Astérisque

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Astérisque, tome 266 (2000), Séminaire Bourbaki,
exp. nº 854, p. 113-136
<http://www.numdam.org/item?id=SB_1998-1999__41__113_0>

Séminaire BOURBAKI Novembre 1998
51ème année, 1998-99, n° 854

CHAOTIC BEHAVIOUR IN THE SOLAR SYSTEM [following J. LASKAR]

by **Stefano MARMI**

0. INTRODUCTION

I. Newton certainly believed that the Solar System is topologically unstable. In his view the perturbations among the planets were strong enough to destroy the stability of the Solar System. He even made the hypothesis that God controls the instabilities so as to insure the existence of the Solar System: "but it is not to be conceived that mere mechanical causes could give birth to so many regular motions .... This most beautiful system of the sun, planets, and comets, could only proceed from the counsel and dominion of an intelligent powerful Being" [N, p. 544]. The problem of Solar System stability was (and for many aspects still is) a real one: Halley was able to show, by analyzing the Chaldean observations transmitted by Ptolemy, that Saturn was moving away from the Sun while Jupiter was getting closer. A crude extrapolation leads to a possible collision in 6 million years (Myrs) in the past.

From a mathematical point of view arguments supporting the long–time stability of the orbits of the planets were given by Lagrange, Laplace and Poisson who proved the absence of secular evolution (polynomial increase in time) of the semi–major axis of the planets up to third order in the planetary masses.

On the contrary the researches of Poincaré [P] and Birkhoff [B] showed that instabilities might occur in the dynamics of the planets and that the phase space must have a quite complicated structure.

In the course of the year 1954, Kolmogorov [K] stated his famous theorem of persistence of quasiperiodic motions in near to integrable Hamiltonian systems, and first suggested that the picture may be twofold: stability in the sense of measure theory conjugated with topological instability. Arnold moreover proved [Ar1] that bounded orbits have positive measure in the planar three–body problem and claimed that the same result must be true for the n–body problem (provided that the masses of the planets are sufficiently – unrealistically – small). He also first proved in the general context of Hamiltonian systems with many degrees of freedom the existence of orbits which drift (or diffuse) along resonances so as to change by a finite amount their action [Ar2][1], even if this process is very slow [Ne]. This reinforced the belief that "the time after which chaos manifests itself under a sufficiently small perturbation of the initial state is large in comparison with the time of existence of the Solar System" [Ar3, p.82].

Following Herman [He2] one can legitimately ask the following question: *If one of the masses $m_0 = 1$ and all the other masses $m_j \ll 1$ are sufficiently small, are there wandering domains*[2] *in any neighborhood of fixed distinct circular orbits around the mass m_0 and moving in the same direction in a plane?*

Quite recently some progress has been made in the heuristic understanding of the dynamics of the planets of the Solar System, due largely to the help provided by computers but also due to a better understanding of the underlying dynamics, resulting from the great progress in the overall field of Dynamical Systems. Modern computers allow extensive analytic calculations and numerical integrations of realistic models over very long times, even if the shortness of the step–size needed for the computation has for many years limited the investigation to the outer planets of the Solar System (Jupiter, Saturn, Uranus, Neptune and Pluto) [CMN], [SW]. Indeed, the faster the orbital movement of the planet is, the shorter is the step–size required (from approximately 40 days for Jupiter to 12 hours for Mercury). As a result, until 1991 the only available numerical integration of a realistic model of the full Solar System was spanning only 44 centuries. For this reason the analytical approach, which makes use of perturbation theory, is needed.

J. Laskar replaced the full Newtonian equations of the motion by the so–called *secular system* introduced by Lagrange where the fast angular variables are eliminated. This system, instead of giving the fast motion of the planets in space, describes the slow deformation of the planets' orbits. In this way Laskar reduced the number of degrees of freedom of the system and achieved an impressing reduction of the step–size required (most of the computational time in traditional numerical integrations is actually spent in the numerical solution of Kepler's problem). In fact he was able to

[1] "Thus, even if the motion of a planet or an asteroid is regular, an arbitrarily small perturbation of the initial state is sufficient to make it chaotic" [Ar3, p.82].

[2] i.e. an open set V and $t_0 > 0$ such that $f^t(V) \cap V = \emptyset$ for all $t > t_0$, where f^t denotes the Hamiltonian flow.

use a step–size of 500 years. The numerical integration of this system shows that *the inner Solar System* (Mercury, Venus, Earth and Mars) *is chaotic* with a Lyapounov time of 5 Myrs. This measures the rate of the exponential growth of the distance in phase space between the orbits of two points initially close [Yo]. As a consequence it is not possible to compute ephemeris for the position of the Earth over 100 Myrs: an error of 15 meters in the initial position of the Earth may grow to an error of 150 million kilometers (i.e. its present distance to the Sun) after 100 Myrs. This kind of strong instability could even result in the escape of Mercury in 3.5 billion years (Gyrs). The deformations of the planets' orbits is responsible for an external forcing on the Hamiltonian describing the evolution of the obliquity of each planet. The obliquity is the angle between the equator and the orbital plane. Laskar shows that it can undergo dramatic variations on a time scale very short in geological terms.

In what follows we will describe these results and the ideas underlying Laskar's approach, mainly coming from the theory of Hamiltonian systems and classical perturbation theory. We will also briefly discuss the technique of the numerical analysis of the fundamental frequency developed by Laskar to study the mixed phase space structure of quasi–integrable Hamiltonian systems.

The style of this exposition will be quite informal, partially because most of the results reported here lack a rigorous justification (and sometimes even a good mathematical formulation). In the last section we will try to formulate some open problems inspired by Laskar's work.

Acknowledgments. In the preparation of this review I have extensively used references [Ar1], [L1995a], [L1996] and some unpublished seminar notes of M. Herman [He3]. I have also benefited a lot from many discussions with A. Chenciner, J. Laskar, D. Sauzin and J.–C. Yoccoz.

0.1. Hamiltonian systems, integrable systems, quasiperiodic motions [Ar4]

Usually in mechanics the equations of the motion of a conservative system with phase space $M = T^*N$ (here the configuration space N is an f–dimensional riemannian manifold) are given in Hamiltonian form: $\dot{p}_i = -\frac{\partial H}{\partial q_i}$, $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $1 \leq i \leq f$. Here the "generalized coordinates" q_i and their "conjugate momenta" p_i are a system of local canonical coordinates in M and $H : M \to \mathbb{R}$ is smooth (the Hamiltonian of the system). Note that in many problems arising from celestial mechanics the flow is not complete due to the unavoidable occurence of collisions (see [Ch2] for a recent review). The symplectic form on M is $\omega = \sum_{i=1}^{f} dp_i \wedge dq_i$ and maintains this expression in all canonical systems of coordinates (they form an atlas by Darboux' theorem). Two functions $F, G \in \mathcal{C}^\infty(M, \mathbb{R})$ are *in involution* if their Poisson bracket $\{F, G\} = 0$, i. e. when their Hamiltonian flows commute.

An important extension of the Hamiltonian formalism is obtained considering time–dependent Hamiltonian functions $H : M \times \mathbb{R} \to \mathbb{R}$. These are especially useful, as we will see, for modeling non–isolated systems, i. e. mechanical systems under the action of some external forcing.

