Y, D)$ is calculated by the right-hand side of (4.1).*

(3) *Birational invariance: let $Y', D'$ and $Y, D$ be pairs, and $\varphi\colon Y' \to Y$ a birational morphism such that $K_{Y'} - D' = \varphi^*(K_Y - D)$; then*

$$h(Y', D') = h(Y, D).$$

(4) *If $X = M/G$ is as in Assumption 1.2, $Y \to X$ a normal crossing resolution, and $D$ the discrepancy, then*

$$h_{\text{string}}(X) = h(Y, D) = \sum_{[H] \subset G} \big[X^H\big] \cdot \sum_{[g] \in H} \mathbb{L}^{\operatorname{age} g}, \tag{4.5}$$

*where the range of summation is as in (3.2), and the second sum is over conjugacy classes in $H$.*

---

(1) The papers [DL1] and [C1] have the exponent $\mathbb{L}^{-n(k+1)}$. This is just a normalising convention, giving $h(Y, D) = [Y] \cdot \mathbb{L}^{-n}$ in Theorem 4.4, (1), and making the motive of $Y$ 0-dimensional. I prefer my version.

*Discussion of proof.* — I give some indications, leaving most of the proof as references to [DL1] and [DL2]. Alternatively, do them as exercises (see [Homework] for more hints). The key point of the proof is that, whatever its substance, (4.4) has the formal properties of an integral, and is subject to the same kind of *change of variables* formula. In the words of the Master:

"La théorie consiste pour l'essentiel dans des questions de variance" ([H], Introduction). Note first that the condition in (3) says that $D' - D = \operatorname{div}(\operatorname{Jac}\varphi)$ is the divisor of zeros of the Jacobian determinant of $\varphi$ (I omit $\varphi^*$ from now on). Composition defines a map $j_\varphi\colon J_\infty Y' \to J_\infty Y$, and, unless it falls entirely in the locus of indeterminacy of $\varphi^{-1}$, an arc in $Y$ has a birational transform as an arc in $Y'$; in other words, away from subsets of measure zero, $j_\varphi$ is a bijection on the infinite jet spaces. For (3), it remains only to stratify the finite jet spaces $J_kY'$ and $J_kY$ so that the corresponding morphism $j_k\colon J_kY' \to J_kY$ is a $\mathbb{C}^t$-bundle on each stratum with $F_{D'-D}(\gamma) = \operatorname{div}(\operatorname{Jac}\varphi)\cdot\gamma = t$ (see [DL1], Lemma 3.4 and [Homework]).

(2) is proved in [DL1], Proposition 6.3.2, [Ba2], Theorem 6.28, and worked out in detail in [C1], Theorem 1.16. The proof of (4) consists of two steps, relating to the two morphisms $\pi\colon M \to X$ and $\varphi\colon Y \to X$ of Assumption 1.2.

*Step I.* — We translate the twisted sectors of [DHVW] into the language of formal arcs, obtaining the stratification (4.6) below.

Let $y \in M^H$ be a point with $\operatorname{Stab} y = H$ and $x = \pi(y) \in X^H$. As at the start of Section 2, suppose that $r$ is an integer divisible by the order of each $g \in H$, and choose an $r$th root $\varepsilon$ of 1 and an $r$th root $\zeta = z^{1/r}$ of the parameter used for formal arc, so that a formal arc $\gamma$ at $x \in X$ parametrised by $z$ lifts to a formal arc at $y \in M$ parametrised by $\zeta$. Unless $\gamma$ falls entirely in the branch locus of $\pi\colon M \to X$, there is a unique conjugacy class $g \in H$ defined by $\gamma(\varepsilon\zeta) = g\gamma(\zeta)$. Here $g$ is the *twisted sector*, the conjugacy class of $\gamma$ in the local fundamental group $H$ (where $\gamma$ is viewed as a little loop in $X$ minus the branch locus).

This argument shows that, after we delete the subset of arcs falling entirely in the branch locus (which has infinite codimension, so measure zero) the infinite jet space $J_\infty X$ is a disjoint union

$$J_\infty X = \coprod_{[H]\subset G} \coprod_{[g]\in H} J_\infty^{H,g} Y, \tag{4.6}$$

where $H, g$ are as in (3.2), and $J_\infty^{H,g}Y$ is the set of arcs with $\gamma(0) \in X^H$ in the twisted sector $g$.

