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THE SEIBERG–WITTEN INVARIANTS OF SYMPLECTIC FOUR–MANIFOLDS [after C. H. Taubes]

by **Dieter KOTSCHICK**

Over the last ten years, the applications of gauge theory to four–dimensional differential topology and the theory of pseudo–holomorphic curves in symplectic geometry have developed in parallel. The techniques used in the two theories were strikingly similar, but the results were unrelated. Nevertheless, there was speculation that Donaldson's invariants constructed using the moduli spaces of anti–self–dual Yang–Mills connections should be non–trivial for symplectic manifolds. Donaldson proved this for Kähler manifolds, but his arguments did not seem to have a viable extension to arbitrary symplectic manifolds.

While it has been known for decades that the class of symplectic four–manifolds is strictly larger than that of Kähler surfaces, it seemed possible until 1992 that there would only be a few symplectic non–Kähler four–manifolds, and perhaps no simply connected ones. This changed drastically when Gompf introduced his symplectic sum construction and used it to show that there are slews of symplectic four–manifolds which cannot be Kähler, including many simply connected ones, and including examples which realize all finitely presentable groups as fundamental groups. Certain properties of Gompf's examples, and the partial calculations of Donaldson invariants that were carried out for a few of them, suggested that symplectic manifolds might be the building blocks for all simply connected smooth four–manifolds.

These speculations set the stage for the work of Taubes that we report on. Immediately after the introduction of the Seiberg–Witten invariants, which are close cousins of the Donaldson invariants, Taubes [18] proved that they are non–trivial for all symplectic four–manifolds. This implies some topological constraints which one expected to prove using Donaldson invariants. For example, if a symplectic four–manifold splits as a smooth connected sum, then one of the summands must have a negative definite intersection form. But there are also geometric consequences which one did not expect from Donaldson theory, for example the conclusion that a symplectic four–manifold admitting a metric of positive scalar curvature must have an almost definite intersection form.

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Easy properties of the Seiberg–Witten invariants, together with this first result of Taubes, imply strong consequences for the classification of symplectic structures. For example, on a four–manifold with $b_1 = 0$ and $b_2^+ > 1$ there are at most finitely many homotopy classes of almost complex structures underlying symplectic structures, and in some cases this finiteness result can be sharpened to determine exactly which almost complex structures come from symplectic ones.

More recently, Taubes [21, 22, 23, 24] has shown that the Seiberg–Witten invariants of symplectic four–manifolds, suitably normalized, coincide with certain Gromov invariants defined by counting pseudo–holomorphic curves. This is an extremely surprising and far–reaching result, the consequences of which have not yet been fully worked out. Assuming the conjectured equivalence of the Seiberg–Witten invariants and the Donaldson invariants, Taubes's result ties up and explains the parallel developments of Yang–Mills theory and the theory of pseudo–holomorphic curves.

We will discuss here only one part of the equivalence of the Seiberg–Witten and the Gromov invariants, namely the fact that every non–trivial Seiberg–Witten invariant of a symplectic four–manifold leads to the existence of a pseudo–holomorphic curve in a specific homology class [21, 22]. This result has two kinds of consequences. On the one hand, the topological constraints on symplectic four–manifolds can be strengthened, and the place of symplectic manifolds in four–dimensional differential topology can be elucidated further than was possible using Taubes's first theorem only. For example, we will see that many minimal symplectic four–manifolds do not admit any non–trivial connected sum decompositions at all. On the other hand, there are consequences in symplectic geometry, without reference to differential topology. For example, Taubes's existence theorem for pseudo–holomorphic curves combined with a result of Gromov [10] shows that the symplectic structure of the complex projective plane $\mathbb{C}P^2$ is essentially unique.

1. REVIEW OF THE SEIBERG–WITTEN INVARIANTS

In this section we recall the definition of the Seiberg–Witten invariants and introduce our notation.

Consider a smooth closed oriented 4–manifold X equipped with a Riemannian metric g. The metric and orientation define a Hodge star operator

$$*\colon \Omega^k(X) \longrightarrow \Omega^{4-k}(X)$$

by setting

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle dvol_g \ ,$$

where on the right–hand–side we use the natural inner product induced by g. The $*$ operator is an involution on Ω^2, and we denote by $\Omega^2_\pm$ the ± 1 eigenspaces. Accordingly we split each 2–form into its self–dual and anti–self–dual parts: $\alpha = \alpha_+ + \alpha_-$.

Applying this splitting to the harmonic 2–forms, we obtain a direct sum decomposition of $H^2(X,\mathbb{R})$ into maximal subspaces on which the cup product is positive respectively negative definite. The dimensions of these subspaces are denoted by b_2^+ and b_2^-.

The group $SO(4) \times U(1)$ admits a unique non–trivial double cover

$$\rho^c\colon Spin^c(4) \to SO(4) \times U(1)$$

for which the preimage of each factor is connected. In fact,

$$Spin^c(4) = (SU(2) \times SU(2) \times U(1))/\mathbb{Z}_2 \ . \tag{1}$$

Let P be the orthonormal frame bundle of (X,g) and suppose Q is a principal $U(1)$ bundle over X with $c_1(Q) \equiv w_2(X) \pmod 2$. A $Spin^c$ structure on X with auxiliary bundle Q is a double covering of $P \times Q$ by a principal $Spin^c(4)$–bundle $\tilde{P}$, such that the covering map is ρ^c–equivariant for the principal bundle actions[1].

If one varies the metric g continuously, the frame bundle changes continuously. As the space of Riemannian metrics is path–connected and contractible, one can canonically identify the $Spin^c$ structures defined with respect to different metrics. The equivalence classes of $Spin^c$ structures on X form a principal homogeneous space $Spin^c(X)$ for the group $H^2(X,\mathbb{Z})$.