An especially interesting case is provided by the manifold $\mathbb{R}^f \times \mathbb{T}^f$ which can be identified with the cotangent bundle of the f–dimensional torus $\mathbb{T}^f = \mathbb{R}^f/(2\pi\mathbb{Z})^f$. This manifold has a natural symplectic structure defined by the closed 2–form $\omega = \sum_{i=1}^f dJ_i \wedge d\vartheta_i$ where $(J_1, \ldots J_f, \vartheta_1, \ldots \vartheta_f)$ is a point on $\mathbb{R}^f \times \mathbb{T}^f$. Let U denote an open connected subset of $\mathbb{R}^f$. Whenever an Hamiltonian system can be reduced by a symplectic change of coordinates to a function $H : U \times \mathbb{T}^f \to \mathbb{R}$ which *does not depend on the angular variables* ϑ one says that the system is *completely canonically integrable* and the variables J are called *action variables.* Note that in this case Hamilton's equations take the particularly simple form $\dot{J}_i = -\frac{\partial H}{\partial \vartheta_i} = 0$, $\dot{\vartheta}_i = \frac{\partial H}{\partial J_i}$, $i = 1, \ldots, f$. Let $\nu_i(J) = \frac{\partial H}{\partial J_i}$, $i = 1, \ldots, f$. The associated flow $t \mapsto (J(t) \equiv J(0)$, $\vartheta(t) = \vartheta(0) + t\nu(J(0)))$, $t \in \mathbb{R}$, is *linear* and leaves invariant the f–dimensional torus $J = J(0)$. The motion is therefore bounded and *quasiperiodic* if the $\mathbb{Z}$–module $\{k \in \mathbb{Z}^f, k \cdot \nu(J(0)) = 0\}$ has dimension at most $f - 2$. Otherwise the motion is periodic. The Hamiltonian is *non degenerate* (i.e. satisfies the "twist condition") if $\det\left(\frac{\partial^2 H}{\partial J_i \partial J_k}(J)\right) \neq 0$ for all $J \in U$, thus the "frequency map"

$$J \mapsto \nu(J) = \frac{\partial H}{\partial J}(J) \in \mathbb{R}^f \tag{0.1}$$

is a local diffeomorphism. This condition is generic, but in many applications and especially in the two–body problem (see below) the Hamiltonian H is *properly degenerate*: $\det\left(\frac{\partial^2 H}{\partial J_i \partial J_k}(J)\right) = 0$ for all $J \in U$. In this case the linear flow on the invariant tori can be described by only $f_0 < f$ frequencies, where $f_0 = \operatorname{rank}\left(\frac{\partial^2 H}{\partial J_i \partial J_k}\right)$, by suitably choosing new action–angle coordinates $J' = (A^T)^{-1}J$, $\vartheta' = A\vartheta$, $A \in \mathrm{GL}\,(f, \mathbb{Z})$.

0.2. The two–body problem and action–angle variables. If the mutual attraction of the planets is neglected, each planet is attracted only by the Sun. This leads to the two–body problem whose Hamiltonian $\mathcal{H}_0 : T^*(\mathbb{R}^3 \setminus \{0\}) \mapsto \mathbb{R}$ in the center of mass frame is[3]

$$\mathcal{H}_0(\mathbf{p},\mathbf{q}) = \frac{1}{2\mu}\|\mathbf{p}\|^2 - \frac{m_0 m}{\|\mathbf{q}\|} \tag{0.2}$$

where m_0 is the mass of the Sun, m is the mass of the planet and $\mu = m_0 m/(m_0 + m)$ is the reduced mass of the system. $\mathcal{H}_0$ is called the *Kepler Hamiltonian.* For negative energy, the solutions are ellipses with one focus at the origin (i.e. the center of mass). These are called *Keplerian orbits.* The shape and the position of the ellipse in space are determined from the knowledge of the semi–major axis a, the eccentricity e, the angle of inclination i of its plane w.r.t. the horizontal plane $q_3 = 0$, the argument of the perihelion ω and the longitude of the ascending node Ω (fig. 1). The position of the planet along the ellipse is determined by the mean anomaly l which is proportional to the area swept by the position vector $\mathbf{q}$ of the planet starting from the perihelion.

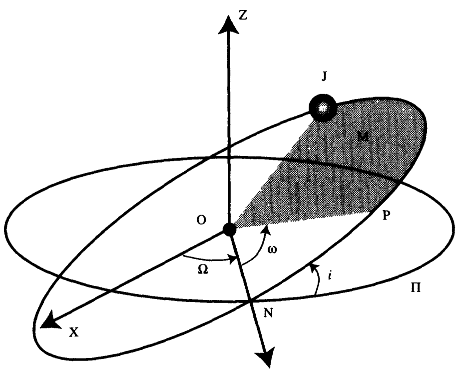

Fig. 1. Elliptical elements of a Keplerian orbit (from [L1995a])

The system admits 5 *independent* first integrals: the total energy H, the three components of the angular momentum $\mathbf{q}\wedge\mathbf{p}$ and one of the components of the Laplace vector $\mathbf{A} = \frac{1}{m}\mathbf{p}\wedge(\mathbf{q}\wedge\mathbf{p}) - \frac{m_0 m}{\|\mathbf{q}\|}\mathbf{q}$. Among these integrals one can choose three integrals in involution and construct the canonical transformation to action–angle variables (see [P2], Ch. III and [Ch1]). The other two integrals are responsible for the proper complete degeneration of the Kepler problem: for all initial conditions the orbit is periodic and the period depends only on the energy. A set of action–angle

[3] We have set the universal gravitational constant $= 1$.

coordinates $(L, G, \Theta, l, g, \theta) \in \mathbb{R}^2_+ \times \mathbb{R} \times \mathbb{T}^3$ such that two of the three frequencies of the linear flow on the invariant tori vanish is given by *Delaunay variables.* They are related to the orbital elements as follows:

$$L = \mu\sqrt{(m_0+m)a}\ , \quad G = L\sqrt{1-e^2}\ , \quad \Theta = G\cos i\ , \quad l\ , \quad g = \omega\ , \quad \theta = \Omega\ .$$

Note that G is the modulus of angular momentum $\mathbf{q} \wedge \mathbf{p}$, thus Θ is its projection along the q_3-axis. One has the obvious limitation $|\Theta| \leq G$. The new Hamiltonian reads $\mathcal{H}_0 = -\frac{\mu^3(m_0+m)^2}{2L^2}$ thus $\nu_g = \frac{\partial \mathcal{H}_0}{\partial G} = \nu_\theta = \frac{\partial \mathcal{H}_0}{\partial \Theta} = 0$. The Delaunay variables are not suitable for the description of the orbits of the planets of the Solar System since they are singular at circular orbits ($e = 0$, thus $L = G$ and the argument of the perihelion g is not defined) and at horizontal orbits ($i = 0$ or $i = \pi$, thus $G = \Theta$ and the longitude of the ascending node θ is not defined). But all the planets of the Solar System have almost circular orbits (with the exception of Mercury and Mars) and small inclinations (see Table 1).