*Step II.* — Using change of variables as in the proof of (3), one calculates that $J_\infty^{H,g}Y$ contributes $X^H \cdot L^{g^{-1}}$ to $h(Y,D)$ ([DL2], Lemma 4.3). The difference in appearance of the formulas here and in [DL2] is explained by two trivial shifts of notation: as explained in the footnote on page 63, my measure is $\mathbb{L}^n$ times theirs; and they diagonalise $g$ as $\varepsilon^{e_i}$ with $1 \le e_i \le r$, defining $w(g) = \frac{1}{r}\sum e_i = n - \operatorname{age}(g^{-1})$.

*Remark 4.5.* — Statement (4) is an exact analogue of the [DHVW] formula (3.2), saying that the stratum $X^H$ appears in the stringy homology of $Y$ multiplied by the set of conjugacy classes in $H$.

As discussed in Definition 1.3, the discrepancy $D = \operatorname{div} s_X$ is the divisor of zeros of $s_X$, the global basis of $\Omega^n_{\operatorname{NonSing} X}$. In the normal course of events, integrating *functions* on $Y$ requires a volume form; here we take $s_X$ as a holomorphic volume form, viewing its zeros on $D$ as scaling down the contribution from neighbourhood of the discrepant exceptional divisors. This is what produces a birationally invariant answer.

## 5. HILBERT SCHEMES OF $G$-ORBITS

This section explains the definition of the $G$-orbit Hilbert scheme $G\text{-Hilb}\, M$, and Nakamura's idea of using it to resolve certain quotient singularities. We know by general results (especially Hironaka's theorems) that the singularities of a quotient variety $X = M/G$ can be resolved somehow-or-other, but the construction of an actual resolution is messy, involves lots of choices, and will probably have almost nothing to do with the group action. Around 1995, Ito and Nakamura observed that in the case of $G \subset \mathrm{SL}(2,\mathbb{C})$, *the Hilbert scheme $G\text{-Hilb}\,\mathbb{C}^2$ of $G$-clusters is a crepant resolution of the quotient $\mathbb{C}^2/G$.* Nakamura conjectured that this continues to hold for $G \subset \mathrm{SL}(3,\mathbb{C})$, and this has since been confirmed and extended to some other cases by work of Bridgeland and others (see [BKR] and Theorem 6.1).

First, a *cluster* in a variety $M$ (say, quasiprojective and nonsingular) is a 0-dimensional subscheme $Z \subset M$, defined by an ideal $\mathcal{I}_Z \subset \mathcal{O}_M$, so that the cokernel $\mathcal{O}_Z = \mathcal{O}_M/\mathcal{I}_Z$ is a finite dimensional $\mathbb{C}$-vector space. The *degree* of $Z$ is the dimension of $\mathcal{O}_Z$. Like the intersection of two plane curves in Bezout's theorem, a cluster $Z$ may consist of reduced points $Z = P_1 + \cdots + P_N$, or may have a nonreduced structure; in the latter case, we *keep track of the ideal $\mathcal{I}_Z \subset \mathcal{O}_M$*, as a way of using algebraic equations to keep information about the relative positions when some of the points $P_i$ come together. For example,

$$(x^2, xy, y^2) \quad \text{and} \quad (x - ay - by^2, y^3) \quad \text{for any } a, b \in \mathbb{C}$$

are clusters of degree 3 supported at $0 \in \mathbb{C}^2$.

LEMMA 5.1. — *All clusters $Z \subset M$ of given degree $N$ in $M$ are parametrised by a quasiprojective scheme $\operatorname{Hilb}^N M$, which is a fine moduli space.*

*Proof.* — The assertion is quite elementary. $M$ is quasiprojective; choose an embedding $M \subset \mathbb{P}^s$. Every ideal $\mathcal{I}_Z \subset \mathcal{O}_M$ of codimension $N$ defines and is defined by a codimension $N$ vector subspace

$$H^0(\mathbb{P}^s, \mathcal{I}_Z(N)) \subset H^0(\mathbb{P}^s, \mathcal{O}_{\mathbb{P}^s}(N)),$$

the forms of degree $N$ vanishing on $Z$ (same $N$). Subspaces of given codimension are parametrised by a Grassmann variety, and the condition that a space of forms defines a cluster of degree $N$ in $M$ is a locally closed condition. (It can be written in terms of rank of a matrix $= N$.) $\square$