Associated with every metric g and $Spin^c$ structure $\tilde{P}$ there are bundles $V_\pm$ of spinors of positive and negative chirality. These are the $U(2)$–bundles associated with the projections $Spin^c(4) \to U(2)$ corrsponding to the two $SU(2)$ factors in (1). We think of the spinor bundles as complex rank 2 vector bundles equipped with Hermitian metrics. The determinant bundle $det_{\mathbb{C}}(V_\pm) = L$ is the Hermitian line bundle associated with Q.

Every connection A on Q, together with the Levi–Civita connection of P, induces a unique connection on $\tilde{P}$. The corresponding covariant derivatives on $V_\pm$ are denoted by ∇_A. As the bundles $V_\pm$ are bundles of spinors, we can follow the covariant derivative by Clifford multiplication to obtain the two Dirac operators

$$D_A\colon V_+ \longrightarrow V_- \quad , \quad D_A^*\colon V_- \longrightarrow V_+ \ .$$

They are formal L^2 adjoints of each other.

The Seiberg–Witten equations are a family of pairs of coupled equations in which the variables are a connection A on Q and a section ϕ of V_+, and the parameter in the equations is a self–dual 2–form η on X with purely imaginary values. We denote by

$$\rho\colon \Omega^2_+(X,\mathbb{C}) \longrightarrow End_{\mathbb{C}}(V_+)$$

the bundle map induced by the action of the self–dual 2–forms by Clifford multiplication on V_+. Further, we denote by

$$\sigma\colon V_+ \otimes V_+ \longrightarrow End_{\mathbb{C}}(V_+)$$

[1] $Spin^c$ structures always exist on 4–manifolds, because the second Stiefel–Whitney class $w_2(X)$ always lifts to an integral class.

the bundle map induced by the $U(2)$–equivariant map

$$\begin{aligned}\sigma\colon \mathbb{C}^2\times\mathbb{C}^2 &\longrightarrow End_{\mathbb{C}}(\mathbb{C}^2)\\ (\phi,\psi)&\longmapsto \phi\otimes\psi^*-\frac{1}{2}(Tr(\phi\otimes\psi^*))Id\ .\end{aligned}$$

The Seiberg–Witten equations are:

$$D_A\phi = 0\ , \tag{2}$$

$$\rho(F_A^+) = \sigma(\phi,\phi)+\rho(\eta)\ . \tag{3}$$

We shall refer to these equations as the Dirac and curvature equations respectively.

To see that the curvature equation makes sense, observe that the Lie algebra of $U(1)$ is $i\mathbb{R}$, so the curvature F_A of A is a purely imaginary 2–form. Its self–dual part is mapped to the trace–less endomorphisms of V_+ by ρ. Moreover, the natural real structure on $End_{\mathbb{C}}(\mathbb{C}^2)$ is that of $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}$, so $\rho^{-1}\sigma(\phi,\phi)$ is also a purely imaginary self–dual 2–form.

A configuration (A,ϕ) is called reducible if $\phi\equiv 0$, and is called irreducible otherwise.

The gauge group $\mathcal{G} = Map(X, U(1)) = Aut(Q)$ acts on pairs (A,ϕ) by

$$u(A,\phi) = ((u^2)^*A, u^{-1}\phi)\ ,$$

and on $(\eta,\psi)\in\Omega^2_+(i\mathbb{R})\times\Gamma(V_-)$ by

$$u(\omega,\psi) = (\omega, u^{-1}\psi)\ .$$

FACT 1.1. — *The Seiberg–Witten equations are equivariant with respect to this action. The moduli space of solutions modulo gauge equivalence is denoted by $\mathcal{M}$. It is a subvariety of $\mathcal{B}$, the space of configurations (A,ϕ) modulo the action of $\mathcal{G}$.*

FACT 1.2 ([14]). — *For every fixed g and η, the moduli space $\mathcal{M}$ is compact.*

Proof. The Weitzenböck formula[2] for the Dirac operator is

$$D_A^*D_A\phi = \nabla_A^*\nabla_A\phi+\frac{1}{4}s\phi+\frac{1}{2}\rho(F_A^+)\phi\ , \tag{4}$$

where s is the scalar curvature function of g. Using this and the equations, the maximum principle gives a C^0 bound on ϕ in terms of s and η, for every solution of the Seiberg–Witten equations for (g,η). Compactness follows from this a priori estimate using standard arguments (Sobolev embedding and multiplication theorems, bootstrap using the equations). □

FACT 1.3. — *When considered on $\mathcal{B}$, the Seiberg–Witten equations are elliptic.*

One can thus apply the usual techniques of non–linear analysis to study $\mathcal{M}$. For example, the Sard–Smale theorem implies:

[2]The sign of the curvature term $\rho(F_A^+)$ is wrong in [14], and in [5]. Therefore, to get compactness, those papers change the sign of $\sigma(\phi,\phi)$ in the curvature equation (3).

FACT 1.4. — *If $b_2^+(X) > 0$, then for a generic metric g and a generic g–self–dual form η, the moduli space $\mathcal{M}$ is either empty, or is a smooth submanifold of $\mathcal{B}^*$, the space of gauge equivalence classes of irreducible configurations.*

Smoothness means that $\mathcal{M}$ is cut out transversely by the equations. When $\mathcal{M}$ is non-empty and smooth, its dimension is given by the Fredholm index of the linearized equations. This index can be calculated from the Atiyah–Singer index theorem:

FACT 1.5. — *If $\mathcal{M}$ is non–empty and smooth, then it has dimension*

$$d(Q) = \frac{1}{4}(\langle c_1^2(Q), [X]\rangle - (2\chi(X) + 3\sigma(X))) ,$$

where χ and σ denote the Euler characteristic and signature.