Planet	ϖ	Ω	i	e	λ	a
Mercury	77.4561	48.3309	7.0050	0.205632	252.2509	0.387104
Venus	131.5637	76.6799	3.3947	0.006772	181.9798	0.723307
Earth	102.9373			0.016709	1000.4664	1.000012
Mars	336.0602	49.5581	1.8497	0.093401	355.4333	1.523711
Jupiter	374.3313	100.4644	1.3033	0.048495	34.3515	5.210216
Saturn	93.0568	113.6655	2.4889	0.055509	50.0775	9.538070
Uranus	173.0052	74.0060	0.7732	0.046296	314.0550	19.183302
Neptune	48.1237	131.7841	1.7700	0.008989	304.3487	30.055144

TABLE 1: Orbital elements of the planets of the Solar System (without Pluto): ϖ, Ω, i and λ are given in degrees, a is given in Astronomical Units (1 A.U. $= 1.5 \cdot 10^8$ km).

This problem is solved by introducing a new set of action–angle variables $(\Lambda, H, Z, \lambda, h, \zeta) \in \mathbb{R}^3 \times \mathbb{T}^3$: $\Lambda = L$, $H = L - G$, $Z = G - \Theta$, $\lambda = l + g + \theta$, $h = -g - \theta$, $\zeta = -\theta$ (the variable λ is called the mean longitude, $-h$ is the longitude of the perihelion) then considering the couples (H, h) and (Z, ζ) as polar symplectic coordinates:

$$\text{(0.3)} \quad \xi_1 = \sqrt{2H}\cos h\ , \quad \eta_1 = \sqrt{2H}\sin h\ , \qquad \xi_2 = \sqrt{2Z}\cos\zeta\ , \quad \eta_2 = \sqrt{2Z}\sin\zeta\ .$$

The variables $(\Lambda, \xi, \lambda, \eta) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{T}^1 \times \mathbb{R}^2$ are called *Poincaré variables.* They are well defined also in the case of circular ($H = 0$) or horizontal ($Z = 0$) orbits. The new Hamiltonian reads $\mathcal{H}_0 = -\frac{\mu^3(m_0+m)^2}{2\Lambda^2}$ thus $\dot{\lambda} = \frac{\mu^3(m_0+m)^2}{\Lambda^3}$ whereas Λ, ξ and η are constant. The relation between Poincaré variables and the original momentum–position $(\mathbf{p}, \mathbf{q})$ variables is much more subtle and will not be discussed here (see [P2], Ch. III, [L1989a]). Note however that Λ is proportional to $\sqrt{a}$,

$$\sqrt{\xi_1^2 + \eta_1^2} \simeq \sqrt{\Lambda} e(1 + \mathcal{O}(e^2)) \ , \quad \sqrt{\xi_2^2 + \eta_2^2} \simeq \sqrt{\Lambda} i(1 + \mathcal{O}(i^2) + \mathcal{O}(e^2)) \ .$$

Astronomers actually use the non–canonical coordinates given by the orbital elements themselves.

0.3. KAM theory, Nekhoroshev theorem, Arnol'd diffusion. Following Poincaré, the *fundamental problem of dynamics* is the study of quasi–integrable Hamiltonian systems, i.e. Hamiltonians $\mathcal{H} : U \times \mathbb{T}^f \mapsto \mathbb{R}$ (smooth or analytic) of the form

$$\mathcal{H}(J, \chi) = h_0(J) + \varepsilon h_1(J, \chi) \ , \tag{0.4}$$

where ε is a small real parameter. As we will see in Sect. 1.1, the Solar System Hamiltonian has this form with $f = 3n$, where n is the number of the planets. h_0 is equal to the sum of n independent Kepler Hamiltonians and ε is of the order of the planetary masses. The Keplerian orbits give an n–dimensional torus invariant for the flow associated to h_0.

Most results have been obtained under the assumption that the unperturbed Hamiltonian h_0 is non degenerate. The general picture is provided by KAM [Bo, Y1] and Nekhoroshev [Ne, Lo] theorems: if ε is sufficiently small, most initial conditions (w.r.t. Lebesgue measure) lie on invariant f–dimensional Lagrangian tori carrying quasiperiodic motions with diophantine frequencies. The action variables corresponding to these KAM orbits will remain ε–close to their initial values for all times. The complement of this set is open and dense and it is connected if $f \geq 3$. It contains a connected ($f \geq 3$) web $\mathcal{R}$ of resonant zones corresponding to $\mathbb{Z}^f$–linearly dependent frequencies: $\cup_{k \in \mathbb{Z}^f} \{J \in U \, , \nu_0(J) \cdot k = 0\} \times \mathbb{T}^f$. Motion along these resonances cannot be excluded, resulting in a variation of $\mathcal{O}(1)$ of the actions in a finite time[4]. But if the Hamiltonian is *analytic* and h_0 is *steep* in the sense of [Ne] (for example convex) then this variation is very slow: it takes a time at least $\mathcal{O}\left(\exp\left(\frac{1}{\varepsilon^a}\right)\right)$ to change the

[4] It is conjectured [Ar4, p. 189] that generically quasi–integrable Hamiltonians with more than two degrees of freedom are topologically unstable.

actions by $\mathcal{O}(\varepsilon^b)$, where a and b are two positive constants. Moreover each invariant torus $\mathcal{T}$ has a neighborhood filled in with trajectories which remain close to it for an even longer time [MG]. Indeed assume that h_0 is convex and the frequencies ν of the linear flow on $\mathcal{T}$ satisfy a diophantine condition (see [Y1]) of exponent $\tau \geq n-1$, i.e. there exists $\gamma > 0$ such that

$$|k \cdot \nu| \geq \gamma |k|^{-\tau} , \quad \forall k \in \mathbb{Z}^f \setminus \{0\} , \tag{0.5}$$

where $|k| = |k_1| + \ldots + |k_f|$. Then all trajectories starting at a distance of order $\rho < \rho^*$ from $\mathcal{T}$ will remain close to it for a time $\mathcal{O}\left(\exp\left(\exp\left(\frac{\rho^*}{\rho}\right)^{1/(\tau+1)}\right)\right)$.

Unfortunately the proper degeneration of the Kepler Hamiltonian makes the application of these results to the Solar System problematic. One can introduce the secular system of Lagrange and use the perturbation to remove the degeneracy. However, this leads to a small twist which is of the order of ε and the application of KAM theory to the Solar System Hamiltonian is a very delicate task [He3].

A much simpler but still important case is given by the problem of stability of Lagrange's equilateral equilibrium solutions in the restricted three–body problem [Li1,Li2,Pol,SM,Wi]. In the Sun–Jupiter situation computer–assisted proofs can be made so accurate to prove the practical stability of some Trojan asteroids for 10 Gyrs [GDFGS,GS].