*Remark 5.2.* — The map $\operatorname{Hilb}^N M \to S^N M$ to the symmetric product, defined at the level of sets by $Z \mapsto \operatorname{Supp} Z$, is a morphism of schemes, the Hilbert–Chow morphism (see [GIT], Chapter 5, §4). For a curve, $\operatorname{Hilb}^N C$ is just the symmetric product $S^N C$, which is itself already nonsingular. For a surface, the symmetric product $S^N S$ is singular at the diagonals, and $\operatorname{Hilb}^N S \to S^N S$ is a crepant resolution, in fact, a symplectic resolution; see [IN2], §6. But $\operatorname{Hilb}^N M$ is singular as soon as $\dim M \geq 3$ and $N = \deg Z \geq 4$, and usually even has components of excess dimension.

PROPOSITION–DEFINITION 5.3 (Ito and Nakamura). — *Let $G$ be a finite group of order $N$ acting faithfully on an algebraic manifold $M$; consider the action of $G$ on* $\operatorname{Hilb}^N M$ *and its fixed locus* $(\operatorname{Hilb}^N M)^G$. *This has a unique irreducible component that contains a general orbit $G \cdot y$ of $G$ on $M$. This component is defined to be the* $G$-Hilbert scheme, *and denoted by* $G\text{-Hilb}\, M$. *The composite* $G\text{-Hilb}\, M \hookrightarrow \operatorname{Hilb}^N M \to S^N M$ *induces a Hilbert–Chow morphism* $G\text{-Hilb}\, M \to M/G$ *which is proper and birational.*

*A cluster* $Z \in G\text{-Hilb}$ *is* $G$*- invariant, and is called a* $G$-cluster*; its defining ideal* $\mathcal{I}_Z$ *is* $G$*-invariant, and as a representation of* $G$*, the quotient* $\mathcal{O}_Z = \mathcal{O}_M/\mathcal{I}_Z$ *is the regular representation* $\mathbb{C}[G]$.

See also [CR], 4.1 for a rival definition and a comparison between the two.

*Proof.* — The general orbit $G \cdot y$ consists of $N$ points permuted simply transitively by $G$, so is a $G$-invariant cluster in $(\operatorname{Hilb}^N M)^G$. These orbits fill out an irreducible open set in $(\operatorname{Hilb}^N M)^G$, because a small $G$-invariant deformation of $G \cdot y$ is clearly still a set of $N$ distinct points permuted by $G$ and disjoint from any fixed locus. The closure of this component is $G\text{-Hilb}\, M$ by definition. The composite $G\text{-Hilb}\, M \hookrightarrow \operatorname{Hilb}^N M \to S^N M$ is a morphism; by definition, a dense open set of $G\text{-Hilb}\, M$ consists of general orbits $G \cdot y$, and these maps to orbits in $S^N M$, that is, to $M/G$.

Finally, the quotient sheaves $\mathcal{O}_Z$ for $Z \in G\text{-Hilb}\, M$ fit together as a locally free sheaf $\mathcal{O}_{\mathcal{Z}}$ over $G\text{-Hilb}\, M$, with a $G$-action that makes it the regular representation on a dense open set. Its isotypical decomposition under the idempotents of $\mathbb{C}[G]$ is a direct sum, so each component must also vary as a locally free sheaf, therefore $\mathcal{O}_Z \cong \mathbb{C}[G]$ for every $Z \in G\text{-Hilb}\, M$ (since $G\text{-Hilb}\, M$ is defined to be irreducible). $\square$

The $G$-Hilbert scheme is a crepant resolution for finite groups $G \subset \mathrm{SL}(3, \mathbb{C})$. The general case of this is proved by Bridgeland and others [BKR] using derived category methods and a homological characterisation of regularity. For a diagonal Abelian group, $A\text{-Hilb}\, \mathbb{C}^3$ is a completely explicit construction of Nakamura (see [N] and [CR]):

the monomial $xyz$ is $A$-invariant, and every $G$-cluster $Z$ is defined by 7 (possibly redundant) equations of the form

$$\begin{array}{lll} x^{a+1} = \lambda y^d z^g & y^{d+1} z^{g+1} = \alpha x^a & \\ y^{b+1} = \mu z^e x^h & z^{e+1} x^{h+1} = \beta y^b & \text{and} \quad xyz = \xi \\ z^{c+1} = \nu x^f y^i & x^{f+1} y^{i+1} = \gamma z^c & \end{array}$$

for appropriate exponents $a, \dots, i$ and coefficients $\alpha, \dots, \xi$ satisfying $\alpha\lambda = \beta\mu = \gamma\nu = \xi$. The monomial basis of $\mathcal{O}_Z$ forms a tripod shaped Newton polygon in the plane lattice $\mathbb{Z}^2$ of Laurent monomials modulo $xyz$; this lattice is naturally the universal cover of the McKay quiver and the tripod is a choice of fundamental domain for the covering group (see [N] and [R] for pictures). The explicit calculations remain an interesting challenge in the non-Abelian cases, e.g., in the trihedral case.