FACT 1.6. — *If $b_2^+(X) > 0$, then for a generic metric g and a generic g–self–dual form η, there are only finitely many $Spin^c$ structures for which the moduli space $\mathcal{M}$ is non–empty.*

Proof. Suppose that for a given $Spin^c$ structure there is a solution of the Seiberg–Witten equation. Using the a priori estimate obtained in the proof of the compactness of the moduli space, the Chern–Weil formula for $\langle c_1^2(L), [X]\rangle$ and the assumption $d(Q) \geq 0$, one sees that there are only finitely many choices for $c_1(L) = c_1(Q)$. For each fixed Q, there are only finitely many $\tilde{P}$. □

FACT 1.7. — *The moduli space $\mathcal{M}$ is orientable, and an orientation is specified by choosing a homology orientation of X, i.e., an orientation α_X of the line*

$$H^0(X, \mathbb{R}) \otimes (\Lambda^{b_1(X)} H^1(X, \mathbb{R}))^* \otimes \Lambda^{b_2^+(X)} H_+^2(X, \mathbb{R}) ,$$

where $H_+^2(X, \mathbb{R})$ denotes a maximal positive definite subspace of $H^2(X, \mathbb{R})$ for the cup product.

Varying the choice of a maximal positive definite subspace does not affect the definition of α_X, as the space of maximal positive definite subspaces is connected and contractible.

Putting all these properties together we have a definition of the Seiberg–Witten invariants SW_{X,α_X}, by sending a $Spin^c$ structure to the fundamental homology class of $\mathcal{M}$:

THEOREM 1.8. — *Let X be a smooth closed oriented 4–manifold with $b_2^+(X) > 1$; and let α_X be a homology orientation. Then*

$$\begin{aligned} SW_{X,\alpha_X} \colon Spin^c(X) &\longrightarrow H_*(\mathcal{B}_X^*, \mathbb{Z}) \\ \tilde{P} &\longmapsto [\mathcal{M}] \end{aligned}$$

is a diffeomorphism invariant of X.

More precisely, if $f \colon X \to Y$ is an orientation–preserving diffeomorphism, then

$$SW_{X,f^*\alpha_Y} \circ f^* = f^* \circ SW_{Y,\alpha_Y} .$$

Proof. By Fact 1.4, for a generic choice of (g, η) the moduli space $\mathcal{M}$ is a smooth submanifold of $\mathcal{B}^*$ (possibly empty). As it is also compact and oriented, it has a fundamental class. The assumption that $b_2 hX) > 1$ ensures that the space of generic pairs (g, η) is path–connected. The parametrised moduli space of solutions for a generic path connecting two generic choices provides a homology between the fundamental classes of the moduli spaces corresponding to those two choices. Thus $[\mathcal{M}]$ is independent of choices and depends only on (X, α_X) and on the $Spin^c$ structure under consideration.

To understand the diffeomorphism invariance, notice that $\mathcal{B}^*_X$ has the weak homotopy type of a classifying space for the gauge group, which is a homotopy invariant of X. (And is independent of the $Spin^c$ structure.) Thus f, like every homotopy equivalence, induces an isomorphism between $H_*(\mathcal{B}^*_X, \mathbb{Z})$ and $H_*(\mathcal{B}^*_Y, \mathbb{Z})$. The map SW commutes with f^* as stated because one can pull back to X a generic pair (g, η) used to calculate an invariant of Y. This then gives the same calculation on X. □

FACT 1.9. — *The description of $\mathcal{B}^*_X$ as a classifying space for the gauge group shows that it is weakly homotopy equivalent to $\mathbb{C}P^\infty \times J(X)$, where $J(X) = H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$ is the Jacobean torus of X.*

Fact 1.6 implies:

LEMMA 1.10. — For every smooth closed oriented 4–manifold there are at most finitely many $Spin^c$ structures with a non–trivial Seiberg–Witten invariant.

Here is a general vanishing theorem for the Seiberg–Witten invariants.

THEOREM 1.11 (Witten [28]). — *Let X be a smooth closed oriented 4–manifold with $b_2^+ > 1$. If X admits a metric of positive scalar curvature, or if it admits a smooth connected sum decomposition $X = X_1 \# X_2$ with $b_2^+(X_i) > 0$ for both $i = 1, 2$, then SW_{X,α_X} is the zero map.*

Proof. If X admits a metric g of positive scalar curvature, then the Weitzenböck formula (4) shows that all solutions to the Seiberg–Witten equations with $\eta = 0$ must be reducible. But $s > 0$ is an open condition in the space of metrics, so we may assume that g is generic and there are no reducible solutions, cf. Fact 1.4.

The proof of the vanishing theorem for connected sums is completely analogous to, but simpler than, the corresponding result for Donaldson invariants. See [5]. □

2. THE SEIBERG–WITTEN EQUATIONS ON AN ALMOST COMPLEX MANIFOLD

2.1. Arbitrary almost complex manifolds

Let X be a smooth oriented 4–manifold equipped with an almost complex structure J. We consider a J–invariant Riemannian metric g, called an almost Hermitian metric.

