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Recent advances in the theory of holonomy

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by **Robert BRYANT**

1. INTRODUCTION

1.0. OUTLINE. This report is organized as follows:

1. Introduction
2. Riemannian Holonomy
3. Torsion-free non-metric connections: the irreducible case

In a short lecture of this nature, it is impossible to describe the history of the subject in any depth, but the reader can find more information on the Riemannian (and pseudo-Riemannian) case by consulting [Bes], [Sa], and the forthcoming, much-anticipated [Jo3], especially for the exceptional cases. For the non-metric case, aside from the survey [Br3], the expository papers [MS2] and [Schw] provide a valuable account of both the representation theoretic and twistor theoretic approaches to the study of holonomy.

1.1. HISTORICAL REMARKS. According to the Oxford English Dictionary, it was Heinrich Hertz in 1899 who introduced the words *holonomic* and *nonholonomic* to describe a property of velocity constraints in mechanical systems.

Velocity constraints are *holonomic* if they force a curve in state space to stay in a proper subspace. As an example, the condition $\mathbf{p}\cdot d\mathbf{p} = 0$ for a vector particle $\mathbf{p} \in \mathbb{R}^n$ forces $\mathbf{p}$ to have constant length, while the constraint $\mathbf{p}\wedge d\mathbf{p} = 0$ forces $\mathbf{p}$ to move on a line.

Nonholonomic constraints, on the other hand, imply no such 'finite' constraints. A classical example is that of a ball rolling on a table without slipping or twisting. The state space is $B = \mathrm{SO}(3)\times\mathbb{R}^2$, where the $\mathrm{SO}(3)$ records the orientation of the ball and the $\mathbb{R}^2$ records its contact point on the plane. The rolling constraint is expressed as the set of differential equations

$$\alpha := a^{-1}\,da + a^{-1}\begin{pmatrix} 0 & 0 & -dx \\ 0 & 0 & -dy \\ dx & dy & 0 \end{pmatrix} a = 0 \tag{0.1}$$

for a curve $\big(a(t); x(t), y(t)\big)$ in B. The curves in B satisfying this constraint are those tangent to the 2-plane field $D = \ker\alpha$ that is transverse to the fibers of the projection $\mathrm{SO}(3)\times\mathbb{R}^2 \to \mathbb{R}^2$. It is not difficult to show that any two points of B can be joined by a curve tangent to D.

Such constraints and their geometry have long been of considerable interest in the calculus of variations and control theory. For recent results, see the foundational work on Carnot-Carathéodory geometries by Gromov [Gr].

1.2. THE HOLONOMY GROUP. Élie Cartan [Ca2] introduced the *holonomy group* in the context of differential geometry. It measures the failure of the parallel translation associated to a connection to be holonomic.

The data are a connected manifold M, a Lie group H with Lie algebra $\mathfrak{h}$, a principal right H-bundle $\pi : B \to M$, and a connection θ on B, i.e., θ is an $\mathfrak{h}$-valued 1-form on B that pulls back to each π-fiber to be the canonical left invariant 1-form on H and that satisfies the equivariance relation $R_h^*(\theta) = \mathrm{Ad}(h^{-1})(\theta)$. (The nonholonomic example described above with $(M, H, B, \theta) = \big(\mathbb{R}^2,\ \mathrm{SO}(3),\ \mathrm{SO}(3)\times\mathbb{R}^2,\ \alpha\big)$ is an example.)

A piecewise C^1 curve $\gamma : [0,1] \to B$ is said to be *θ-horizontal* or *θ-parallel* if $\gamma^*(\theta) = 0$. The *θ-holonomy* $B_u^\theta \subset B$ of $u \in B$ is defined to be the set of values of $\gamma(1)$ as $\gamma : [0,1] \to B$ ranges over the θ-horizontal curves with $\gamma(0) = u$. The $\mathrm{Ad}(H)$-equivariance of θ (coupled with the connectedness of M) implies that there is a subgroup $H_u^\theta \subset H$ so that B_u^θ is an H_u^θ-subbundle of B and that $H_{u\cdot h}^\theta = h^{-1}\,H_u^\theta\,h$ for $h \in H$. Consequently, the conjugacy class of $H_u^\theta \subset H$ depends only on θ. The group H_u^θ (or, more informally, its conjugacy class in H) is called the *holonomy* of θ.

It is a theorem of Borel and Lichnerowicz [BL] that H_u^θ is a Lie subgroup of H. By a theorem of Ambrose and Singer [AS], the Lie algebra $\mathfrak{h}_u^\theta$ of H_u^θ is spanned by the set

$$\{\ \Theta(x,y)\ |\ x, y \in T_vB\,,\ v \in B_u^\theta\ \}$$

where $\Theta = d\theta + \frac{1}{2}[\theta,\theta]$ is the *curvature* of θ. The identity component $(H_u^\theta)^0 \subset H_u^\theta$ is known as the *restricted holonomy* of θ. There is a well-defined surjective homomorphism $\rho^\theta : \pi_1\big(M, \pi(u)\big) \to H_u^\theta/(H_u^\theta)^0$ that satisfies $\rho^\theta\big([\pi\circ\gamma]\big) = \gamma(1)(H_u^\theta)^0$ for every θ-horizontal curve γ with $\gamma(0) = u$ and $\gamma(1) \in u\cdot H$.

Using these results, it can be shown [KNo] that, when $\dim M > 1$, a Lie subgroup $G \subset H$ can be the holonomy group of a connection on B if and only if B admits a structural reduction to a G-bundle. Thus, the set of possible holonomies of connections on B is determined topologically.

1.3. H-STRUCTURES AND TORSION. A common source of connections in geometry is that of H-structures on manifolds. Suppose $\dim M = n$ and let $\mathfrak{m}$ be a reference vector space of dimension n. An ($\mathfrak{m}$-valued) *coframe* at $x \in M$ is a linear isomorphism $u : T_xM \to \mathfrak{m}$. The set $F(M,\mathfrak{m})$ of $\mathfrak{m}$-valued coframes at the points of M is naturally a principal right $\mathrm{GL}(\mathfrak{m})$-bundle over M, with basepoint projection $\pi : F(M,\mathfrak{m}) \to M$. There is a tautological $\mathfrak{m}$-valued 1-form ω on $F(M,\mathfrak{m})$ defined by the formula $\omega(v) = u\bigl(\pi'(v)\bigr)$ for all $v \in T_uF(M,\mathfrak{m})$.

Let $H \subset \mathrm{GL}(\mathfrak{m})$ be a subgroup. An *H-structure* on M is an H-subbundle B such that $B \subset F(M,\mathfrak{m})$. When H is a closed subgroup of $\mathrm{GL}(\mathfrak{m})$ (the only case I will consider today), the set of H-structures on M is the set of sections of the bundle $F(M,\mathfrak{m})/H \to M$. The problem of determining whether there exists an H-structure on M is a purely topological one.

Most of the familiar geometric structures on M can be described as H-structures. For example, when $H = \mathrm{O}(Q) \subset \mathrm{GL}(\mathfrak{m})$ is the group of linear transformations preserving a quadratic form Q of type (p,q) on $\mathfrak{m}$, a choice of H-structure on M is equivalent to a choice of pseudo-Riemannian metric of type (p,q) on M. When $H = \mathrm{Sp}(S) \subset \mathrm{GL}(\mathfrak{m})$ is the group of linear transformations preserving a nondegenerate skewsymmetric bilinear form S on $\mathfrak{m}$, a choice of H-structure on M is equivalent to a choice of a nondegenerate 2-form σ on M, i.e., an almost symplectic structure.

If $\pi : B \to M$ is an H-structure on M, the tautological form ω pulled back to B will also be denoted ω when there is no chance of confusion. If θ is a connection on B, then the *first structure equation of Cartan* says that there exists an H-equivariant function $T : B \to \mathrm{Hom}\bigl(\Lambda^2(\mathfrak{m}),\mathfrak{m}\bigr) \simeq \mathfrak{m} \otimes \Lambda^2(\mathfrak{m}^*)$ so that

$$d\omega + \theta \wedge \omega = \tfrac{1}{2}\, T(\omega \wedge \omega). \tag{0.2}$$

The function T is the *torsion function* of θ, and θ is *torsion-free* if $T = 0$. Any connection θ' on B is $\theta + p(\omega)$ where $p : B \to \mathrm{Hom}(\mathfrak{m},\mathfrak{h})$ is an H-equivariant function. Its torsion function is $T' = T + \delta(p)$, where $\delta : \mathfrak{h} \otimes \mathfrak{m}^* \to \mathfrak{m} \otimes \Lambda^2(\mathfrak{m}^*)$ is defined via the inclusion $\mathfrak{h} \subset \mathfrak{m} \otimes \mathfrak{m}^*$ and the skewsymmetrizing map $\mathfrak{m} \otimes \mathfrak{m}^* \otimes \mathfrak{m}^* \to \mathfrak{m} \otimes \Lambda^2(\mathfrak{m}^*)$.