1. FREQUENCY MAP ANALYSIS

1.1. The frequency map analysis. Let $N_0 \subset \mathbb{R}^f$ be the image of U under the frequency map $\nu = \frac{\partial h_0}{\partial J} : U \to N_0$ and assume that this map is a diffeomorphism. One of the consequences of KAM theorem is the existence, for sufficiently small values of ε, of a Cantor set $N_\varepsilon \subset N_0$ of values of the frequencies ν for which the Hamiltonian system (0.4) has smooth invariant tori with linear flow with frequencies ν. This Cantor set corresponds to diophantine frequencies (0.5) with $\tau > n-1$ and $\gamma = \gamma_\varepsilon$. Moreover there exists a diffeomorphism

$$F_\varepsilon : N_0 \times \mathbb{T}^f \to U \times \mathbb{T}^f , \quad (\nu, \varphi) \mapsto (J, \vartheta) \tag{1.1}$$

which on $N_\varepsilon \times \mathbb{T}^f$ "straightens out all these invariant tori at the same time" ([Pö], p. 655), i.e. transforms Hamilton's equations of motion of (0.4) into $\dot{\varphi} = \nu$, $\dot{\nu} = 0$. This diffeomorphism is ε–close to $(\frac{\partial h_0}{\partial J}, \mathrm{id}_{\mathbb{T}^n})$ and restricted to $N_\varepsilon \times \mathbb{T}^f$ is smooth in the sense of Whitney w.r.t. the first factor and analytic w.r.t. the second (if the

Hamiltonian (0.4) is analytic). Let us fix $\vartheta = 0$ and consider the diffeomorphism $\mathcal{F}_\varepsilon : N_0 \to U$ induced by (1.1). If the initial conditions $(J(0), 0)$ lie on a KAM torus with frequency $\nu \in N_\varepsilon$ each component z_r of $(J_1 e^{i\vartheta_1}, \ldots, J_f e^{i\vartheta_f})$ will be an analytic quasiperiodic function of time ε–close to $J_r(0)e^{i\nu_r t}$. Given $t_0, T \in \mathbb{R}$, $T > 0$, the *frequency map analysis* [L1993, L1995b] is the numerical construction of an approximate inverse $\mathcal{F}^{-1}_{\varepsilon,T,t_0}$ of $\mathcal{F}_\varepsilon$ *on the whole actions space* U from the datum of $\{z_r(t)\,, r = 1, \ldots, f\,,\ t \in [t_0, t_0 + 2T]\}$. As $T \to +\infty$, this approximate inverse converges to $\mathcal{F}_\varepsilon^{-1}$ on the set of KAM tori.

1.2. Construction of the approximate frequency map on the set of KAM tori. Assume for simplicity that $t_0 = -T$ and that z_r has a Fourier series of the form $z_r(t) = e^{i\nu_r t} + \sum_{k\in\mathbb{Z}^f, k\neq e_r} \hat{z}_k e^{ik\cdot\nu t}$ with $|\hat{z}_k| < 1$ (here e_r is the r–th canonical basis vector of $\mathbb{Z}^f$, thus $e_r \cdot \nu = \nu_r$). Laskar looks for a quasiperiodic approximation to z_r of the form of a finite sum $\zeta_r^{(N)}(t) = \sum_{s=1}^N \hat{\zeta}_r^{(s)} e^{i\mu_r^{(s)} t}$, where the coefficients $\hat{\zeta}_r^{(s)}$ have decreasing amplitude with s, $\mu_r^{(1)} \simeq \nu_r$ and $\mu_r^{(s)} \simeq k^{(r,s)} \cdot \nu$ for some suitably chosen $k^{(r,s)} \in \mathbb{Z}^f$, $s = 2, \ldots N$.

This quasiperiodic approximation is chosen in the following way.

1. Consider the weighted L^2 scalar product

$$\langle z, w\rangle_{T,p} = \frac{1}{2T}\int_{-T}^{T} z(t)\overline{w(t)}\chi_p\left(\frac{t}{T}\right) dt\,,$$

   where z and w are smooth quasiperiodic functions. Here χ_p is the so–called Hanning window filter: $\chi_p(t) = \frac{2^p (p!)^2}{(2p)!}(1 + \cos \pi t)^p$ but one may consider more general situations. Let $\hat{\chi}_p(x) = \langle e^{ixt}, 1\rangle_{1,p} = \frac{(-1)^p \pi^{2p} (p!)^2 \sin x}{x(x^2-\pi^2)\ldots(x^2-p^2\pi^2)}$. Note that $\hat{\chi}_p(x) = 1 - c_p x^2 + \mathcal{O}(x^3)$ as $x \to 0$, for some $c_p > 0$, $\hat{\chi}_p(x) = \mathcal{O}(x^{-2p-1})$ as $x \to \infty$ as well as all its derivatives.
2. Let $y = (\nu_r - x)T$. Then

$$f(y,T) = \langle z_r, e^{ixt}\rangle_{T,p} = \hat{\chi}_p(y) + \sum_{k\in\mathbb{Z}^f, k\neq e_r} \hat{z}_k \hat{\chi}_p(y + (k\cdot\nu - \nu_r)T)$$

   verifies $\lim_{T\to+\infty}[f(y,T) - \hat{\chi}_p(y)] = 0$ uniformly w.r.t. $y \in K$, where K is any compact subset of $\mathbb{R}$. Applying the implicit function theorem to the function $(y,T) \mapsto \frac{\partial}{\partial y}|f(y,T)|^2$ in a neighborhood of the point $(0, +\infty)$ one obtains the existence of a unique value x_r of x which maximizes $x \mapsto |\langle z_r, e^{ixt}\rangle_{T,p}|$ for sufficiently large T.

3. Let $\mu_r^{(1)} = x_r$, $r = 1, \dots, f$, and define $\mathcal{F}_{\varepsilon,T,t_0}^{-1}(J(0)) = \mu = (\mu_1^{(1)}, \dots, \mu_r^{(1)})$.

The pointwise convergence of $\mathcal{F}_{\varepsilon,T,t_0}^{-1}$ restricted to the set of KAM tori to $\mathcal{F}_\varepsilon^{-1}$ follows from the following asymptotic estimate proved in [L1995b]: there exists $c_{p,\varepsilon} > 0$ such that $|\nu_r - \mu_r^{(1)}| \leq c_{p,\varepsilon}(p!)^2 T^{-2p-2}$. This error term actually is the first term of an asymptotic series in powers of $\frac{1}{T}$ which is Gevrey. Applying the classical method of summation of the series to the smallest term one may choose $p = p(T)$ so as to obtain an exponentially small error term but for the time being this seems to be more mathematically entertaining than numerically useful.

To obtain $\zeta_r^{(N)}$ one iterates the above procedure: once the first periodic approximation $e^{i\mu_r^{(1)}t}$ is obtained, $\hat{\zeta}_r^{(1)}$ is computed by orthogonal projection and the process is started again on the remaining part $z_{r,1} = z_r - \hat{\zeta}_r^{(1)} e^{i\mu_r^{(1)}t}$ of the function.[5]

1.3. Applications of the frequency map analysis. Outside the set of KAM tori one can still apply the method described above but the result lacks a rigorous justification. This is mainly due to the fact that the dynamics "between" KAM tori is far from being understood. Keeping T and the initial condition $(J(0), \vartheta(0))$ fixed one computes $\mu(t) = \mathcal{F}_{\varepsilon,T,t}^{-1}(J(0))$ for different values of t. The time–dependence of μ is then used by Laskar to measure the "diffusion" of the orbit since, as we have seen, for KAM orbits μ is constant.

This method was first introduced by Laskar [L1990] in his study of the Solar System in order to estimate the size of the chaotic zones. Outside the domain of celestial mechanics it has been applied to many different dynamical situations. For exact area preserving twist maps, applying the classical theory of Birkhoff for KAM curves [He1] (these must be graphs of Lipschitz functions) one can derive a very practical criterion for the non existence of KAM curves [LFC]. The frequency map analysis has also been succesfully applied to the study of realistic models of particle accelerators [LRo].