Define the fundamental 2–form ω corresponding to J and g by

$$\omega(X,Y) = g(JX,Y) \ . \tag{5}$$

This is a J–invariant g–self–dual form of length $\sqrt{2}$. Any two elements of the triple (J, g, ω) determine the third one. For example, one can recover J from g and ω because for every 1–form a we have

$$J(a) = *(\omega \wedge a) \ .$$

We extend J complex linearly to $TX \otimes_{\mathbb{R}} \mathbb{C}$ and $T^*X \otimes_{\mathbb{R}} \mathbb{C}$ and split these bundles into the $\pm i$ eigenbundles. For example, we write

$$\Lambda^1 \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^{1,0} \oplus \Lambda^{0,1}$$

and decompose every 1–form accordingly as $a = a^{1,0} + a^{0,1}$. This splitting induces a decomposition of all the complex differential forms into types, just like in the case of complex manifolds. In particular, we obtain:

$$\Lambda^2_+ \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}\omega \oplus \Lambda^{2,0} \oplus \Lambda^{0,2} \ , \tag{6}$$

which corresponds to the real decomposition $\Lambda^2_+ = \mathbb{R} \oplus K^{\pm 1}$, where $K = \Lambda^{2,0}$ is the canonical bundle of the almost complex structure.

Like in the case of complex manifolds, one defines operators ∂, $\overline{\partial}$ by setting $\partial\alpha = (d\alpha)^{p+1,q}$ and $\overline{\partial}\alpha = (d\alpha)^{p,q+1}$ for a form $\alpha \in \Lambda^{p,q}$. In general, $d \neq \partial \oplus \overline{\partial}$, as $d\alpha$ can also have components of type $(p-1, q+2)$ and $(p+2, q-1)$, denoted $\overline{N}\alpha$ and $N\alpha$ respectively. The operator N is of order zero and can be identified with the Nijenhuis tensor of J; it vanishes if and only if J is integrable. Observe that $d^2 = 0$ does not imply $\overline{\partial}^2 = 0$, unless J is integrable. For example, on functions we have

$$\overline{\partial}^2 f = -N(\partial f) \ . \tag{7}$$

An almost complex structure defines a homotopy class of reductions of the structure group of TX to $U(2)$. The map $i \times det \colon U(2) \to SO(4) \times U(1)$ lifts to $Spin^c(4)$, which implies:

FACT 2.1. — *An almost complex manifold has a canonical $Spin^c$ structure with $L = det_{\mathbb{C}}(V_\pm) = det_{\mathbb{C}}(TX, J) = K^{-1} = \Lambda^{0,2}$.*

This is given concretely by setting $V_+^{\text{can}} = \Lambda^{0,0} \oplus \Lambda^{0,2}$ and $V_-^{\text{can}} = \Lambda^{0,1}$ and by defining Clifford multiplication by a 1–form a on $(\alpha, \beta) \in \Gamma(\Lambda^{0,0} \oplus \Lambda^{0,2})$ by

$$\gamma(a)(\alpha, \beta) = \sqrt{2}(a^{0,1} \wedge \alpha - *(a^{1,0} \wedge *\beta)) \ ,$$

and on $\psi \in \Gamma(\Lambda^{0,1})$ by

$$\gamma(a)(\psi) = \sqrt{2}(- * (a^{1,0} \wedge *\psi) \ , \ a^{0,1} \wedge \psi) \ .$$

Using this $Spin^c$ structure defined by the almost complex structure as a reference, we identify $Spin^c(X) = H^2(X, \mathbb{Z})$. The spinor bundles for the $Spin^c$ structure corresponding to E are given by $V_+^{\mathrm{can}} \otimes E = E \oplus (E \otimes K^{-1})$ and by $V_-^{\mathrm{can}} \otimes E = \Lambda^{0,1} \otimes E$. Thus we write the Seiberg–Witten invariant as a map

$$SW_{X,\alpha_X} \colon H^2(X, \mathbb{Z}) \longrightarrow H_*(\mathcal{B}_X^*, \mathbb{Z}) \ .$$

This map is no longer diffeomorphism invariant, because an orientation–preserving self–diffeomorphism of X will not usually fix the given almost complex structure, or its associated $Spin^c$ structure. Compare Theorem 1.8.

If ϕ is a spinor with components (α, β) with respect to the splitting $V_+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$, then $\sigma(\phi, \phi)$ is given by

$$\begin{pmatrix} \frac{1}{2}(|\alpha|^2 - |\beta|^2) & \alpha\overline{\beta} \\ \overline{\alpha}\beta & -\frac{1}{2}(|\alpha|^2 - |\beta|^2) \end{pmatrix}$$

Applying ρ^{-1} and comparing this formula with the decomposition of Λ_+^2 given in (6), we find that the curvature equation for a connection A on K^{-1} and a pair $(\alpha, \beta) \in \Gamma(V_+^{\mathrm{can}})$ is equivalent to the following two equations:

$$F_A^\omega = \frac{i}{4}(|\alpha|^2 - |\beta|^2)\omega + \eta^\omega \tag{8}$$

$$F_A^{0,2} = \frac{1}{2}\overline{\alpha}\beta + \eta^{0,2} \ , \tag{9}$$

where in the first equation F_A^ω denotes the part of F_A parallel to ω, which is the same as the $(1,1)$ component of F_A^+.

Fix a connection A_0 on $\Lambda^{0,2}$ and consider the connection induced on $V_\pm^{\mathrm{can}}$ by lifting from $SO(4) \times U(1)$ to $Spin^c(4)$, as explained in Section 1.

We obtain a connection on $V_+(E) = V_+^{\mathrm{can}} \otimes E$ as the product of the connection induced by A_0 with some connection B on E. We often think of B as one of the variables in the Seiberg–Witten equations, instead of thinking of the corresponding connection $A = A_0 \otimes B^2$ on the determinant $L = K^{-1} \otimes E^2$. Notice that if we take $E = 0$, then the device of twisting by B is just the same as varying the connection for the canonical $Spin^c$ structure.

In the twisted case, we again decompose $\phi = (\alpha, \beta)$, but now the forms α and β take values in E. However, the formula for $\sigma(\phi, \phi)$ is still true for the bundle–valued forms.