Evidently, the reduced map $\bar{T} : B \to \mathrm{coker}(\delta)$ is independent of the choice of θ and is the obstruction to choosing a torsion-free connection on B. For example, when $H = \mathrm{Sp}(S)$ as above, it is not difficult to see that $\mathrm{coker}(\delta) \simeq \Lambda^3(\mathfrak{m}^*)$ and that the reduced torsion represents the exterior derivative $d\sigma$ of the associated almost symplectic form $\sigma \in \Omega^2(M)$. Thus, an $\mathrm{Sp}(S)$-structure admits a torsion-free connection if and only if the almost symplectic structure on M is actually symplectic. Furthermore, $\ker(\delta) \simeq \mathrm{S}^3(\mathfrak{m}^*)$, so that the torsion function T does not determine a unique connection θ. By contrast, when $H = \mathrm{O}(Q)$, the map δ is an isomorphism, which is merely the fundamental lemma of Riemannian geometry: Every pseudo-Riemannian metric on M possesses a unique compatible, torsion-free connection.

The reduced torsion is the first order local invariant for H-structures on M and H-structures satisfying $\bar{T} = 0$ are usually referred to as *torsion-free* or 1-*flat*. Many (but not all) of the H-structures that have received the most attention in differential geometry are 1-flat. All pseudo-Riemannian metric structures are 1-flat, an almost symplectic structure is 1-flat if and only if it is symplectic (Darboux' Theorem), an almost complex structure is 1-flat if and only if it is complex (Newlander-Nirenberg Theorem), an Hermitian structure is 1-flat if and only if it is Kähler. Contact structures and Carnot-Carathéodory structures, on the other hand, are definitely not 1-flat, as the nondegeneracy of the reduced torsion is an essential part of the geometry.

1.4. Criteria for holonomy in the torsion-free case. The condition $\bar{T} = 0$ is a $\mathrm{Diff}(M)$-invariant, first order equation for H-structures on M. Given a torsion-free H-structure $B \subset F(M, \mathfrak{m})$, one can ask about the possibilities for the holonomy group of a torsion-free connection θ on B. When $\ker(\delta) = 0$, a very common situation, the torsion-free connection, if it exists, will be unique, so that it makes sense to speak of the holonomy of B itself.

The vanishing of the torsion implies nontrivial restrictions on the holonomy. Assuming $T = 0$ and differentiating (0.2) yields $\Theta \wedge \omega = 0$. Thus, the *curvature function* $R : B \to \mathfrak{h} \otimes \Lambda^2(\mathfrak{m}^*)$ of θ, for which the *second structure equation of Cartan*

$$\Theta = d\theta + \tfrac{1}{2}\,[\theta, \theta] = \tfrac{1}{2} R(\omega \wedge \omega) \tag{0.3}$$

holds, takes values in the kernel $K(\mathfrak{h})$ of the map $\delta : \mathfrak{h} \otimes \Lambda^2(\mathfrak{m}^*) \to \mathfrak{m} \otimes \Lambda^3(\mathfrak{m}^*)$ defined by the same methods[1] as the previous δ. In particular, if there is a proper subalgebra $\mathfrak{g} \subset \mathfrak{h}$ so that $K(\mathfrak{h}) \subseteq \mathfrak{g} \otimes \Lambda^2(\mathfrak{m}^*)$, then, by the Ambrose-Singer holonomy

[1] The maps I am denoting by δ (as well as the ones to follow) are part of the *Spencer complex* associated to the inclusion $\mathfrak{h} \subset \mathfrak{m} \otimes \mathfrak{m}^*$, see [Br3]. These maps are $\mathfrak{h}$-module maps and so the various kernels and cokernels to be introduced are $\mathfrak{h}$-modules as well.

theorem, the (restricted) holonomy of θ must lie in the connected Lie group $G \subset H$ whose Lie algebra is $\mathfrak{g}$.

The intersection of the subspaces $\mathfrak{s} \subset \mathfrak{h}$ that satisfy $K(\mathfrak{h}) \subseteq \mathfrak{s} \otimes \Lambda^2(\mathfrak{m}^*)$ is an ideal $\mathfrak{g} \subset \mathfrak{h}$. Thus, a necessary condition that there exist a torsion-free connection with holonomy $H \subset \mathrm{GL}(\mathfrak{m})$ is that $K(\mathfrak{h}) \neq K(\mathfrak{g})$ for any proper ideal $\mathfrak{g} \subset \mathfrak{h}$. This is usually referred to as *Berger's first criterion* [Ber, Br3].

This criterion is very restrictive: If $\mathfrak{h}$ is semi-simple, there are, up to equivalence, only a finite number of representations $\mathfrak{h} \hookrightarrow \mathfrak{gl}(\mathfrak{m})$ without trivial summands satisfying it. As a simple example, if $\mathfrak{h} \simeq \mathfrak{sl}(2,\mathbb{R})$, and $V_k \simeq \mathrm{S}^k(V_1)$ denotes the irreducible $\mathfrak{sl}(2,\mathbb{R})$-representation of dimension $k+1$, then the $\mathfrak{sl}(2,\mathbb{R})$-representations $\mathfrak{m}$ without V_0-summands that satisfy Berger's first criterion are V_1, $V_1 \oplus V_1$, V_2, V_3, and V_4.

Symmetric examples. One large class of examples where Berger's first criterion is satisfied is provided by the following construction: Suppose that there is a surjective skewsymmetric pairing $\mathfrak{m} \times \mathfrak{m} \to \mathfrak{h}$ so that, together with the Lie algebra bracket on $\mathfrak{h}$ and the $\mathfrak{h}$-module pairing $\mathfrak{h} \times \mathfrak{m} \to \mathfrak{m}$, it defines a Lie algebra on $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then the pair $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair of Lie algebras [KN]. Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$ and let $\tilde{H} \subset G$ be the (necessarily closed) connected subgroup corresponding to the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let $H = \mathrm{Ad}_{\mathfrak{m}}(\tilde{H}) \subset \mathrm{GL}(\mathfrak{m})$ be its almost faithful image. Then $M = G/\tilde{H}$ is an affine symmetric space in a canonical way, and the coset projection $G \to M$ covers a torsion-free H-structure on M with connection whose holonomy is H (by the assumption $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$ and the Ambrose-Singer holonomy theorem.) The classification of the symmetric Lie algebra pairs $(\mathfrak{g}, \mathfrak{h})$ is a (rather involved) algebra problem. It was solved by Berger [Ber2] in the case that $\mathfrak{g}$ is semi-simple or when $\mathfrak{h}$ acts irreducibly on $\mathfrak{m} \simeq \mathfrak{g}/\mathfrak{h}$.

The case where θ is a locally symmetric connection with holonomy H is characterized on the H-structure B by the condition that $R : B \to K(\mathfrak{h})$ be constant. In fact, differentiating (0.3) yields $0 = (dR + \theta.R)(\omega \wedge \omega)$ (where $\theta.R$ is the result of the $\mathfrak{h}$-module pairing $\mathfrak{h} \times K(\mathfrak{h}) \to K(\mathfrak{h})$). Equivalently, $dR = -\theta.R + R'(\omega)$ where $R' : B \to K(\mathfrak{h}) \otimes \mathfrak{m}^*$ must take values in the kernel $K^1(\mathfrak{h})$ of the natural map $\delta : K(\mathfrak{h}) \otimes \mathfrak{m}^* \to \mathfrak{h} \otimes \Lambda^3(\mathfrak{m}^*)$. Now, R' vanishes identically exactly when R is parallel, which is exactly when the pair (B, θ) defines a locally symmetric affine structure on M. Thus, if H can be the holonomy of a torsion-free affine connection that is not locally symmetric, then $K^1(\mathfrak{h}) \neq 0$. This is *Berger's second criterion.*

For example, when $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{m} = V_4$, one has $K^1(\mathfrak{h}) = 0$. Thus, this 5-dimensional representation of $\mathrm{SL}(2,\mathbb{R})$ could occur as holonomy of a torsion-free connection on M^5 only when that connection is locally symmetric. In fact, it occurs as the holonomy of the symmetric spaces $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(2,1)$ and $\mathrm{SU}(2,1)/\mathrm{SO}(2,1)$ and in no other way. The other four cases with $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{R})$ satisfying Berger's first criterion also satisfy Berger's second criterion. Moreover, each does occur as the holonomy of a torsion-free connection that is not locally symmetric.