*Example 5.4.* — These results are known to fail for finite $G \subset \mathrm{SL}(4, \mathbb{C})$. In the first place, most quotient singularities $X = \mathbb{C}^4/G$ do not have any crepant resolution. For example, the series of cyclic quotient singularities $\mathbb{C}^4/(\mathbb{Z}/r)$ of type $\frac{1}{r}(1, r-1, i, r-i)$ have no junior elements, so are terminal; compare Example 4.1. These examples motivated the initial exploration of stringy homology in [BD].

Next, even when a crepant resolution exists, the $G$-Hilbert scheme may be singular or discrepant or both. A simple example is the quotient singularity $\mathbb{C}^4/G$ by the maximal diagonal subgroup $(\mathbb{Z}/2)^{\oplus 3} \subset \mathrm{SL}(4, \mathbb{C})$ of exponent 2. The junior simplex $\Delta$ has all the midpoints of the edges $\frac{1}{2}(1, 1, 0, 0)$ etc., as lattice points. This has several subdivisions into basic simplexes, giving crepant resolutions, but none that is symmetric under permuting the coordinates – the only symmetric thing you can do is chop off the 4 basic simplexes at the corners, leaving a terminal simplex of volume 2. On the other hand, $G$-Hilb $\mathbb{C}^4$ is obviously symmetric.

## 6. COHERENT DERIVED CATEGORY

Grothendieck and Verdier introduced the derived category $\mathrm{D}(X)$ of coherent sheaves on a variety $X$ in the 1960s as a technical convenience in homological algebra; it has enjoyed an unfortunate reputation for technicality and abstraction ever since then. Recently, however, it has been increasingly used as a geometric characteristic of $X$ similar to K theory: whereas K theory works with the group of bundles or sheaves modulo the relation $F = F' + F''$ for every short exact sequence $0 \to F' \to F \to F'' \to 0$, the derived category $\mathrm{D}(X)$ consists of complexes $F^\bullet$ modulo the relation of quasi-isomorphism (defined at the start of the theory, and thankfully never referred to again). Following Mukai's pioneering work [Mu] for Abelian varieties, Orlov and Bondal [O], [BO1] have advocated the idea of considering the derived category $\mathrm{D}(X)$ (up to isomorphism of triangulated categories) as a geometric characteristic of $X$. From this point of view, $\mathrm{D}(X)$ behaves like an enriched version of K theory.

A variety $X$ with $\pm K_X$ ample can be reconstructed from its derived category $\mathrm{D}(X)$ (as a triangulated category) [BO1], but if $K_X = 0$ (notably for an Abelian variety or a K3 surface), the same triangulated category may occur as $\mathrm{D}(X)$ for different $X$, or there may be infinitely many symmetries of $\mathrm{D}(X)$ not arising from automorphisms of $X$. Isomorphisms $\mathrm{D}(X) \cong \mathrm{D}(Y)$ arise as *Fourier–Mukai transforms* $\Phi^{\mathcal{F}}_{X\to Y}$ corresponding to a sheaf $\mathcal{F}$ on $X \times Y$, defined as the composite of the functors $p_X^*$, $\otimes\mathcal{F}$ and $q_{Y*}$ (more precisely, their derived functors); for an up-to-date treatment, see [Br] and the references given there. In practice, $Y$ is most frequently a moduli space of coherent sheaves on $X$ and $\mathcal{F}$ the universal sheaf over $X \times Y$, so that $Y$ parametrises sheaves $F_y$ on $X$; in very good cases, the apparatus of moduli functors, stable bundles, and deformation theory gives essentially for free that the $F_y$ have orthonormality properties under Ext functors (formally analogous to those of trig functions in the theory of Fourier transform).