FACT 2.2. — *For $A = A_0 \otimes B^2$ and $\phi = (\alpha, \beta)$, the curvature equation* (3) *is equivalent to the following two equations:*

$$F_B^\omega = \frac{i}{8}(|\alpha|^2 - |\beta|^2)\omega - \frac{1}{2}F_{A_0}^\omega + \frac{1}{2}\eta^\omega \tag{10}$$

$$F_B^{0,2} = \frac{1}{4}\overline{\alpha}\beta - \frac{1}{2}F_{A_0}^{0,2} + \frac{1}{2}\eta^{0,2} \ . \tag{11}$$

We will not write out the Dirac equation on an arbitrary almost complex manifold. Notice however that, by definition, the symbol of the Dirac operator is the same as the symbol of the Dolbeault operator $\sqrt{2}(\overline{\partial}+\overline{\partial}^*)$, which means that the two differ only by a term of order zero. This zero order term can be written down explicitely, see [7].

2.2. Symplectic manifolds

The fundamental two–form ω of an almost Hermitian metric is always non–degenerate. Thus, if it is also closed, then it is a symplectic form. Conversely, given a symplectic structure on a manifold X, it determines a homotopy class of almost complex structures J. In this homotopy class we can select a J which makes X almost Hermitian with the metric defined by (5). We say that such an almost complex structure calibrates the symplectic structure.

DEFINITION 2.3. — *An almost Hermitian structure is called almost Kähler if* $d\omega = 0$.

All the symplectic manifolds we consider are assumed to be equipped with almost Kähler metrics.

It is important in the proof of Taubes's theorems that on an almost Kähler manifold a number of identities involving the natural differential operators derived from the metric hold, just like in the case of Kähler manifolds. For example, in Weil's book [27], the "Kähler identities" are proved for almost Kähler manifolds, although in the more recent literature this is not usually done. An exception is Donaldson's article [4], which contains the rudiments of harmonic theory on almost Kähler manifolds.

Here is a useful Weitzenböck formula familiar in the Kähler case:

LEMMA 2.4. — Let s be a smooth section of a Hermitian line bundle E over an almost Kähler manifold X. Let B be a Hermitian connection on E, and let $\overline{\partial}_B$ be the $\overline{\partial}$ operator of the almost Kähler structure coupled to B. Then

$$\overline{\partial}_B^*\overline{\partial}_B s = \frac{1}{2}(d_B^* d_B s - i\Lambda F_B s) \ , \tag{12}$$

where Λ denotes the contraction with the fundamental 2–form ω.

Proof. By the definition of $\overline{\partial}$, we have

$$\overline{\partial}_B s = (d_B s)^{0,1} = \frac{1}{2}(d_B s + iJd_B s) = \frac{1}{2}(d_B s + i*(\omega\wedge d_B s)) \ ,$$

and thus

$$\begin{aligned}\overline{\partial}_B^*\overline{\partial}_B s &= \frac{1}{2}d_B^*(d_B s + i*(\omega\wedge d_B s)) = \frac{1}{2}(d_B^* d_B s - i*d_B(\omega\wedge d_B s))\\ &= \frac{1}{2}(d_B^* d_B s - i*(\omega\wedge d_B^2 s)) = \frac{1}{2}(d_B^* d_B s - i*(\omega\wedge F_B s))\\ &= \frac{1}{2}(d_B^* d_B s - i\Lambda F_B s) \ .\end{aligned}$$

□

Formula (12) will be applied to α in the decomposition $\phi = (\alpha, \beta)$ discussed in the previous subsection. We shall also need a Weitzenböck formula for β.

On the line bundle $\Lambda^{0,2}$ there is a Hermitian connection A_1, unique up to gauge equivalence, such that $\nabla^{1,0}_{A_1} = \partial$. Twisting by a Hermitian line bundle E with connection B, we have $\nabla^{1,0}_{A_1 \otimes B} = \partial_B$. On $(0,2)$ forms the Kähler identities give

$$\partial^* \partial = i[\Lambda, \overline{\partial}]\partial = -i\overline{\partial}\Lambda\partial = \overline{\partial}(-i[\Lambda, \partial]) = \overline{\partial}\overline{\partial}^* .$$

On the other hand, we can calculate $\partial^* \partial \beta$ like in the proof of the previous Lemma. We obtain:

LEMMA 2.5. — Let β be a $(0,2)$ form with values in a Hermitian line bundle E with connection B. Then:

$$\overline{\partial}_B \overline{\partial}^*_B \beta = \frac{1}{2}(\nabla^*_{A_1 \otimes B} \nabla_{A_1 \otimes B} \beta + i\Lambda(F_{A_1} + F_B)\beta) . \tag{13}$$

We remarked in the previous subsection that the Dirac operator for the canonical $Spin^c$ structure of an almost Hermitian manifold differs from its Dolbeault operator only by a term of order zero. In the almost Kähler case, this zero order term vanishes for some choice of connection A_0 on $\Lambda^{0,2}$. Again, this is familiar in the Kähler case, where A_0 is the holomorphic connection induced by the Levi–Civita connection. One can always choose local holomorphic coordinates in which the Kähler metric is standard up to second order, and thus the equality of the two operators must be true because it is true on flat space.