1.5. Classification. In the case where $\mathfrak{m}$ is an irreducible $\mathfrak{h}$-module (implying that $\mathfrak{h}$ is reductive), Berger's fundamental works [Ber1, Ber2] went a long way towards classifying the subalgebras $\mathfrak{h} \subset \mathfrak{gl}(\mathfrak{m})$ that satisfy his first and second criteria. The methods involved heavy use of representation theory and ultimately reduced to a painstaking, elaborate case analysis. This list was refined and completed by the combined work of several people: Alekseevskii [Al1], myself, and most recently and importantly, the combined work of Chi, Merkulov, and Schwachhöfer. I will discuss this further in §3.

Berger's work provided a (partial) list of candidates for the irreducibly acting holonomy groups of torsion-free connections that are not locally symmetric. He divided the list into two parts: The first part consists of $\mathfrak{h} \subset \mathfrak{gl}(\mathfrak{m})$ that lie in some $\mathfrak{so}(Q)$ for some nondegenerate quadratic form Q on $\mathfrak{m}$, so that the associated H-structure defines a pseudo-Riemannian structure on M. For these cases, the injectivity of the map $\delta : \mathfrak{h} \otimes \mathfrak{m}^* \to \mathfrak{m} \otimes \Lambda^2(\mathfrak{m}^*)$ implies that the torsion-free connection is unique, so that it makes sense to speak of the holonomy of the underlying H-structure itself. I will refer to this part as the *metric list*. The second part consists of $\mathfrak{h} \subset \mathfrak{gl}(\mathfrak{m})$ that do not lie in any $\mathfrak{so}(Q)$ and hence will be referred to as the *nonmetric list*. For many of the subalgebras on the nonmetric list, the map δ is not injective, so that the associated H-structure does not determine the torsion-free connection.

These two lists are Tables I and II, essentially. I have taken the liberty of modifying Berger's lists slightly, dropping the entries in the original list that did not actually satisfy Berger's two criteria ($\mathrm{SO}^*(2n) \simeq \mathrm{SO}(n,\mathbb{H})$, which does not satisfy the first criterion[2], and the $\mathrm{Spin}(9,\mathbb{C})$-type entries, which do not satisfy the second criterion[3]) and including $\mathrm{Sp}(p,\mathbb{R})\cdot\mathrm{SL}(2,\mathbb{R})$, the inadvertently omitted 'split form' of the quaternionic Kähler case. In the nonmetric case, I have amplified the list somewhat by making explicit the various real forms as well as the fact that, except for $\mathrm{CO}(p,q)$, each of the entries on Berger's nonmetric list represents only the semi-simple part of H, to which one can add an arbitrary subgroup of the (abelian) commuting subgroup to make the full group H.

[2] An observation due to R. McLean.

[3] Observed independently by Alekseevskii [Al1] and Brown and Gray [BG].

I. Pseudo-Riemannian, Irreducible Holonomies in $\mathbb{R}^n$

n	H	Local Generality*
$p+q \geq 2$ $2p$	$\mathrm{SO}(p,q)$ $\mathrm{SO}(p,\mathbb{C})$	$\frac{1}{2}n(n-1)$ of n $\frac{1}{2}p(p-1)^{\mathbb{C}}$ of $p^{\mathbb{C}}$
$2(p+q) \geq 4$ $2(p+q) \geq 4$	$\mathrm{U}(p,q)$ $\mathrm{SU}(p,q)$	1 of n 2 of $n-1$
$4(p+q) \geq 8$	$\mathrm{Sp}(p,q)$	$2(p+q)$ of $(2p+2q+1)$
$4(p+q) \geq 8$ $4p \geq 8$ $8p \geq 16$	$\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$ $\mathrm{Sp}(p,\mathbb{R})\cdot\mathrm{SL}(2,\mathbb{R})$ $\mathrm{Sp}(p,\mathbb{C})\cdot\mathrm{SL}(2,\mathbb{C})$	$2(p+q)$ of $(2p+2q+1)$ $2p$ of $(2p+1)$ $2p^{\mathbb{C}}$ of $(2p+1)^{\mathbb{C}}$
7 7 14	G_2 G_2' $G_2^{\mathbb{C}}$	6 of 6 6 of 6 $6^{\mathbb{C}}$ of $6^{\mathbb{C}}$
8 8 16	$\mathrm{Spin}(7)$ $\mathrm{Spin}(4,3)$ $\mathrm{Spin}(7,\mathbb{C})$	12 of 7 12 of 7 $12^{\mathbb{C}}$ of $7^{\mathbb{C}}$

*Counted modulo diffeomorphism. The notation "d of ℓ" means "d functions of ℓ variables" and a superscript $\mathbb{C}$ means 'holomorphic'.

Exotic Holonomies. The nonmetric list supplied by Berger was never claimed to be a complete list, though it was supposed to have omitted at most a finite number of possibilities. The full list of omissions, comprising Tables III and IV and nowadays referred to as the *exotic list*, was recently compiled by a combination of the efforts of Chi, Merkulov, Schwachhöfer, and myself. This will be reported on in §3, along with the reasons for the division into two lists.

1.6. LOCAL EXISTENCE. Berger's lists (suitably modified) provide possibilities for irreducibly acting holonomy groups, but to verify that these possibilities actually can occur requires methods beyond representation theory. Most of the methods that have been employed can be grouped into a small number of categories:

Explicit construction. This is the simplest method, when it is available. For the metric list, there are locally symmetric examples with every holonomy except the special Kähler cases, where the holonomy is $\mathrm{SU}(p,q) \subset \mathrm{GL}(\mathbb{C}^{p+q})$; the hyper-Kähler cases, where the holonomy is $\mathrm{Sp}(p,q) \subset \mathrm{GL}(\mathbb{H}^{p+q})$; and the 'exceptional' holonomies, which comprise the groups $\mathrm{G}_2^{\mathbb{C}} \subset \mathrm{GL}(\mathbb{C}^7)$ and $\mathrm{Spin}(7,\mathbb{C}) \subset \mathrm{GL}(\mathbb{C}^8)$ and certain of their real forms. Of course, one would like to know that the locally symmetric examples are not the only ones.

II. The 'Classical' Non-Metric, Irreducible Holonomies

$\mathrm{H}^{\diamond}$	$\mathfrak{m}$	Restrictions$^{\nabla}$
$G_{\mathbb{R}}\cdot\mathrm{SL}(n,\mathbb{R})$	$\mathbb{R}^n$	$n \geq 2$
$G_{\mathbb{C}}\cdot\mathrm{SL}(n,\mathbb{C})$	$\mathbb{C}^n$	$n \geq 1$
$G_{\mathbb{R}}\cdot\mathrm{SL}(n,\mathbb{H})$	$\mathbb{H}^n$	$n \geq 1$
$\mathrm{Sp}(n,\mathbb{R})$	$\mathbb{R}^{2n}$	$n \geq 2$
$\mathrm{Sp}(n,\mathbb{C})$	$\mathbb{C}^{2n}$	$n \geq 2$
$\mathbb{R}^{+}\cdot\mathrm{Sp}(2,\mathbb{R})$	$\mathbb{R}^4$	
$\mathbb{C}^{*}\cdot\mathrm{Sp}(2,\mathbb{C})$	$\mathbb{C}^4$	
$\mathrm{CO}(p,q)$	$\mathbb{R}^{p+q}$	$p+q \geq 3$
$G_{\mathbb{C}}\cdot\mathrm{SO}(n,\mathbb{C})$	$\mathbb{C}^n$	$n \geq 3$
$G_{\mathbb{R}}\cdot\mathrm{SL}(p,\mathbb{R})\cdot\mathrm{SL}(q,\mathbb{R})$	$\mathbb{R}^{pq} \simeq \mathbb{R}^p \otimes_{\mathbb{R}} \mathbb{R}^q$	$p \geq q \geq 2,\ (p,q) \neq (2,2)$
$G_{\mathbb{C}}\cdot\mathrm{SL}(p,\mathbb{C})\cdot\mathrm{SL}(q,\mathbb{C})$	$\mathbb{C}^{pq} \simeq \mathbb{C}^p \otimes_{\mathbb{C}} \mathbb{C}^q$	$p \geq q \geq 2,\ (p,q) \neq (2,2)$
$G_{\mathbb{R}}\cdot\mathrm{SL}(p,\mathbb{H})\cdot\mathrm{SL}(q,\mathbb{H})$	$\mathbb{R}^{4pq} \simeq \mathbb{H}^p \otimes_{\mathbb{H}} \mathbb{H}^q$	$p \geq q \geq 1,\ (p,q) \neq (1,1)$
$G_{\mathbb{R}}\cdot\mathrm{SL}(p,\mathbb{C})$	$\mathbb{R}^{p^2} \simeq \left(\mathbb{C}^p \otimes_{\mathbb{C}} \overline{\mathbb{C}^p}\right)^{\mathbb{R}}$	$p \geq 3$
$G_{\mathbb{R}}\cdot\mathrm{SL}(p,\mathbb{R})$	$\mathbb{R}^{p(p+1)/2} \simeq \mathsf{S}^2_{\mathbb{R}}(\mathbb{R}^p)$	$p \geq 3$
$G_{\mathbb{C}}\cdot\mathrm{SL}(p,\mathbb{C})$	$\mathbb{C}^{p(p+1)/2} \simeq \mathsf{S}^2_{\mathbb{C}}(\mathbb{C}^p)$	$p \geq 3$
$G_{\mathbb{R}}\cdot\mathrm{SL}(p,\mathbb{H})$	$\mathbb{R}^{p(2p+1)} \simeq \mathsf{S}^2_{\mathbb{H}}(\mathbb{H}^p)$	$p \geq 2$
$G_{\mathbb{R}}\cdot\mathrm{SL}(p,\mathbb{R})$	$\mathbb{R}^{p(p-1)/2} \simeq \Lambda^2_{\mathbb{R}}(\mathbb{R}^p)$	$p \geq 5$
$G_{\mathbb{C}}\cdot\mathrm{SL}(p,\mathbb{C})$	$\mathbb{C}^{p(p-1)/2} \simeq \Lambda^2_{\mathbb{C}}(\mathbb{C}^p)$	$p \geq 5$
$G_{\mathbb{R}}\cdot\mathrm{SL}(p,\mathbb{H})$	$\mathbb{R}^{p(2p-1)} \simeq \Lambda^2_{\mathbb{H}}(\mathbb{H}^p)$	$p \geq 3$