Let $M$ be a nonsingular quasiprojective $n$-fold with $K_M = 0$, and $G$ a finite group acting on $M$, with trivial action on $K_M$. Set $Y = G\text{-}\mathrm{Hilb}\, M$. Since $Y$ is a fine moduli space for $G$-clusters $Z \subset M$, there is a universal $G$-cluster $\mathcal{Z} \subset Y \times M$, fitting in a diagram

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{q} & M \\ p\downarrow & & \downarrow\pi \\ Y & \xrightarrow{\varphi} & X \end{array} \tag{6.1}$$

Bridgeland and others [BKR] prove the following theorem.

THEOREM 6.1. — *Suppose that the inverse image of the diagonal $(\varphi \times \varphi)^{-1}(\Delta_X)$ has dimension $\leq n+1$ (automatic for $n = 3$). Then $Y$ is a crepant resolution of $X$ and the Fourier–Mukai functor $\Phi = \mathbb{R}q_* \circ p^*\colon\ \mathrm{D}(Y) \to \mathrm{D}^G(M)$ is an equivalence of categories.*

Once we know that $Y$ is a crepant resolution, $\omega_M$ is trivial as a $G$-sheaf and $\omega_Y$ is trivial, so that both the derived categories $\mathrm{D}^G(M)$ and $\mathrm{D}(Y)$ have Serre duality functors; the remainder of the proof is then standard Fourier–Mukai technology. However, the surprising thing here is Bridgeland's derivation of the nonsingularity of $Y$ from the famous theorem of commutative algebra known for a long time as Serre's "Intersection conjecture".

## 7. FIN DE PARTIE

Samuel Beckett's play of the same title has the wonderful line:
"Personne au monde n'a jamais pensé aussi tordu que nous."

This seems to reflect a truth about math research: progress beyond the obvious takes really twisted thinking. In this spirit, let me raise all the open questions I can think of.

There are two basic flavours of McKay correspondence:

(1) conjugacy classes of $G \leftrightarrow$ homology of $Y$ (or stringy homology); and

(2) representations of $G \leftrightarrow$ derived category $\mathrm{D}(Y)$ or K theory of $Y$.

Is there a "bivariant" version of the correspondence containing both (1) and (2) at the same time? For example, in some contexts, $\mathcal{D}$-modules or perverse sheaves manage to accommodate both coherent and topological cohomology. Note that (1) and (2) achieve a well posed question in completely different ways: (1) takes accounts of discrepancy systematically, whereas (2) currently only works under the very strict condition that $Y = G$-Hilb is a crepant resolution.

The representation theory of finite groups has two ingredients, conjugacy classes and irreducible representations, and a character table, which is a nonsingular matrix making them "dual" (I apologise to group theorists for this gratuitous vulgarity). Although in substance very different, the homology and K theory of a variety $Y$ could be described in similar terms. In cases when McKay holds, is there any direct relation?

All the different approaches to McKay described here have one thing in common: none of them seems to say anything very useful about multiplicative structures. The following questions seem most likely to be approachable: can tensor product of $G$-modules and tensor product in K theory of $Y$ be related? Can you reconstruct the McKay quiver in $\mathrm{D}(Y)$ or $K_0Y$?

Motivic integration takes a fraction of the homology of a discrepant exceptional divisor, say, half the homology of the exceptional $\mathbb{P}^3$ for the quotient singularity $\mathbb{C}^4/(\mathbb{Z}/2)$ (the cone on the second Veronese embedding $v_2(\mathbb{P}^3)$). In contrast, half of a derived category is something no-one has ever seen. In the case of $v_2(\mathbb{P}^3)$, the Gonzalez-Sprinberg–Verdier sheaves corresponding to the characters $\pm 1$ are $\mathcal{O}_Y$ and $\mathcal{O}_Y(1)$. Breaking up the derived category $\mathrm{D}(Y)$ into two bits, one of which will correspond to the representations of $\mathbb{Z}/2$, doesn't seem to make any sense. On the other hand, in this case we can extend the action of $\mathbb{Z}/2$ to the action $\frac{1}{4}(1,1,1,1)$ of $\mathbb{Z}/4$, whose quotient does have a crepant resolution.

Another general problem area: resolutions of Gorenstein quotient singularities give a collection of examples of Calabi–Yau 3-folds with very nice properties: the homology of the resolution is well defined (independent of the choice of resolution), and the homology and K theory are closely related by something like a duality. Do these properties hold for Calabi–Yau 3-folds more generally? It seems very likely that birational Calabi–Yau 3-folds have isomorphic derived categories, but so far this only seems to be established when they are related by classic flops [BO2].