In general, the zero order operator which is the difference between the Dirac and Dolbeault operators is a bundle homomorphism $V_+ \longrightarrow V_-$, i.e., a section of

$$(\Lambda^{0,0} \oplus \Lambda^{2,0}) \otimes \Lambda^{0,1} = \Lambda^{0,1} \oplus \Lambda^{2,1} .$$

The deviation of the almost Kähler metric from being Kähler is measured by the torsion $\nabla\omega$, where ∇ is the connection induced on Λ^2_+ by the Levi–Civita connection. As ω has constant length, $\nabla\omega$ is orthogonal to ω, thus $\nabla\omega \in \Lambda^1 \otimes (\Lambda^{2,0} \oplus \Lambda^{0,2})$. Decomposing the tensor product we find

$$(\Lambda^1 \otimes \mathbb{C}) \otimes (\Lambda^{2,0} \oplus \Lambda^{0,2}) = \Lambda^{2,1} \oplus \Lambda^{1,2} \oplus \Lambda^{3,0} \oplus \Lambda^{0,3} \oplus A \oplus \overline{A} ,$$

where A is a $U(2)$–module not containing any $\Lambda^{p,q}$. (Notice that because X is 4–dimensional, $\Lambda^{3,0} = \Lambda^{0,3} = 0$.) Now $d\omega = 0$ is equivalent to $\nabla\omega$ having no component which can act as a homomorphism from $(\Lambda^1 \otimes \mathbb{C}) \otimes \Lambda^{0,\mathrm{even}}$ to $\Lambda^{0,\mathrm{odd}}$. Thus, the Dolbeault and Dirac operators for the canonical $Spin^c$ structure are equal for a connection A_0 for which the difference is a linear function of the torsion $\nabla\omega$. This will still be true after twisting by a line bundle E with connection B, because the space of homomorphisms $Hom(V_+, V_-)$ is canonically the same for all $Spin^c$ structures.

To find the connection A_0 we can proceed as follows. The identification $V_+^{\mathrm{can}} = \Lambda^{0,0} \oplus \Lambda^{0,2}$ can be made explicit by fixing a section $u_0 \in \Gamma(V_+^{\mathrm{can}})$ of constant length, and then letting the 0– and $(0,2)$–forms act on u_0 by Clifford multiplication. We fix A_0 (up to gauge equivalence) by requiring that $\nabla_{A_0} u_0 \in \Lambda^1 \otimes \Lambda^{0,2}$. Then applying the Dirac operator D_{A_0} to the equation $\rho(\omega)u_0 = -2iu_0$ implies, because ω is self–dual harmonic, that $D_{A_0}u_0 = 0$, which in turn implies, via the Leibnitz rule, that $D_{A_0}(\alpha u_0) = \sqrt{2}\overline{\partial}\alpha$. Similarly, $D_{A_0}\beta = \sqrt{2}\overline{\partial}^*\beta$.

We have shown that the first Seiberg–Witten equation, the Dirac equation (3), can be written as

$$\overline{\partial}_B\alpha + \overline{\partial}_B^*\beta = 0 \ . \tag{14}$$

3. TAUBES'S THEOREMS

3.1. Symplectic manifolds with $b_2^+ > 1$

Following Taubes, we shall calculate Seiberg–Witten invariants of symplectic 4–manifolds using almost Kähler metrics. But first, we make a remark following Witten [28]:

FACT 3.1. — *In the notation of the previous section, $SW_{X,\alpha_X}(E) = \pm SW_{X,\alpha_X}(K - E)$.*

Proof. The Seiberg–Witten equations with $\eta = 0$ are invariant under the involution $E \mapsto K \otimes E^{-1}$ using $(B, \alpha, \beta) \mapsto (A_0^{-1} \otimes B^{-1}, -\overline{\beta}, \overline{\alpha})$.

In the Kähler case this is a manifestation of Serre duality. □

The following theorem is a combination of theorems which first appeared in [18] and [19], with a more complicated proof. The argument below (in different notation) was sketched by Taubes in [20], and it is also implicit in his later paper [21]. However, our presentation is closer to [5].

THEOREM 3.2 (Taubes). — *Let X be a closed symplectic 4–manifold with $b_2^+(X) > 1$, and let K be its canonical class. Then $SW_{X,\alpha_X}(0)$ and $SW_{X,\alpha_X}(K)$ are both equal to $\pm 1 \in H_0(\mathcal{B}_X^*, \mathbb{Z})$.*

Moreover, if E is any class with $SW_{X,\alpha_X}(E) \neq 0$, then

$$0 \leq E \cdot [\omega] \leq K \cdot [\omega] \ ,$$

with equality if and only if $E = 0$ or $E = K$.

Proof. Using the symmetry property above, it suffices to prove $SW_{X,\alpha_X}(0) = \pm 1$, and $0 \leq E \cdot [\omega]$ for all E with a non–trivial invariant, together with the characterisation of equality in this case.

We consider the Seiberg–Witten equations for the canonical $Spin^c$ structure twisted by E, and we choose η such that the equations become

$$\overline{\partial}_B\alpha = -\overline{\partial}_B^*\beta \ , \tag{15}$$

$$F_B^\omega = \frac{i}{8}(|\alpha|^2 - |\beta|^2 - r)\omega \tag{16}$$

$$F_B^{0,2} = \frac{1}{4}\overline{\alpha}\beta \ , \tag{17}$$

with $r \in \mathbb{R}$ a constant.

Notice that if $E = 0$ and $r \geq 0$, then these equations always admit a solution with B the trivial connection, $\beta = 0$ and α a constant section of squared length r. The main point of the proof of the theorem is to show that this is the only solution if $c_1(E) \cdot [\omega] = 0$.