$^{\diamond}$ $G_{\mathbb{F}}$ is any connected subgroup of $\mathbb{F}^*$. $^{\nabla}$ To avoid repetition or reducibility.

Sometimes constructing examples is easy: The generic pseudo-Riemannian metric has holonomy is $\mathrm{SO}(p,q)$.

In other cases, simple underlying geometric structures can be used as a starting point. For example, all complex structures are flat, and the general (pseudo-)Kähler metric can be described in the standard background complex structure by means of a (pseudo-)Kähler potential. For the generic choice of such a potential, the holonomy will be $\mathrm{U}(p,q) \subset \mathrm{SO}(2p,2q)$. Another example is the special Kähler case, where one can start with a background complex structure with a specified holomorphic volume form and then find a Kähler metric preserving this volume form by requiring that the Kähler potential satisfy a single second order, elliptic equation.

One can also find examples by looking for those with a large symmetry group. For the hyper-Kähler case, one can start with the complex symplectic structure on

III. Exotic Conformal Holonomies

$H^\diamond$	$\mathfrak{m}$
$\mathbb{R}^+\cdot SL(2,\mathbb{R})$ $\mathbb{C}^*\cdot SL(2,\mathbb{C})$	$\mathbb{R}^4 \simeq S^3(\mathbb{R}^2)$ $\mathbb{C}^4 \simeq S^3(\mathbb{C}^2)$
$G_{\mathbb{C}}\cdot SL(2,\mathbb{R})$ [a] $G_{\mathbb{C}}\cdot Sp(1)$ [b]	$\mathbb{C}^2 \simeq \mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}^2$ $\mathbb{C}^2 \simeq \mathbb{C}\otimes_{\mathbb{C}}\mathbb{H}$
$G_{\mathbb{R}}\cdot Spin(5,5)$ $G_{\mathbb{R}}\cdot Spin(1,9)$ $G_{\mathbb{C}}\cdot Spin(10,\mathbb{C})$	$\mathbb{R}^{16}$ $\mathbb{R}^{16}$ $\mathbb{C}^{16}$
$G_{\mathbb{R}}\cdot E_6^1$ $G_{\mathbb{R}}\cdot E_6^4$ $G_{\mathbb{C}}\cdot E_6^{\mathbb{C}}$	$\mathbb{R}^{27}$ $\mathbb{R}^{27}$ $\mathbb{C}^{27}$

$^\diamond$ $G_{\mathbb{F}}$ is any connected subgroup of $\mathbb{F}^*$

Restrictions:

[a] $G_{\mathbb{C}} \not\subseteq \mathbb{R}^*$ (for irreducibility).
[b] $G_{\mathbb{C}} \not\subseteq S^1$ (to be nonmetric).
[c] $p+q \geq 3$ (for irreducibility).
[d] $n \geq 3$ (for irreducibility).
[e] $n \geq 2$ (to be nonmetric).

IV. Exotic Symplectic Holonomies

H	$\mathfrak{m}$
$SL(2,\mathbb{R})$ $SL(2,\mathbb{C})$	$\mathbb{R}^4 \simeq S^3(\mathbb{R}^2)$ $\mathbb{C}^4 \simeq S^3(\mathbb{C}^2)$
$SL(2,\mathbb{R})\cdot SO(p,q)$ [c] $SL(2,\mathbb{C})\cdot SO(n,\mathbb{C})$ [d] $Sp(1)\cdot SO(n,\mathbb{H})$ [e]	$\mathbb{R}^2\otimes\mathbb{R}^{p+q}$ $\mathbb{C}^2\otimes\mathbb{C}^n$ $\mathbb{H}^n$
$Sp(3,\mathbb{R})$ $Sp(3,\mathbb{C})$	$\mathbb{R}^{14}\simeq\Lambda_0^3(\mathbb{R}^6)$ $\mathbb{C}^{14}\simeq\Lambda_0^3(\mathbb{C}^6)$
$SL(6,\mathbb{R})$ $SU(1,5)$ $SU(3,3)$ $SL(6,\mathbb{C})$	$\mathbb{R}^{20}\simeq\Lambda^3(\mathbb{R}^6)$ $\mathbb{R}^{20}\simeq\Lambda^3(\mathbb{C}^6)^{\mathbb{R}}$ $\mathbb{R}^{20}\simeq\Lambda^3(\mathbb{C}^6)^{\mathbb{R}}$ $\mathbb{C}^{20}\simeq\Lambda^3(\mathbb{C}^6)$
$Spin(2,10)$ $Spin(6,6)$ $Spin(12,\mathbb{C})$	$\mathbb{R}^{32}$ $\mathbb{R}^{32}$ $\mathbb{C}^{32}$
E_7^5 E_7^7 $E_7^{\mathbb{C}}$	$\mathbb{R}^{56}$ $\mathbb{R}^{56}$ $\mathbb{C}^{56}$

the cotangent bundle of certain Hermitian symmetric spaces and look for a Kähler potential compatible with this complex symplectic structure that is invariant under the action of the isometry group, thereby reducing the problem to solving an ordinary differential equation. This was Calabi's method for constructing a hyper-Kähler metric on $T^*\mathbb{CP}^n$, the first known example in general dimensions. In the quaternionic Kähler case, where the holonomy is $Sp(p,q)\cdot Sp(1) \subset GL(\mathbb{R}^{4(p+q)})$, Alekseevskii found homogeneous nonsymmetric examples on certain solvable Lie groups. Even for the exceptional holonomies, there are explicit examples of cohomogeneity one [Br1], [BS].

Examples in the hyper-Kähler and quaternionic Kähler cases can also be constructed by the method of *reduction*, which takes advantage of descriptions of these structures in terms of multi-symplectic geometry, generalizing the well-known method of Marsden-Weinstein reduction in symplectic geometry so as to handle the multi-symplectic case. For an account, see [Bes, Addendum E].

Twistor Methods. After Penrose's description of the self-dual metrics in dimension 4, Hitchin and Salamon [Sa], among others, were able to generalize this method to describe the hyper-Kähler and quaternionic Kähler metrics in terms of natural holomorphic geometric structures on the moduli space of rational curves with certain simple normal bundles in a complex manifold.

In fact, it was the study [Br2] of the moduli space of rational curves on a complex surface with normal bundle $\mathcal{O}(3)$ that turned up the first known examples (the first two entries in each of Tables III and IV) of omissions[4] from Berger's nonmetric list. Moreover, in the holomorphic category, it was shown that any connection with one of these holonomies could be constructed as the natural connection on the four-dimensional component of the moduli space of Legendrian rational curves in a complex contact three-fold.

Following this, Merkulov [Me1] showed that this approach could be generalized to cover the geometry of the moduli space of Legendrian deformations of certain complex homogeneous spaces and began to discover more exotic examples. This and its further developments will be reported on in §3.