[5] Note the analogy between this procedure and the epicycloid theory of Ptolemy where one adds up uniform circular motions to represent the motion of the planets: see [St] for a very beautiful exposition.

2. CHAOTIC BEHAVIOUR IN THE SOLAR SYSTEM

2.1. Planetary theory in Poincaré's canonical heliocentric coordinates.[6] Let us consider $n+1$ bodies with masses $m_0, \dots m_n$. Let O denote the center of mass of the system and $u_i \in \mathbb{R}^3$ the coordinate of the i–th body in a fixed barycentric reference frame. The conjugate momenta are $v_i = m_i \dot{u}_i$ and the Hamiltonian of the system is simply given by $\mathcal{H}(v,u) = \sum_{i=0}^{n} \frac{v_i^2}{2m_i} - \sum_{0 \le i < j \le n} \frac{m_i m_j}{\|u_i - u_j\|}$. Clearly $\mathcal{H}$ is defined on $T^*(\mathbb{R}^{3(n+1)} \setminus \Delta)$ where $\Delta = \{(u_0, \dots, u_n) \in \mathbb{R}^{3(n+1)} \;,\; \exists i \neq j \;,\; u_i = u_j\}$. Heliocentric canonical coordinates are induced by the linear transformation in configuration space $q_0 = u_0 \;, \quad q_i = u_i - u_0 \;, \quad 1 \le i \le n$ thus the conjugate momenta become $p_0 = \sum_{i=0}^{n} v_i \;, \quad p_i = v_i \;, \quad 1 \le i \le n$. By Noether's Theorem the classical first integrals of the system can be deduced from the symmetries of the Hamiltonian: the invariance with respect to uniform translations in configuration space implies the conservation of the total momentum p_0, thus it is not restrictive to fix $p_0 = 0$ (reduction of the center of mass). Conservation of energy is a consequence of the independence of the Hamiltonian from time and its invariance with respect to rotations of $\mathbb{R}^3$ implies the conservation of total angular momentum $C = \sum_{i=1}^{n} q_i \wedge p_i$. After reduction of the center of mass the number of degrees of freedom becomes $3n$ and in canonical heliocentric coordinates the Hamiltonian is given by $\mathcal{H}(p,q) = \mathcal{H}_0(p,q) + \mathcal{H}_1(p,q)$ with

$$\begin{aligned} \mathcal{H}_0(p,q) &= T_0(p) + V_0(q) = \sum_{i=1}^{n} \frac{\|p_i\|^2}{2\mu_i} - \sum_{i=1}^{n} \frac{m_0 m_i}{\|q_i\|} \\ \mathcal{H}_1(p,q) &= T_1(p) + V_1(q) = \sum_{0<i<j\le n} \frac{p_i \cdot p_j}{m_0} - \sum_{0<i<j\le n} \frac{m_i m_j}{\|q_i - q_j\|} \end{aligned} \tag{2.1}$$

where $\frac{1}{\mu_i} = \frac{1}{m_i} + \frac{1}{m_0}$. $\mathcal{H}_0$ is the Hamiltonian of a collection of n independent Kepler problems each relative to a planet of mass μ_i attracted by a fixed point at the origin of mass $m_0 + m_i$. In the case considered by Laskar $n = 8$ (he did not consider Pluto whose mass is negligible), the phase space has 48 dimensions and is foliated with 8–dimensional tori invariant for the flow associated to the Kepler Hamiltonian $\mathcal{H}_0$.

[6] Actually Laskar in his numerical computations did use the same set of non canonical heliocentric coordinates traditionally used by astronomers since Laplace which allowed him comparison with the semianalytical theory of the Solar System of Bretagnon [Br]. This is irrelevant for our purposes. In the words of Poincaré [P, tome II, p. 37] "Les équations où s'introduisent les crochets de Lagrange prennent ainsi une forme en apparence plus compliquée. Mais cette différence n'a rien d'essentiel".

Choosing units $m_0 = 1$, $m_i = \varepsilon M_i$, $i = 1, \dots n$, where $\varepsilon \le 10^{-3}$ for the Solar System, the conformally symplectic coordinate change $p = \varepsilon P$, $q = Q$, $\mathcal{H} = \varepsilon \hat{\mathcal{H}}$ shows that $\hat{\mathcal{H}}$ has the form (0.4) of a quasi integrable properly degenerate Hamiltonian system and that $\mathcal{H}_1$ is formally of the first order (w.r.t. $\mathcal{H}_0$) in the planetary masses.

In order to study the long term behaviour of the planetary orbits it is convenient to introduce for each planet Poincaré's variables $(\Lambda_i, \xi_i, \lambda_i, \eta_i)$, $i = 1, \dots n$, $\xi_i = (\xi_{1i}, \xi_{2i}) \in \mathbb{R}^2$, $\eta_i = (\eta_{1i}, \eta_{2i}) \in \mathbb{R}^2$. The Hamiltonian becomes

$$\mathcal{H}(\Lambda, \lambda, \xi, \eta) = \mathcal{H}_0(\Lambda) + \mathcal{H}_1(\Lambda, \lambda, \xi, \eta) \tag{2.2}$$

where $(\Lambda, \xi, \lambda, \eta) \in \mathbb{R}_+^n \times \mathbb{R}^{2n} \times \mathbb{T}^n \times \mathbb{R}^{2n}$ and $\mathcal{H}_0(\Lambda) = -\sum_{i=1}^n \frac{\mu_i^3 (m_0 + m_i)^2}{2\Lambda_i^2}$. The three components of the total angular momentum are

$$C_1 - iC_2 = \sum_{k=1}^n \sqrt{G_k^2 - \Theta_k^2} e^{i\zeta_k} \ , \quad C_3 = \sum_{k=1}^n \Lambda_k - \frac{1}{2}(\xi_{1k}^2 + \xi_{2k}^2 + \eta_{1k}^2 + \eta_{2k}^2) \tag{2.3}$$

where $G_k^2 - \Theta_k^2 = \frac{1}{2}(\xi_{2k}^2 + \eta_{2k}^2)\left(\Lambda_k - \xi_{1k}^2 - \eta_{1k}^2 - \frac{1}{2}(\xi_{2k}^2 + \eta_{2k}^2)\right)$.