If E has a non-trivial Seiberg–Witten invariant, then for all metrics and all choices of η (here: all choices of r) there must exist a solution to the equations. Let (B, α, β) be a solution. Using the Weitzenböck formula (12) applied to α and taking the L^2 inner product with α, we find:

$$\int_X |d_B\alpha|^2 dvol_g = \int_X (\langle 2\overline{\partial}_B^* \overline{\partial}_B \alpha, \alpha\rangle + i\Lambda F_B |\alpha|^2) dvol_g \ . \tag{18}$$

We calculate the two terms on the right–hand–side separately using the equations. Using (15), (7) and (17), the first term is:

$$\begin{aligned}\int_X \langle 2\overline{\partial}_B^* \overline{\partial}_B \alpha, \alpha\rangle dvol_g &= \int_X -2\langle \overline{\partial}_B^* \overline{\partial}_B^* \beta, \alpha\rangle dvol_g = \int_X -2\langle \beta, \overline{\partial}_B^2 \alpha\rangle dvol_g \\ &= \int_X (-2\langle \beta, F_B^{0,2}\alpha\rangle + 2\langle \beta, N(\partial_B \alpha)\rangle) dvol_g \\ &= \int_X (-\frac{1}{2}|\alpha|^2|\beta|^2 + 2\langle \beta, N(\partial_B\alpha)\rangle) dvol_g \ .\end{aligned}$$

Using (16), the second term is:

$$\int_X i\Lambda F_B |\alpha|^2 dvol_g = \int_X -\frac{1}{4}(|\alpha|^2 - |\beta|^2 - r)|\alpha|^2 dvol_g \ .$$

Substituting back in equation (18) and rearranging, we have

$$\int_X (|d_B\alpha|^2 + \frac{1}{4}|\alpha|^2|\beta|^2 + \frac{1}{4}(|\alpha|^2 - r)^2 + \frac{1}{4}r(|\alpha|^2 - r)) dvol_g = \int_X 2\langle \beta, N(\partial_B\alpha)\rangle dvol_g \ . \tag{19}$$

Now we express the last term on the left–hand–side in a different way using the Chern–Weil formula:

$$c_1(E) \cdot [\omega] = \int_X \frac{i}{2\pi} F_B \wedge \omega = -\frac{1}{8\pi}\int_X (|\alpha|^2 - |\beta|^2 - r) dvol_g \ . \tag{20}$$

We can also estimate the right–hand–side of equation (19) using the Peter Paul inequality:

$$\int_X 2\langle \beta, N(\partial_B\alpha)\rangle dvol_g \leq \int_X (\frac{1}{2}|d_B\alpha|^2 + C|\beta|^2) dvol_g \ , \tag{21}$$

where $C > 0$ is a constant which depends on the geometry of the almost Kähler metric, but does not depend on (B, α, β), or on r. Substituting back in equation (19), we obtain

$$\int_X (\frac{1}{2}|d_B\alpha|^2 + \frac{1}{4}|\alpha|^2|\beta|^2 + \frac{1}{4}(|\alpha|^2 - r)^2 + \frac{1}{4}r|\beta|^2) dvol_g - 2\pi r c_1(E) \cdot [\omega] \leq C \int_X |\beta|^2 dvol_g \ . \tag{22}$$

Choosing $r > 4C$, we conclude that $c_1(E) \cdot [\omega] \geq 0$. Moreover, if $c_1(E) \cdot [\omega] = 0$ then the last inequality shows that $\beta = 0$, $|\alpha|^2 = r$ and $d_B\alpha = 0$. This shows that B is the trivial connection, and $E = 0$. All the solutions of this form are clearly gauge equivalent to one another. For a generic r which is large enough, this solution is cut out transversely by the Seiberg–Witten equations. Thus $SW_{X,\alpha_X}(0) = \pm 1$.

This completes the proof of the theorem, because of the symmetry 3.1. □

Thus, $SW_{X,\alpha_X}(E) \neq 0$ implies $E \cdot [\omega] \geq 0$, with equality only if $E = 0$. Analysing the above argument for $E \neq 0$ leads, after a lot of work, to the following:

THEOREM 3.3 (Taubes [21, 22]). — *Let X be a closed symplectic 4-manifold with $b_2^+(X) > 1$, and let $E \in H^2(X, \mathbb{Z})$ be a class with $E \neq 0$ and $SW_{X,\alpha_X}(E) \neq 0$. Then, for the almost complex structure associated with an almost Kähler metric on X, there is a pseudo-holomorphic curve Poincaré dual to E.*

Sketch of proof. Due to lack of space and time, we cannot hope to do justice to the proof. We will only make a few general remarks on the structure of the argument, and indicate how to rewrite Taubes's proof [22] in the notation of Section 2 above.

One considers the same perturbation of the equations as in the proof of Theorem 3.2, and makes the parameter $r \in \mathbb{R}$ large. But now $c_1(E) \cdot [\omega] > 0$, so the inequality (22) does not imply $\beta = 0$, no matter how large r is. All we get, is that

$$\int_X |d_B\alpha|^2 dvol_g\ ,\ \int_X |\alpha|^2|\beta|^2 dvol_g\ ,\ \int_X (|\alpha|^2 - r)^2 dvol_g\ ,\ \int_X (r - C)|\beta|^2 dvol_g$$

are bounded above by uniform multiples of $rc_1(E) \cdot [\omega]$. The bound on the third term implies that, as $r \to \infty$, $\frac{1}{\sqrt{r}}\alpha$ approaches a section of E of constant length $= 1$. (In the proof of Theorem 3.2, this happened for finite large r.) However, the estimate above is only in L^2, and we know that because E is non-trivial the section α must vanish somewhere.

Consider now the Weitzenböck formula (13). Taking the L^2 inner product with β and using the equations as in the proof of Theorem 3.2, we obtain:

$$(23)\quad \int_X (|\nabla_{A_1 \otimes B}\beta|^2 + \frac{1}{4}(|\alpha|^2 + |\beta|^2 + r)|\beta|^2 + f|\beta|^2) dvol_g = \int_X 2\langle N(\partial_B\alpha, \beta\rangle dvol_g\ ,$$

where the term with the unknown function f corresponds to the curvature of A_1 in the Weitzenböck formula. This function is independent of (B, α, β) and independent of r. Estimating the right–hand–side using the Peter Paul inequality we find:

$$(24)\quad \int_X (|\nabla_{A_1 \otimes B}\beta|^2 + \frac{1}{4}(|\alpha|^2 + |\beta|^2)|\beta|^2 + (\frac{1}{8}r + f)|\beta|^2) dvol_g \leq \frac{C'}{r} \int_X |\partial_B\alpha|^2 dvol_g\ ,$$

where C' is a constant depending on the geometry of the base manifold which is independent of (B, α, β) and independent of r.