Exterior differential systems. Another approach is to treat the equation $\bar{T} = 0$ directly as a system of PDE for sections of the bundle $F(M, \mathfrak{m})/H \to M$. In nearly all cases, this method leads to the study of an overdetermined system of PDE, so that Cartan-Kähler machinery must be brought to bear. For definitions and results regarding Cartan-Kähler theory, the reader can consult [BCG].

The general approach can be summarized as follows: Consider the structure equations derived so far for a torsion-free connection θ on an H-structure $B \subset F(M, \mathfrak{m})$,

$$\begin{aligned} d\omega &= -\theta \wedge \omega \\ d\theta &= -\tfrac{1}{2}[\theta, \theta] + \tfrac{1}{2} R(\omega \wedge \omega) \\ dR &= -\theta.R + R'(\omega) \end{aligned} \tag{0.4}$$

where $R : B \to K(\mathfrak{h})$ and $R' : B \to K^1(\mathfrak{h}) \subset K(\mathfrak{h}) \otimes \mathfrak{m}^*$ are as defined before. There are two things that need to be checked in order to be able to apply Cartan's general existence theorem for coframings at this level: First, the inclusion $K^1(\mathfrak{h}) \subset K(\mathfrak{h}) \otimes \mathfrak{m}^*$ should be an involutive tableau in Cartan's sense. Second, there should be a quadratic map $Q : K(\mathfrak{h}) \to K^1(\mathfrak{h}) \otimes \mathfrak{m}^*$ so that the exterior derivative of the third structure equation in (0.4) can be written in the form $\big(dR' + \theta.R' - Q(R)(\omega)\big)(\omega) = 0$. (This is the familiar 'vanishing torsion' condition in exterior differential systems.)

When these two conditions are satisfied, Cartan's existence theorem asserts that, up to local diffeomorphism, the real analytic torsion-free connections with holonomy

[4] These examples were referred to as 'exotic' in [Br2] and the term has been adopted to describe any nonmetric subgroup $H \subset \mathrm{GL}(\mathfrak{m})$ that satisfies Berger's criteria but that does not appear on Berger's original nonmetric list.

lying in $H \subset \mathrm{GL}(\mathfrak{m})$ depend on s_q functions of q variables, where s_q is the last nonzero Cartan character of $K^1(\mathfrak{h}) \subset K(\mathfrak{h}) \otimes \mathfrak{m}^*$. Moreover, for any $(R_0, R_0') \in K(\mathfrak{h}) \times K^1(\mathfrak{h})$, there exists a torsion-free connection θ on an H-structure $B \subset F(\mathfrak{m}, \mathfrak{m})$ and a $u_0 \in B$ for which the curvature functions R and R' satisfy $R(u_0) = R_0$ and $R'(u_0) = R_0'$.

When one can choose the element $R_0 \in K(\mathfrak{h}) \subset \mathfrak{h} \otimes \Lambda^2(\mathfrak{m}^*)$ so that it is surjective[5] as a map $R_0 : \Lambda^2(\mathfrak{m}) \to \mathfrak{h}$, it will then follow from the Ambrose-Singer holonomy theorem that the holonomy of θ contains the identity component of H. If, moreover, one can choose R_0' to be nonzero, such a connection θ will not be locally symmetric.

This analysis applies successfully to each of the entries of Tables I, II, and III. By contrast, for each of the entries of Table IV, the tableau $K^1(\mathfrak{h}) \subset K(\mathfrak{h}) \otimes \mathfrak{m}^*$ is not involutive. Further discussion of this point will be given in §3.

Generally, this method is good only for local analysis, but it has the distinct advantage that it not only proves existence of connections with a given holonomy, but provides their 'degree of generality' in Cartan's sense. For example, Table I gives the degree of generality of each of the possible pseudo-Riemannian, irreducible holonomies. For a similar discussion of the nonmetric list, see the survey [Br3], where various simplifications of the general argument are introduced to shorten the exposition.

This method was first used to prove the existence of metrics with holonomy G_2 and Spin(7) and is still the only method that constructs the general local solution and describes its degree of generality. This is also the only known method for analyzing Entries 3 and 4 of Table III.

Poisson Constructions. The examples H in Table IV are subgroups of $\mathrm{Sp}(S) \subset \mathrm{GL}(\mathfrak{m})$ for a nondegenerate skewsymmetric bilinear form S on $\mathfrak{m}$. Hence the corresponding H-structures (when they exist) have an underlying symplectic structure.

For the first two examples from Table IV, each torsion-free connection (M, B, θ) of these types was analyzed and reconstructed in [Br2] from its derived curvature map $J = (R, R') : B \to K(\mathfrak{h}) \oplus K^1(\mathfrak{h})$. This reconstruction involved a number of seemingly miraculous identities, but since only these two examples were known, it did not seem worthwhile to look for an interpretation of these identities. However, when Chi, Merkulov, and Schwachhöfer [CMS] found other exotic symplectic examples, they noticed that this reconstruction technique generalized and they were able to explain it in the context of Poisson geometry in a very beautiful way. This, too, will be reported on in §3.

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5 Actually, one only needs that the image $R_0(\Lambda^2(\mathfrak{m})) \subset \mathfrak{h}$ generates $\mathfrak{h}$ as an algebra.

1.7. COMPACT RIEMANNIAN EXAMPLES. The history of compact Riemannian manifolds with reduced holonomy groups is long and complex, so I will not attempt a full account here. For more details on the cases I only mention, the reader can consult the relevant chapters of [Bes] and the references cited therein.

Kähler manifolds. This subject has the longest history, predating even Berger's classification. Every smooth algebraic variety carries a Kähler structure and this accounts for their importance in algebraic geometry.

Special Kähler manifolds. The major milestone here is, of course, Yau's solution in the mid 1970s of the Calabi conjecture, showing that every compact Kähler manifold with trivial canonical bundle carries a special Kähler structure. For this reason, compact manifolds endowed with such a structure are usually referred to as Calabi-Yau manifolds.

Hyper-Kähler manifolds. In the early 1980s, Beauville and Mukai were each able to construct compact, simply connected Kähler manifolds M^{4p} that carried a nondegenerate complex symplectic form (Fujiki had constructed examples for $p = 2$ slightly earlier). These necessarily had trivial canonical bundle, so by Yau's solution of the Calabi conjecture, these carried special Kähler structures. By an argument of Bochner, the complex symplectic form had to be parallel with respect to this special Kähler structure, which forced the holonomy to lie in $\mathrm{Sp}(p)$. Further arguments showed that these examples were not products of lower dimensional complex manifolds and this implied that the holonomy had to actually be $\mathrm{Sp}(p)$. These were the first known compact examples.

Quaternion Kähler manifolds. In this case, all known compact examples are locally symmetric, but we know of no reason why this should be true, except for dimension 8 [PoS]. Of course, a great deal is known about the possible geometry of such examples, see [Sa].

G_2 *and* Spin(7) *manifolds.* The remarkable recent work of Joyce [Jo1,2] establishes the existence of compact manifolds with these holonomies. This will be reported on in §2.

2. RIEMANNIAN HOLONOMY

In this section, g will denote a smooth Riemannian metric on a connected, smooth manifold M^n. The reference space $\mathfrak{m}$ will be taken to be $\mathbb{R}^n$ with its standard inner product, and $\mathrm{O}(\mathfrak{m}) \subset \mathrm{GL}(\mathfrak{m})$ will denote its orthogonal group. The $\mathrm{O}(\mathfrak{m})$-structure $B \subset F(M, \mathfrak{m})$ consisting of the coframes $u : T_xM \to \mathfrak{m}$ that are isometries of vector spaces is the orthonormal coframe bundle and the Levi-Civita connection on B will be denoted θ. This is, of course, the unique torsion-free $\mathrm{O}(\mathfrak{m})$-connection on B. When there is no danger of confusion, I will simply write H_u and B_u instead of H^θ_u and B^θ_u.

One thing that makes the Riemannian case simpler to deal with than others is the *de Rham Splitting Theorem* [Bes, KNo], which occurs in a local form and a global form. The local form asserts that if $(H_u)^0$ acts reducibly on $\mathfrak{m}$, say, preserving irreducible orthogonal subspaces $\mathfrak{m}_i \subset \mathfrak{m}$ for $1 \leq i \leq k$, then $(H_u)^0$ is the direct product of its subgroups $(H_u)^0_i$, where $(H_u)^0_i$ is the subgroup that acts trivially on $\mathfrak{m}_j$ for $j \neq i$. Moreover, the metric g locally splits as a product in a corresponding way[6]. The global form asserts that if, in addition, M is simply connected and the metric g is complete, then M globally splits as a Riemannian product

$$(M, g) = (M_1, g_1) \times \cdots \times (M_k, g_k).$$

so that $(H_u)^0_i = (H_u)_i$ is the holonomy of (M_i, g_i).