2.2. Averaging and the secular system. In order to study the behaviour of the orbits in a neighborhood of a fixed Keplerian orbit we translate $\Lambda = \Lambda_0 + \tilde{\Lambda}$. Neglecting constant terms and the terms of order at least three in $\tilde{\Lambda}$ we obtain a new Hamiltonian $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0 + \tilde{\mathcal{H}}_1$, where $\tilde{\mathcal{H}}_0 = N \cdot \tilde{\Lambda}$ and

$$\tilde{\mathcal{H}}_1(\tilde{\Lambda}, \lambda, \xi, \eta) = \mathcal{H}_1(\Lambda_0, \lambda, \xi, \eta) + f_1(\Lambda_0, \lambda, \xi, \eta) \cdot \tilde{\Lambda} + f_2(\Lambda_0, \lambda, \xi, \eta)\tilde{\Lambda} \cdot \tilde{\Lambda} \ .$$

In these formulas $N_j = \frac{\partial \mathcal{H}_0}{\partial \Lambda_j}(\Lambda_0) = \frac{\mu^3 (m_0 + m_j)^2}{\Lambda_{0j}^3}$ is the mean motion of the j-th reference Keplerian orbit, $(f_1)_i = \frac{\partial \mathcal{H}_1}{\partial \Lambda_i}$, $(f_2)_{ij} = \frac{1}{2}\left(\frac{\partial^2 \mathcal{H}_0}{\partial \Lambda_i \Lambda_j} + \frac{\partial^2 \mathcal{H}_1}{\partial \Lambda_i \Lambda_j}\right)$, $i, j = 1, \dots, n$. Note that $\tilde{\mathcal{H}}$ is already an approximation of (2.2) since we have neglected all the terms $\mathcal{O}(\tilde{\Lambda}^3)$. Following Poincaré [P] we now look for a canonical transformation to new variables $(\Lambda', \lambda', \xi', \eta')$ *formally* close to the identity such that the new Hamiltonian $\mathcal{H}'$ does not depend on λ' when terms of the third order in the planetary masses are neglected

$$\mathcal{H}'(\Lambda', \xi, \eta) = \mathcal{H}_0'(\Lambda') + \mathcal{H}_1'(\Lambda', \xi', \eta') + \mathcal{H}_2'(\Lambda', \xi', \eta') \tag{2.4}$$

where $\mathcal{H}_1'$ and $\mathcal{H}_2'$ are of the first and of the second order respectively in the planetary masses. This "partial averaging procedure" (or resonant normal form) is a standard technique in the study of quasi integrable Hamiltonian systems (see, for example, [Ar4], Ch. 5) and leads to the *secular system* studied by Laskar [L1992]. A first order calculation easily shows that $\mathcal{H}_0'(\Lambda') = N \cdot \Lambda'$ and $\mathcal{H}_1'(\Lambda', \xi', \eta') = \langle \tilde{\mathcal{H}}_1 \rangle = \int_{\mathbb{T}^n} \tilde{\mathcal{H}}_1(\Lambda', \lambda, \xi', \eta') d\lambda$. The frequencies vector $N \in \mathbb{R}^n$ must be non resonant (i.e. linearly independent on $\mathbb{Z}^n$) and this imposes a condition on Λ_0. For the same reason the coordinate change, as well as the Hamiltonian $\mathcal{H}'$, cannot be globally defined in the phase space.

Clearly the new variables Λ' are first integrals of the Hamiltonian flow of $\mathcal{H}'$. They are thus omitted from the numerical integration. Of course, this does not necessarily prove that the original variables Λ (i.e. the actual major semiaxes) are constant in time. Indeed this is not true. Since $\dot{\Lambda} = \{\Lambda, \mathcal{H}_1\}$, at the first order in the planetary masses one has $\{\Lambda, \mathcal{H}_1\} = \{\Lambda', \mathcal{H}_1\}$, whose average w.r.t. λ is zero. This is the classical theorem of Laplace [Lp1, Lp2] and Lagrange [Lg] asserting the absence of secular terms in the time evolution of the major semiaxes at first order in the masses. Poisson [Po] generalized this result to the second order whereas it is false at order three [Ha].

frequency ($''$/year)	period (years)
$g_1 = 5.596$	231000
$g_2 = 7.456$	174000
$g_3 = 17.365$	74600
$g_4 = 17.916$	72300
$g_5 = 4.249$	305000
$g_6 = 28.221$	45900
$g_7 = 3.089$	419000
$g_8 = 0.667$	1940000
$s_1 = -5.618$	230000
$s_2 = -7.080$	183000
$s_3 = -18.851$	68700
$s_4 = -17.748$	73000
$s_5 = 0.000$	
$s_6 = -26.330$	49200
$s_7 = -3.005$	431000
$s_8 = -0.692$	1870000

TABLE 2: Secular frequencies of the solar system (J. Laskar [L1996, p. 167])

$\mathcal{H}_1'$ is an even function of (ξ', η') and has an equilibrium position at $\xi' = \eta' = 0$. By a linear symplectic coordinate change $(\xi'', \eta'') = S(\xi', \eta')$ its quadratic part $\mathcal{H}_{12}'(\xi', \eta')$ can be written

$$\mathcal{H}_{12}'(\xi'', \eta'') = \frac{1}{2}\sum_{k=1}^{n}[g_k((\xi_{1k}'')^2 + (\eta_{1k}'')^2) + s_k((\xi_{2k}'')^2 + (\eta_{2k}'')^2)] \tag{2.5}$$

where the *secular frequencies* $g_k > 0$, $k = 1, \dots n$, $s_5 = 0$, $s_k < 0$, $k \neq 5$ (see Table 2).

The vanishing of s_5 is due to the existence of the total angular momentum integral (2.3). Note that the labeling of the secular frequencies does not correspond to the labeling of the planets since the matrix S is not diagonal. Thus the choice of $s_5 = 0$ is conventional. The solutions of Hamilton's equations associated to (2.5) are

$$\xi_{1k}''(t) + i\eta_{1k}''(t) = \sum_{j=1}^{n} \alpha_{kj} e^{ig_j t} \, , \quad \xi_{2k}''(t) + i\eta_{2k}''(t) = \sum_{j=1}^{n} \beta_{kj} e^{is_j t} \, , \tag{2.6}$$

thus the secular frequencies describe the (small amplitude) quasi periodic time–dependence (in this integrable approximation) of the inclinations and the eccentricity of the planets (fig. 2).

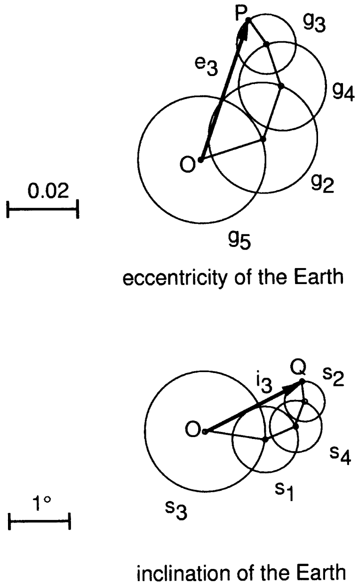

Fig. 2. Quasiperiodic variations of the Earth eccentricity and inclination according to Laplace and Lagrange (from [L1995a])

The Keplerian ellipses are no longer fixed: they are subject to a double precessionary motion (precession of the perihelion and of the line of nodes) with periods ranging from 45000 to 1940000 years. The quite long time scale of these phenomena has allowed Laskar to use a time–step of 500 years in the numerical integrations of Hamilton's equations associated to $\mathcal{H}'$.

2.3. Planetary evolution over millions and billions of years. The actual secular systems numerically integrated by Laskar [L1989, L1990] is accurate up to second order w.r.t. the masses and up to degree 5 in the eccentricities and inclinations (i.e. is obtained retaining polynomial terms of degree up to 6 w.r.t. the variables ξ, η in the Hamiltonian (2.4)). It contains about 150000 polynomial terms and the secular effects of general relativity and the Moon represent a few terms which have been added to the secular system. Its accuracy has also been tested comparing it with a direct numerical integration of the full Solar System which spans 6 Mys [LQT].