Part of motivic integration is the simple idea of using $\varphi^* s_X$ as the volume form, even though it vanishes along the discrepancy divisor $D$ (compare Remark 4.5). Maybe this idea can be used with differentials on $X$ itself (not passing to $J_\infty X$) to get birationally invariant de Rham and Hodge cohomology?

Elliptic cohomology is another area of geometry with an alleged stringy interpretation – as the index of the Dirac operator on the space of loops. Could part of this theory have a rigorous treatment in terms of spaces of formal arcs, like motivic integration in Section 4? If we believe that the elliptic cohomology of $M/G$ has a well defined answer (see Totaro [T] for some evidence) then Principle 1.1 predicts what the answer must look like in a whole pile of substantial cases.

Which Gorenstein quotient singularities admit crepant resolutions? Since 4-fold singularities usually do not have crepant resolutions, those that do are of particular interest; see [DHZ] for examples. How does this relate to complex symplectic geometry? The papers of Verbitsky [Vb] and Kaledin [Ka1], [Ka2] study crepant resolutions and related issues for symplectic quotient singularities. When crepant resolutions exist they are symplectic [Vb], therefore "semismall", giving a complete and elegant solution to the homological form (1) of the McKay correspondence [Ka2]. Is it possible that there is a "special" geometry in 3 complex dimensions (such as complexified imaginary quaternions), like symplectic or hyper-Kähler geometry for complex surfaces or 4-folds, that explain why crepant resolutions exist for 3-folds?

How should we interpret Nakamura's results and conjectures on $G$-Hilb? If a crepant resolution exists, it would be exceedingly convenient to be able to describe it as a fine moduli space of something; $G$- clusters have no especially privileged role, but the requirement that the space be birational to $M/G$ seems to impose some relation with the moduli space of group orbits. Nakamura and Nakajima have raised the question of whether the *other* crepant resolutions (after a flop) can also be interpreted as moduli, for example as Quot schemes; a single convincing example of this would add weight to their suggestion. Do the crepant resolutions in Example 5.4 have interpretations as moduli?

## REFERENCES

[Ba1] V. BATYREV, Birational Calabi–Yau $n$-folds have equal Betti numbers, in New trends in algebraic geometry, Klaus Hulek and others (eds.), CUP, 1999, pp. 1–11.

[Ba2] V. BATYREV, Stringy Hodge numbers of varieties with Gorenstein canonical singularities, in Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 1–32, World Sci. Publishing, River Edge, NJ, 1998.

[Ba3] V. BATYREV, Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs, J. Eur. Math. Soc. **1** (1999), 5–33.

[BD] V. BATYREV and D. DAIS, Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry, Topology **35** (1996), 901–929.

[BO1] A. Bondal and D. Orlov, Reconstruction of a variety from the derived category and groups of autoequivalences, Max Planck Inst. Bonn preprint MPI-97-36, `math.AG/9712029`, 20 pp.

[BO2] A. Bondal and D. Orlov, Semi-orthogonal decomposition for algebraic varieties, preprint `alg-geom/9506012`.

[Br] T. Bridgeland, Equivalences of triangulated categories and Fourier–Mukai transforms, Bull. London Math. Soc. **31** (1999), 25–34.

[BrM] T. Bridgeland and A. Maciocia, Fourier–Mukai transforms for K3 fibrations, preprint `math/9908022`, 18 pp.

[B] J.-L. Brylinski, A correspondence dual to McKay's, preprint `alg-geom/9612003`.

[BKR] T. Bridgeland, A. King and M. Reid, Mukai implies McKay, preprint `math/9908027`, 17 pp.

[C1] A. Craw, An introduction to motivic integration, preliminary draft available from `www.maths.warwick.ac.uk/~craw`, 23 pp.

[C2] A. Craw, $A$-Hilb $\mathbb{C}^3$ and McKay correspondence, work in progress.

[CR] A. Craw and M. Reid, How to calculate $A$-Hilb $\mathbb{C}^3$, preprint `math/9909085`, 29 pp.

[DHVW] L. Dixon, J. Harvey, C. Vafa and E. Witten, Strings on orbifolds. I, Nuclear Phys. B **261** (1985), 678–686. II, same J. **274** (1986), 285–314.

[DHZ] D. I. Dais, M. Henk and G. M. Ziegler, All Abelian quotient c.i. singularities admit crepant resolutions, preprint `alg-geom/9704007`, 35 pp.