2.1. Non-closed holonomy. While the remainder of this report deals only with connected holonomy groups, I cannot pass up the opportunity to mention a recent result of particular interest in Riemannian holonomy. It had been a question for some time whether the holonomy of a compact Riemannian manifold must necessarily be compact. Using the de Rham Splitting Theorem and Berger's holonomy classification in the irreducible Riemannian case, one sees that this is so if M is simply connected. Thus, for any Riemannian manifold, the restricted holonomy group is compact, so it becomes a question of whether the fundamental group can cause the holonomy group to have an infinite number of components even when M is compact.

Very recently, B. Wilking [Wi] has shown that this can indeed occur. He has produced an example of a compact manifold whose holonomy group does have an infinite number of components. His example is of the form $M^5 = \Gamma\backslash(\mathbb{R}^2 \times N^3)$ where Γ is a subgroup of the isometry group of $\mathbb{R}^2 \times N^3$ that acts properly discontinuously and cocompactly.

2.2. Compact manifolds with exceptional holonomy. The most remarkable development in Riemannian holonomy in recent years has been the spectacular con-

[6] What is also true, but not obvious, is that each of the groups $(H_u)^0_i \subset \mathrm{SO}(\mathfrak{m}_i)$ is the holonomy of some Riemannian metric, even when it is not the holonomy of the corresponding local factor of g. See [Bes, Theorem 10.108]

struction by Dominic Joyce of compact 7-manifolds with holonomy G_2 and compact 8-manifolds with holonomy Spin(7). His constructions are full of new ideas and, while it is not difficult to outline these ideas, their successful execution turns out to require very careful, subtle estimates. I will not attempt to explain these, but refer the reader to the original sources [Jo1,2] and to the forthcoming book [Jo3]. Also, I will concentrate on the G_2 case, as the Spin(7) case follows the same spirit, but the details are different.

The fundamental 3*-form.* Let $\mathfrak{m} = \mathbb{R}^7$. It has been known for some time [Br1] that there is an open $\mathrm{GL}(\mathfrak{m})$-orbit $\Lambda^3_+(\mathfrak{m}^*)$ in the 3-forms on $\mathfrak{m}$ so that the stabilizer of any element $\phi \in \Lambda^3_+(\mathfrak{m}^*)$ is a compact connected simple Lie group of dimension 14 and which is therefore isomorphic to G_2. Consequently, for any 7-manifold M, there is an open subbundle $\Lambda^3_+(T^*M) \subset \Lambda^3(T^*M)$ so that the G_2-structures on M are in one-to-one correspondence with the sections $\Omega^3_+(TM)$ of this bundle. Such a section will exist if and only if M is orientable and spinnable [LM], so I assume this from now on.

Associated to any section $\sigma \in \Omega^3_+(M)$, there is a canonical metric g_σ and orientation, inducing a well-defined Hodge star operator $*_\sigma : \Omega^p(M) \to \Omega^{7-p}(M)$. If σ is parallel with respect to the Levi-Civita connection of g_σ then the holonomy of g_σ will be a subgroup of G_2 and the G_2-structure associated to σ will be torsion-free. By a result of Gray, this G_2-structure is torsion-free if and only if both σ and $*_\sigma\sigma$ are closed. Conversely, if a Riemannian metric g on M^7 has holonomy a subgroup of G_2, then there will be a g-parallel section $\sigma \in \Omega^3_+(M)$ for which $g = g_\sigma$.

A metric with holonomy a subgroup of G_2 is known to be Ricci flat [Bes, 10.64], so if M is compact, Bochner vanishing shows that its harmonic 1-forms and its Killing fields must be parallel. Thus, in the compact case, $b_1(M) = 0$ if the holonomy actually equals all of G_2. Combining this information with the Cheeger-Gromoll splitting theorem [Bes, 6.67], one finds that a compact manifold with holonomy G_2 must have finite fundamental group, so one can, without loss of generality, assume that M is simply connected, which I do from now on. Conversely, a compact simply connected Riemannian 7-manifold whose holonomy is a subgroup of G_2 must actually equal G_2 since any proper subgroup of G_2 that can be a holonomy group fixes a nontrivial subspace.

One can now consider the moduli space $\mathcal{M}$ consisting of the sections $\sigma \in \Omega^3_+(M)$ satisfying $d\sigma = d*_\sigma\sigma = 0$ modulo diffeomorphisms of M isotopic to the identity. There is an obvious 'Torelli' map $\tau : \mathcal{M} \to H^3_{\mathrm{dR}}(M)$ defined by $\tau([\sigma]) = [\sigma]_{\mathrm{dR}}$. Joyce's first striking result [Jo1] is the analog of the local Torelli theorem, namely that τ is locally one-to-one and onto, in fact, a local diffeomorphism in the natural smooth structure on $\mathcal{M}$. Thus, the moduli space is said to be 'unobstructed'.

Next, Joyce proves a remarkable existence theorem: If $\sigma \in \Omega^3_+(M)$ is closed, then there is a constant C that depends on the norm of the curvature of the metric g_σ, its volume, and its injectivity radius, so that, if $\left|d(*_\sigma\sigma)\right|_\sigma < C$, then there exists an exact 3-form ϕ so that $\sigma+\phi$ lies in $\Omega^3_+(M)$ and is closed and coclosed. In other words, a closed 3-form in $\Omega^3_+(M)$ that is 'close enough' to being coclosed can be perturbed to a cohomologous 3-form in $\Omega^3_+(M)$ that is both closed and coclosed.

Thus, to prove the existence of a compact Riemannian 7-manifold with holonomy G_2, it suffices to construct a simply connected 7-manifold endowed with a closed 3-form that satisfies such a 'close enough' estimate.

Joyce's idea for doing this is extremely clever: He starts with the flat G_2-structure σ_0 on the 7-torus $T^7 = \mathbb{R}^7/\mathbb{Z}^7$ and passes to a simply connected quotient orbifold $X = \Gamma\backslash T$ where Γ is a finite group of σ_0-symmetries. This provides a flat G_2-orbifold whose singular locus is a finite number of 3-tori T^3, each of which has a neighborhood of the form $T^3 \times B^4/\{\pm\}$ where $B^4/\{\pm\}$ is the standard 4-ball around the origin in $\mathbb{R}^4 = \mathbb{C}^2$ divided by the equivalence relation $v \sim -v$.

Now, it has been known for a long time that $\mathbb{R}^4/\{\pm\}$ is metrically the scaling limit of the SU(2)-holonomy metric on $T^*\mathbb{CP}^1$ constructed by Eguchi and Hanson as one scales the metric to contract the zero section to a point. Because $I_3 \times \mathrm{SU}(2) \subset \mathrm{SO}(7)$ is a subgroup of G_2, it follows that one can regard the flat G_2-structure on X in each singular locus neighborhood $T^3 \times B^4/\{\pm\}$ as the limit of a scaling of a G_2-structure on $T^3 \times T^*\mathbb{CP}^1$. Joyce cuts out these singular neighborhoods and glues back in $T^3 \times N$ where N is a neighborhood of the zero section in $T^*\mathbb{CP}^1$, smoothly joining the flat 3-form with the Eguchi-Hanson-derived (closed) 3-form on the overlaps. This 'surgery' produces a smooth manifold $\hat{X}$, but does not disturb the fundamental group, which remains trivial.

By being very careful (here is where Joyce's estimates are extremely delicate), he shows that he can do this in such a way that the resulting closed 3-form $\sigma \in \Omega^3_+(\hat{X})$ satisfies his estimate. I.e., it is close enough to being coclosed that it can be perturbed to a $\tilde{\sigma} \in \Omega^3_+(\hat{X})$ that is both closed and coclosed. Of course, since X is simply connected, it follows that the resulting torsion-free G_2-structure has holonomy equal to G_2.

By applying this idea to a number of different finite groups Γ, Joyce has been able to construct G_2-metrics on a number of different 7-manifolds.

A similar set of ideas allows Joyce to construct compact 8-manifolds with holonomy Spin(7), once one realizes that Spin(7) can be defined in $\mathrm{GL}(8,\mathbb{R})$ as the stabilizer of a certain 4-form in eight variables. The interested reader should consult [Jo1,2].

3. IRREDUCIBLE TORSION-FREE NON-METRIC CONNECTIONS

3.1. TWISTOR CONSTRUCTIONS. As previously mentioned, I found the first examples of 'exotic' holonomy groups by studying the geometry of the moduli space M of rational curves (i.e., complex curves of genus 0) in a complex surface S with normal bundle $\mathcal{O}(3)$. Following the examples of Hitchin's study of rational curves in a surface S with normal bundle $\mathcal{O}(k)$ for $k = 1$ and 2, I knew that M would have dimension 4 and would have a natural G_3-structure, where $G_3 \subset \mathrm{GL}(4,\mathbb{C})$ is the image of $\mathrm{GL}(2,\mathbb{C})$ acting by linear substitutions on the 4-dimensional space V_3 of cubic polynomials in two variables. I also knew that there would be a canonical G_3-connection from general principles, but I was very surprised to find that this connection was torsion-free.