The first main result obtained by Laskar is that the whole solar system, and more specifically the inner Solar System (Mercury, Venus, Earth and Mars) is chaotic with a very short Lyapounov time of 5 Myrs. Since the typical time scale of the Laplace–Lagrange integrable secular system (2.5) is about 50000 years this Lyapounov time corresponds just to about 100 "periods". This has the conceptually striking consequence that it is practically impossible to predict the motion of the planets beyond 100 Myrs. This chaotic behaviour is essentially due to the presence of two secular resonances among the planets: $\theta = 2(g_4 - g_3) - (s_4 - s_3) \approx -10^{-3}$ $''$/yr, which is related to Mars and the Earth, and $\sigma = (g_1 - g_5) - (s_1 - s_2) \approx -10^{-1}$ $''$/yr, related to Mercury, Venus and Jupiter. They are the two most important small divisors appearing in the numerical integration as the frequency analysis of the solution shows since they appear with a large amplitude in the construction of quasiperiodic approximate interpolations of the orbits. The two corresponding arguments change several time from libration to circulation[7] over 200 Myrs, a phenomenon which is also typical of chaotic orbits (fig. 3).

[7] Near an approximate resonance one can introduce canonical coordinates such that the Hamiltonian is in its first approximation a pendulum Hamiltonian $H(p,q) = p^2 + \cos q$, $(p,q) \in \mathbb{R} \times \mathbb{T}^1$. Libration corresponds to homotopically trivial orbits ($H < 1$), circulation to homotopically non–trivial orbits ($H > 1$).

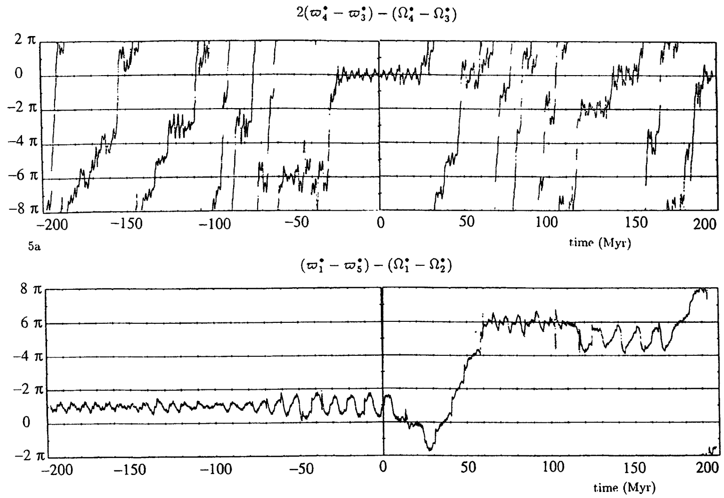

Fig. 3. Transition from libration to circulation of the argument corresponding to the secular resonances θ and σ

It should be stressed that the exponential divergence of the orbits results mostly from these repeated changes from libration to circulation of the resonant precession angles, which leads to a total indeterminacy of the orientation of the orbit in space. The eccentricities and inclinations variations are much much slower and become relevant only on a time scale of billions of years.

The numerical integrations later carried over by Laskar [L1994, L1995a, L1997] on time spans up to 25 Gyrs should be considered as an attempt to explore the chaotic zone where the Solar System evolves so as to have a qualitative description of the possible behaviour of the orbits on a time scale comparable to the age of the universe. The large planets (Jupiter, Saturn, Uranus and Neptun) have always very regular orbits whereas the eccentricities and inclinations of the inner planets show very large and irregular variations. In particular the eccentricity of Mercury reaches 0.5. Indeed exploiting the idea underlying the shadowing lemma [Y2] one can even find an orbit leading Mercury's orbit to intersect Venus' orbit in 3.5 Gyrs, thus leading to a collision or to escape.

2.4. The chaotic obliquity of the planets. Another kind of instabilities manifest themselves in the motion of the Solar System's planets. Because of their equatorial bulge the planets are subject to torques arising from the gravitational attraction of their satellites and of the Sun. This is at the origin of the precession of equinoxes (26000 years for the Earth). Moreover the obliquity of each planet is not fixed but is perturbed by the secular motion of the planet's orbit. Let $I_1 = I_2 < I_3$ be the principal moments of inertia of the planet and assume that the axis of rotation of the planet is also its axis of maximum momentum of inertia I_3. The precession motion of the planet is given by $\frac{d\mathbf{H}}{dt} = \mathbf{L}$, where $\mathbf{H}$ is its spin angular momentum and $\mathbf{L} = \mathbf{L}(\lambda)$ is the torque exerted by the Sun. By averaging over the mean anomaly one obtains the secular equations of precession corresponding to the Hamiltonian[8]

$$\mathcal{H}(X, \psi, t) = \frac{\alpha}{2}(1 - e(t)^2)^{-3/2}X^2 + \sqrt{1 - X^2}(A(t)\sin\psi + B(t)\cos\psi)$$

where $\alpha = \frac{3m_0}{2a^3\nu}\frac{I_3 - I_1}{I_1}$, ν is the rotational angular velocity of the planet, m_0 is the solar mass. $X = \cos\varepsilon$, where ε is the obliquity, ψ is the precession angle. Here $A(t) + iB(t)$ is proportional to $\frac{d}{dt}(\xi_2(t) + i\eta_2(t))$ thus it depends on the change of orientation of the planet's orbital plane (inclination and longitude of the node).

When the effect of the perturbation to the orbital elements due to the other planets is not considered, the eccentricity e is constant and $A \equiv B \equiv 0$. The resulting Hamiltonian describes a free rotator, is clearly completely integrable and the obliquity is constant. X is an action variable and ψ is its canonically conjugated angle variable. If e is kept constant and $(A(t)\sin\psi + B(t)\cos\psi)$ is replaced with a single periodic term $A_0\sin(\nu_0 t + \psi + \phi)$ then the resulting Hamiltonian is once again completely integrable. In the model studied by Laskar and Robutel [LR], $e(t)$ and $A(t) + iB(t)$ take into account the secular perturbations of the whole Solar System, modelled as a system with 15 degrees of freedom. Since the orbital solution [L1990] is not coupled with the precession variables it appears in the Hamiltonian $\mathcal{H}$ as an external time–dependent aperiodic forcing. The result is that all terrestrial planets could have experienced large, chaotic variations in obliquity at some time in the past. As frequency analysis shows, the obliquity of Mars is still in a large chaotic region, ranging from 0° to 60°, Mercury and Venus have been stabilized by tidial dissipation. In the case of Venus the crossing of a large chaotic zone extending from 0° to 90° during the slow–down process of his rotation speed due to dissipative effects may have contributed to the planet's retrograde rotation.

[8] In the case of the Earth the term $\frac{\alpha}{2}(1 - e(t)^2)^{-3/2}X^2$ must be changed because of the asymmetry of the earth ($I_1 < I_2$) and of the torque exerted by the Moon.