[DL1] J. Denef and F. Loeser, Germs of arcs on singular algebraic varieties and motivic integration, Invent. Math. **135** (1999), 201–232.

[DL2] J. Denef and F. Loeser, Motivic integration, quotient singularities and the McKay correspondence, preprint `math/9903187`, 20 pp., to appear in Compositio Math.

[GSpV] G. Gonzalez-Sprinberg and J.-L. Verdier, Construction géométrique de la correspondance de McKay, Ann. Sci. École Norm. Sup. (4) **16** (1983), 409–449.

[H] R. Hartshorne, Residues and duality, L.N.M. **20** Springer, 1966.

[HH] F. Hirzebruch and T. Höfer, On the Euler number of an orbifold, Math. Ann. **286** (1990), 255–260.

[IN] Y. Ito and H. Nakajima, McKay correspondence and Hilbert schemes in dimension three, preprint `math.AG/9803120`, 35 pp., to appear in Topology.

[IN1] Y. Ito and I. Nakamura, McKay correspondence and Hilbert schemes, Proc. Japan Acad. Ser. A Math. Sci. **72** (1996), 135–138.

[IN2] Y. Ito and I. Nakamura, Hilbert schemes and simple singularities, in New trends in algebraic geometry, Hulek and others (eds.), CUP 1999, pp. 155–233.

[IR] Y. Ito and M. Reid, The McKay correspondence for finite subgroups of $\mathrm{SL}(3,\mathbb{C})$, in Higher-dimensional complex varieties (Trento, 1994), 221–240, de Gruyter, Berlin, 1996.

[K] M. Kontsevich, Motivic integration, Legendary lecture at Orsay, Thu 7th Dec 1995.

[Ka1] D. Kaledin, McKay correspondence for symplectic quotient singularities, preprint `math/9907087`, 28 pp.

[Ka2] D. Kaledin, Dynkin diagrams and crepant resolutions of quotient singularities, preprint `math/9903157`, 30 pp.

[Kr] P. B. Kronheimer, The construction of ALE spaces as hyper-Kähler quotients, J. Diff. Geom. **29** (1989), 665–683.

[GIT] D. Mumford, J. Fogarty and F. Kirwan, Geometric invariant theory (3rd edn.), Springer, 1994.

[Mu] S. Mukai, Duality between $\mathrm{D}(X)$ and $\mathrm{D}(\hat{X})$ with its application to Picard sheaves, Nagoya Math. J. **81** (1981), 153–175.

[N] I. Nakamura, Hilbert schemes of Abelian group orbits, to appear in J. Alg. Geom.

[O] D.O. Orlov, Equivalences of derived categories and K3 surfaces, Algebraic geometry, **7** J. Math. Sci. (New York) **84** (1997), 1361–1381, preprint `alg-geom/9606006`, 28 pp.

[YPG] M. Reid, Young person's guide to canonical singularities, in Algebraic Geometry, Bowdoin 1985, ed. S. Bloch, Proc. of Symposia in Pure Math. **46**, A.M.S. (1987), vol. 1, 345–414.

[R] M. Reid, McKay correspondence, in Proc. of algebraic geometry symposium (Kinosaki, Nov 1996), T. Katsura (ed.), 14–41, preprint alg-geom/9702016, 30 pp.

[Homework] Homework sheets will be on my website `www.maths.warwick.ac.uk/~miles`, including examples, exercises, more hints, and errata to this lecture.

[Roan] S.-S. Roan, Orbifold Euler characteristic, in Mirror symmetry, II, AMS 1997, pp. 129–140.

[T] B. Totaro, Chern numbers for singular varieties and elliptic homology, Ann. of Math. **151** (2000), no. 2, 757–791. Preprint available from `www.dpmms.cam.ac.uk/~bt219`.

[V] C. Vafa, String vacua and orbifoldized LG models, Modern Phys. Lett. A **4** (1989), 1169–1185.

[Vb] M. Verbitsky, Holomorphic symplectic geometry and orbifold singularities, preprint `math.AG/9903175`, 17 pp.

[Z] E. Zaslow, Topological orbifold models and quantum cohomology rings, Comm. Math. Phys. **156** (1993), 301–331.



Miles REID

Mathematical Institute
University of Warwick
GB-Coventry CV4 7AL
*E-mail* : `Miles@Maths.Warwick.Ac.UK`