In the examples Hitchin had analyzed, the geometry on the moduli space allowed one to reconstruct the surface S and so I fully expected to be able to do the same in this case. However, it turned out that the story was more subtle than that. In the standard double fibration picture:

$$\begin{array}{ccc} & I & \\ {}^{\lambda}\swarrow & & \searrow^{\rho} \\ M & & S \end{array}$$

where $I \subset M \times S$ is the set of pairs (C,p) where $p \in S$ is a point of the rational curve $C \in M$ and λ and ρ are just the projections onto the factors, each $p \in S$ would correspond to a hypersurface $H_p = \lambda\big(\rho^{-1}(p)\big)$ in M (since it is only one condition for a curve in S to pass through a given point p). The members of this 2-parameter family of hypersurfaces in M would be expected to be the solutions of some differential geometric problem in M, but I was not able to find a geometrically defined 2-parameter family of hypersurfaces in the general 4-manifold carrying a torsion-free G_3-structure.

However, a 3-parameter family Y of 2-dimensional surfaces did present itself. This can be described as follows: By the defining property of a G_3-structure B on M^4, each tangent space T_xM can be thought of as the space of homogeneous cubic polynomials in two variables. This defines a distinguished $\mathbb{P}^1$ of lines, namely the perfect cubes, and a distinguished $\mathbb{P}^1$ of 2-planes, namely the multiples of a perfect square. I called such lines and 2-planes *null*. It was not difficult to show that, when B had a torsion-free connection, each null 2-plane was tangent to a unique totally geodesic 2-surface in M all of whose tangent planes were null. This family Y then fit into a double fibration[7]

$$\begin{array}{ccc} & N^5 & \\ {}^{\lambda}\swarrow & & \searrow^{\rho} \\ M^4 & & Y^3 \end{array}$$

where the fibers of λ are $\mathbb{P}^1$'s. Moreover, I was able to show that Y carried a natural structure as a contact manifold, that the family of $\mathbb{P}^1$'s given by $C_x = \rho(\lambda^{-1}(x))$ for $x \in M$ were all Legendrian curves for this contact structure, and that, moreover, this family was an open set in the space of Legendrian rational curves in Y. (In the surface case that I had started out with, Y turned out to be the projectivized tangent bundle of S.)

I then showed that the picture could be reversed: Starting with a holomorphic contact 3-manifold Y, one could look at the moduli space M of rational Legendrian curves C in Y to which the contact bundle $L \subset T^*Y$ restricts to be isomorphic to $\mathcal{O}(-3)$ and show that it was a smooth moduli space of dimension 4 on which there was a canonical torsion-free G_3-structure.

Merkulov's generalization. In a remarkable series of papers, Merkulov [Me1,2,3] showed that this moduli space and double fibration construction obtains in a very general setting, starting from the data of an irreducibly acting (and therefore reductive) complex subgroup $H \subset \mathrm{GL}(n,\mathbb{C})$, a complex n-manifold M, and a holomorphic H-structure $B \subset F(M,\mathbb{C}^n)$ endowed with a holomorphic torsion-free connection θ.

The semi-simple part $H_s \subset H$ acts irreducibly on $\mathbb{C}^n$. With respect to a Cartan subalgebra of H_s and an ordering of its roots, there will be a unique line $E \subset (\mathbb{C}^n)^*$ spanned by a vector of highest weight. The H_s-orbit $F \subset \mathbb{P}\big((\mathbb{C}^n)^*\big)$ of E is a minimal H_s-orbit in $\mathbb{P}\big((\mathbb{C}^n)^*\big)$, a generalized flag variety of H_s of some dimension $k \leq n-1$, endowed with the hyperplane section bundle $\mathcal{L}$. (In the original case I treated, F is the projectivization of the set of perfect cubes and hence is a $\mathbb{P}^1$. The bundle $\mathcal{L}$ is $\mathcal{O}(-3)$.) The H-structure B provides identifications $T_xM \simeq \mathbb{C}^n$ unique up to an action of H, so there is a subbundle $N \subset \mathbb{P}(T^*M)$ whose fiber $N_x \subset \mathbb{P}(T^*_xM)$ over x corresponds to $F \subset \mathbb{P}\big((\mathbb{C}^n)^*\big)$ via any B-identification.

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[7] Assuming Y to be Hausdorff in its natural topology, which can always be arranged by replacing M by a θ-convex open set in M.

The projectivized cotangent bundle of any manifold is canonically a contact manifold, and the torsion-free condition on θ immediately implies that N is an *involutive* submanifold of $\mathbb{P}(T^*M)$, i.e., the restriction of the contact structure to N is degenerate, with Cauchy leaves of the largest possible dimension, namely $n-k-1$, the codimension of N in $\mathbb{P}(T^*M)$. When it is Hausdorff [8], the leaf space Y of this Cauchy foliation is canonically a contact manifold, yielding the double fibration

$$\begin{array}{ccc} & N^{n+k} & \\ {}^{\lambda}\swarrow & & \searrow^{\rho} \\ M^n & & Y^{2k+1} \end{array}$$

where the manifolds $F_x = \rho\big(\lambda^{-1}(x)\big)$ are Legendrian k-dimensional submanifolds of Y.

Merkulov then goes on to prove that, nearly always, one can recover M as the complete moduli space of the Legendrian immersions $F \subset Y$ that pull back the contact bundle $L \subset T^*Y$ to be $\mathcal{L}$. Moreover, when H_s acts as the full biholomorphism group of F (which, again, is nearly always) one can recover the original H-structure on M up to conformal scaling from the family of submanifolds $S_y = \lambda\big(\rho^{-1}(y)\big)$ for $y \in Y$.

Finally, Merkulov gives representation theoretic criteria on an irreducibly acting subgroup $H \subset \mathrm{GL}(\mathfrak{m})$ with associated generalized flag variety $F \subset \mathbb{P}(\mathfrak{m}^*)$ of dimension k and hyperplane bundle $\mathcal{L}$, which guarantee that taking a $(2k+1)$-dimensional contact manifold Y and considering the moduli space $M(Y)$ of Legendrian embeddings $F \subset Y$ that pull back the contact bundle L to be $\mathcal{L}$ yields a smooth moduli space endowed with an H-structure and a torsion-free connection.

This last step is extremely important, for it provides a way to determine which irreducibly acting subgroups $H \subset \mathrm{GL}(\mathfrak{m})$ can occur as torsion-free holonomy in terms of representation theory, specifically, in terms of the vanishing of certain H-representations constructed functorially from $\mathfrak{m}$. This provides a different approach to solving the torsion-free holonomy problem, one that was carried out successfully by a combination of efforts of Chi, Merkulov, and Schwachhöfer. In particular, this approach led to the discovery of the remaining groups in Table IV and, finally, the proof that Tables I, II, III, and IV exhaust the possibilities for irreducibly acting torsion-free holonomy.

3.2. POISSON CONSTRUCTIONS. The straightforward application of exterior differential systems to the holonomy problem outlined in §0.6 does not work for the entries

[8] This can always be arranged by replacing M by a suitably θ-convex open set in M.

of Table IV, at least at the level described so far. To see where the problem is, recall the structure equations derived so far for a torsion-free connection with holonomy $H \subset \mathrm{GL}(\mathfrak{m})$. They are

$$\begin{aligned} d\omega &= -\theta \wedge \omega \\ d\theta &= -\tfrac{1}{2}[\theta, \theta] + \tfrac{1}{2} R(\omega \wedge \omega) \\ dR &= -\theta.R + R'(\omega) \end{aligned} \tag{3.1}$$

where $R : B \to K(\mathfrak{h})$ and $R' : B \to K^1(\mathfrak{h}) \subset K(\mathfrak{h}) \otimes \mathfrak{m}^*$ are as already defined.

When one considers the first entry of Table IV, where $H = \mathrm{SL}(2, \mathbb{R})$ and $\mathfrak{m} \simeq V_3 = \mathrm{S}^3(V_1)$, it is not difficult to see that $K(\mathfrak{h}) \simeq V_2 = \mathrm{S}^2(V_1)$ has dimension 3 and that $K^1(\mathfrak{h}) \simeq V_3$ has dimension 4. Its Cartan characters are $(s_1, s_2, s_3, s_4) = (3, 1, 0, 0)$ but, as is easily computed, the prolongation $K^2(\mathfrak{h}) \subset K^1(\mathfrak{h}) \otimes \mathfrak{m}^*$ has dimension 1, so the tableau is not involutive and Cartan's existence theorem cannot be applied at this level.