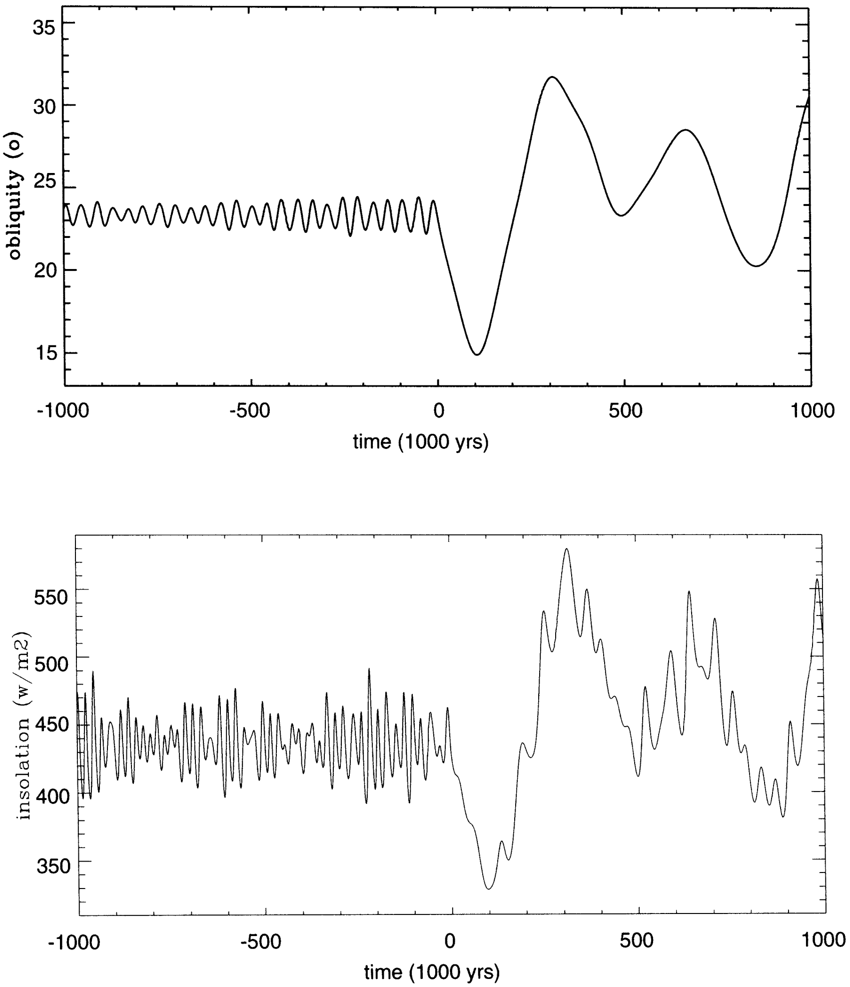

Fig. 4. Variation of the obliquity of the Earth (and of the insolation at northern latitudes) before and after "suppression" of the Moon

In its present state the Earth obliquity is essentially constant, with only small variations of $\pm 1.3°$ around the mean value of $23.3°$. Still, according to Milankovitch theory, this small variation is responsible for the ice ages. This relative stability is due to the fact that the Earth's precession frequency is presently far from being resonant with its secular frequencies, thus avoiding a large chaotic zone which extends from $60°$ to $90°$ in obliquity [LJR]. This zone would be even larger if the Moon were not present, extending from nearly $0°$ up to about $85°$ and leading to dramatic changes in climate (fig. 4).

2.5. Minor bodies in the Solar System. In addition to the 9 planets the Solar System is crowded with thousands of catalogued minor bodies (satellites, asteroids and comets). One can use simplified models, since their masses are negligible, but due to the variety of their canonical elements one needs to understand the dynamics in an even more global way.

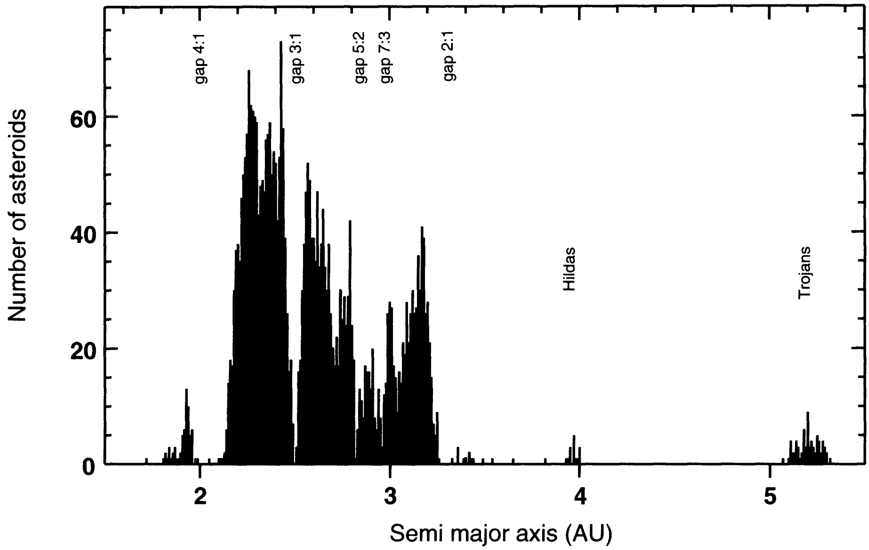

Fig. 5. Histogram of the number of asteroids against their semi–major axes

One of the first examples of chaotic behaviour was given by the chaotic tumbling of Hyperion, a small irregularly shaped satellite of Saturn whose strange rotational behaviour was detected during the encounter of the Voyager spacecraft with Saturn. In this case the dynamics is quite well understood and can be reduced to a pendulum Hamiltonian with a time–periodic perturbation [WPM]. For other satellites of the solar system one has a much more regular behaviour of their rotation since many KAM invariant tori populate the phase space [Ce].

The dynamics of the asteroids also raises a number of interesting problems. Their distribution, observed by plotting the number of asteroids against their semi–major axis length, shows gaps and accumulations first remarked by Kirkwood in 1867 (fig. 5). In this problem two and three–body mean motion resonances with Jupiter, Saturn or Mars play a very important role [Mo]. The chaotic diffusion in the inner asteroidal belt may even be at the origin of Mars and Earth–crossing asteroids$^{\heartsuit}$ [MN].

$^{\heartsuit}$ These are recently most popular among Hollywood's producers.

3. SOME OPEN PROBLEMS

The work of Laskar raises a number of challenging mathematical problems. What follows is a very rough list.

1. Can one give a rigorous justification of the frequency map analysis outside the set of KAM tori[9]? This implies to explore a largely unknown land ([He2], Section 6). At least for area–preserving maps the work of Mather [M] should provide some insight.

2. Prove a realistic stability result in the spirit of Nekhoroshev theorem for the system of the 4 outer planets and for a time comparable to the age of the Solar System.

3. Prove the existence of a KAM solution close to the Jupiter–Saturn system (see [Ro] for a discussion).

3'. Can one do the same for the 4 outer planets?

4. How long do the semi–major axes of the planets remain close to their initial values (see [Ni])?

5. Control the error term in the construction of the secular system numerically integrated by Laskar (is it possible to do it without answering to 4?)

6. Can one prove the existence of a transition chain which can possibly lead to the ejection of Mercury?

7. Can one find a mathematically satisfactory definition of stability for finite times, which reduces to orbital stability for infinite time, and which can be used to prove theorems? (An attempt in a somewhat well–defined context is given in [Gi]).

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