However, there is a quadratic map $Q : K(\mathfrak{h}) \to K^1(\mathfrak{h}) \otimes \mathfrak{m}^*$ so that the exterior derivative of the third equation in (3.1) is $\big(dR' + \theta.R' - Q(R)(\omega)\big)(\omega) = 0$, implying that there is a function $R'' : B \to K^2(\mathfrak{h})$ so that the equation

$$dR' = -\theta.R' + \big(R'' + Q(R)\big)(\omega) \tag{3.2}$$

holds. Moreover, it is possible to choose the quadratic map Q in a unique way so that differentiating this last equation yields $dR''(\omega) = 0$. Now the second prolongation of $K^1(\mathfrak{h})$ vanishes, so this forces the structure equation

$$dR'' = 0. \tag{3.3}$$

Obviously, differentiating this equation will yield no new information.

At this point, Cartan's general existence theorem for coframings satisfying prescribed differential identities (a generalization of Lie's third fundamental theorem) can be applied to the entire ensemble

$$\begin{aligned} d\omega &= -\theta \wedge \omega, \\ d\theta &= -\tfrac{1}{2}[\theta, \theta] + \tfrac{1}{2} R(\omega \wedge \omega), \\ dR &= -\theta.R + R'(\omega), \\ dR' &= -\theta.R' + \big(R'' + Q(R)\big)(\omega), \\ dR'' &= 0. \end{aligned} \tag{3.4}$$

His theorem implies that for every $(R_0, R_0', R_0'') \in K(\mathfrak{h}) \times K^1(\mathfrak{h}) \times K^2(\mathfrak{h})$, there is a torsion-free H-structure $B \subset F(\mathfrak{m}, \mathfrak{m})$, unique up to local diffeomorphism, so that, at a point $u_0 \in B$ one has $R(u_0) = R_0$, $R'(u_0) = R_0'$, and $R''(u_0) = R_0''$. Consequently, the space of diffeomorphism classes of germs of such torsion-free H-structures is finite dimensional.

What Chi, Merkulov, and Schwachhöfer show in [CMS] is that this exact same picture holds for each entry in Table IV. Namely, $K(\mathfrak{h}) \simeq \mathfrak{h}$, $K^1(\mathfrak{h}) \simeq \mathfrak{m}$, $K^2(\mathfrak{h}) \simeq \mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$), and there is a quadratic map $Q : \mathfrak{h} \to \mathfrak{m} \otimes \mathfrak{m}^*$ so that the above analysis of the structure equations goes through exactly the same as for the original two cases (with the obvious interpretations of the pairings).

Cartan's theorem then applies and yields not only the existence of torsion-free connections with these prescribed holonomies, but that the space of germs of such H-structures, modulo diffeomorphism, is finite dimensional.

In the original two cases that I treated, the finer understanding of the moduli space of solutions entailed understanding how the images $(R, R', R'')(B) \subset \mathfrak{h} \times \mathfrak{m} \times \mathbb{F}$ partition $\mathfrak{h} \times \mathfrak{m} \times \mathbb{F}$ into subsets. This analysis would have been hopeless were it not for several (at the time) amazing identities that I found by brute force calculation. They even allowed me to prove existence without using Cartan's existence theorem.

What is shown in [CMS], however, is that these mysterious identities can be explained in terms of a natural Poisson structure on the space $\mathfrak{h} \times \mathfrak{m} \times \mathbb{F}$ (actually, they regard the last factor as a parameter and consider Poisson structures on $\mathfrak{h} \times \mathfrak{m}$). The images $(R, R', R'')(B)$ turn out to be the symplectic leaves of this Poisson structure and this point of view simplifies the reconstruction of the H-structure from the leaf image (though it does not entirely remove some of the global difficulties having to do with the symplectic realizations necessary in their construction).

3.3. Algebraic classification. Once the full list of the irreducible torsion-free holonomies was known, there arose the question of whether this list could be derived through Berger's original approach, i.e., representation theory. Schwachhöfer [Schw] has shown that this can indeed be done. (As is so often the case, knowing the answer in advance helps to organize the proof.) His fully algebraic classification of the irreducibly acting subgroups $H \subset \mathrm{GL}(\mathfrak{m})$ that satisfy Berger's first criterion and the subset of those that also satisfy Berger's second criterion still involves quite a bit of case checking, but the general outline of the argument is clear.

What is particularly intriguing is Ziller's observation (reported in [Schw]) that this list can be constructed by a simple Ansatz starting from the list of Hermitian and quaternionic symmetric spaces. A direct proof of Ziller's Ansatz would be highly desirable.

3.4. TWO LEFTOVER CASES. As of this writing, each entry in the four Tables, save two, Entries 3 and 4 in Table III, has been treated in the literature and shown to occur as holonomy, either by twistor methods or exterior differential systems methods. Existence proofs by twistor methods have some difficulty when $\mathfrak{m}$ is a complex vector space and the group $H \subset \mathrm{GL}(\mathfrak{m})$ is of the form $H = G_{\mathbb{C}} \cdot H_s$ where H_s is the semi-simple part and $G_{\mathbb{C}} \subset \mathbb{C}^*$ is a one-dimensional subgroup of $\mathbb{C}^*$, acting as scalar multiplication on $\mathfrak{m}$. The method of exterior differential systems does not have this problem, but each case does require a separate treatment.

In my survey article [Br3], I left these two entries unsettled in the case where $G_{\mathbb{C}}$ had dimension 1 because, at the time, they did not seem that interesting. Now that they are the last unsettled cases, it seems to be a good idea to resolve them, so I will do that here, though, for lack of space, I will not provide details, just give the results of the Cartan-Kähler analysis.

For the first case, $H = G_{\mathbb{C}} \cdot \mathrm{SL}(2, \mathbb{R}) \subset \mathrm{GL}(2, \mathbb{C})$, one must assume that $G_{\mathbb{C}} \not\subseteq \mathbb{R}^*$, otherwise H will not act irreducibly on $\mathfrak{m} \simeq \mathbb{C}^2$. This leaves a one parameter family of possibilities $H_\lambda = \{e^{(i+\lambda)t} \mid t \in \mathbb{R}\} \subset \mathbb{C}^*$ where λ is any real number. By conjugation, one can assume that $\lambda \geq 0$, so I will do this. It turns out that there are two cases:

If $\lambda = 0$, so that $H_0 = S^1$, it is not difficult to compute that $K(\mathfrak{h}) \simeq V_4 \oplus V_2 \oplus V_0$, while $K^1(\mathfrak{h}) \simeq 2V_5 \oplus 2V_3 \oplus 2V_1$ and is involutive, with characters $(s_1, s_2, s_3, s_4) = (9, 9, 5, 1)$. Moreover, the torsion is absorbable. By Cartan's theorem, solutions exist and depend on one function of four variables.

However, if $\lambda > 0$, so that $H_\lambda \simeq \mathbb{R}^+$, one computes that $K(\mathfrak{h}) \simeq V_4 \oplus V_2$ while $K^1(\mathfrak{h}) \simeq 2V_5 \oplus 2V_3$ and is involutive, with characters $(s_1, s_2, s_3, s_4) = (8, 8, 4, 0)$. Moreover, the torsion is absorbable. By Cartan's theorem, solutions exist and depend on four functions of three variables.

For the second case, $H = G_{\mathbb{C}} \cdot \mathrm{SU}(2) \subset \mathrm{GL}(2, \mathbb{C})$, one must assume that $G_{\mathbb{C}} \not\subseteq S^1$, otherwise $H = \mathrm{U}(2)$ will preserve a metric on $\mathfrak{m} \simeq \mathbb{C}^2$. This leaves a one parameter family of possibilities $J_\lambda = \{e^{(1+i\lambda)t} \mid t \in \mathbb{R}\} \subset \mathbb{C}^*$ where λ is any real number. By conjugation, one can assume that $\lambda \geq 0$, though I won't need to do this. Here there is only one case: One computes that $K(\mathfrak{h}) \simeq V_4^{\mathbb{R}} \oplus V_2^{\mathbb{R}} \simeq \mathbb{R}^8$ while $K^1(\mathfrak{h}) \simeq V_5 \oplus V_3 \simeq \mathbb{C}^{10} \simeq \mathbb{R}^{20}$. The tableau is involutive, with characters $(s_1, s_2, s_3, s_4) = (8, 8, 4, 0)$. Moreover, the torsion is absorbable. By Cartan's theorem, solutions exist and depend on four functions of three variables.

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