Going back to the Weitzenböck formula (13), we have

$$2\int_X |\bar{\partial}_B^*\beta|^2 dvol_g = \int_X (|\nabla_{A_1 \otimes B}\beta|^2 - \frac{1}{4}(|\alpha|^2 - |\beta|^2 - r)|\beta|^2 + f|\beta|^2) dvol_g \leq \frac{2C'}{r} \int_X |\partial_B\alpha|^2 dvol_g\ ,$$

where the last inequality follows from (24). Using the Dirac equation, we have

$$\int_X |\bar{\partial}_B\alpha|^2 dvol_g = \int_X |\bar{\partial}^*_B\beta|^2 dvol_g \leq \frac{C'}{r}\int_X |\partial_B\alpha|^2 dvol_g \ .$$

This shows that as r becomes large, α tends to become holomorphic. (Again, this happened for finite r in the proof of Theorem 3.2.) Note that in the case when (X,ω) is Kähler, the right–hand–side of (23) vanishes, so that for r large one does find that β vanishes identically and α is a holomorphic section of E, whose zero–set is the desired holomorphic curve.

In the almost Kähler situation, it seems impossible to find a finite r for which the α component of the spinor in a solution to the Seiberg–Witten equations is necessarily holomorphic. Therefore, Taubes [22] considers the limiting behaviour of the zero–locus of α as $r \to \infty$ and shows that the zero–locus converges, as a current, to a pseudo–holomorphic curve. This involves delicate pointwise estimates on the solutions. The first of these estimates can be obtained from the Weitzenböck formulae (12) and (13) without integrating over the manifold. Then a term like $\langle d^*_B d_B\alpha, \alpha\rangle$ becomes $\frac{1}{2}\Delta|\alpha|^2 + |d_B\alpha|^2$, and similarly for the equation involving β.

Defining $u = 1 - \frac{1}{r}|\alpha|^2 - \frac{1}{C_1}|\beta|^2 + \frac{C_2}{r}$, the two Weitzenböck formulae together show that the positive constants C_1 and C_2 can be chosen in such a way that $2\Delta u + |\alpha|^2 u \geq 0$ (using Peter Paul judiciously). Therefore, the "minimum principle" implies $u \geq 0$ everywhere on X. This gives the first of a whole series of delicate pointwise estimates that go into controlling the limit of the zero–set of α as $r \to \infty$. $\square$

Remark 3.4. — In the statements of Theorems 3.2 and 3.3 in Taubes's papers [18, 19, 21, 22], more restrictive Seiberg–Witten invariants are considered than we have used here. Taubes only considers the $\mathbb{C}P^\infty$ factor in the homology of $\mathcal{B}^*$, and ignores the Jacobean torus, compare Fact 1.9. The proofs are valid in the generality stated here because they only use the existence of solutions for all metrics and all perturbations, without reference to the homology class of the moduli space. However, we will see in Theorem 4.11 that, as a consequence of Theorem 3.3, the only non–trivial Seiberg–Witten invariants of symplectic manifolds arise from zero–dimensional moduli spaces, so that our statements are, a posteriori, no more general than Taubes's.

3.2. The case of $\mathbb{C}P^2$

There are versions of Theorems 3.2 and 3.3 when the base manifold has $b_2^+ = 1$. We discuss only the case when X has the rational homology the complex projective plane $\mathbb{C}P^2$.

For X with $b_2^+(X) = 1$ and $b_2^-(X) = 0$, the Seiberg–Witten equations (2) and (3) with $\eta = 0$ define a metric–independent invariant

$$SW_{X,\alpha_X}\colon Spin^c(X) \longrightarrow H_*(\mathcal{B}^*_X, \mathbb{Z}) \ . \tag{25}$$

To see this, examine the proof of Theorem 1.8. There the assumption $b_2^+(X) > 1$ only enters to avoid reducible solutions to the equations appearing for a generic 1-parameter family (g, η). For a reducible solution $(A, 0)$ to the equations with $\eta = 0$, F_A is an anti-self-dual harmonic 2-form representing $c_1(Q)$. But if $b_2^-(X) = 0$, then for any metric the only anti-self-dual harmonic form is $= 0$. On the other hand, $c_1(Q)$ is an odd multiple of the generator in $H^2(X, \mathbb{Z})$, in particular it is non-zero. Thus, there are no reducible solutions to the Seiberg-Witten equations with $\eta = 0$. This is still true if η is non-zero but small enough.

As the Fubini-Study metric has positive scalar curvature, the proof of Theorem 1.11 implies:

LEMMA 3.5. — The Seiberg-Witten invariant (25) of $X = \mathbb{CP}^2$ for the $\eta = 0$ equations is identically zero.

On the other hand, the argument in the proof of Theorem 3.2 goes through as before. Namely, given a symplectic form ω on $\mathbb{C}P^2$, for the canonical *Spin*^{c} structure and an almost Kähler metric the Seiberg-Witten invariant calculated from the equations (15), (16), (17) with r large, gives ± 1. (Notice that the curvature of A_0 is some multiple of ω, so we can absorb the term $\frac{1}{2}F_{A_0}$ in Fact 2.2 in the perturbation by $-\frac{i}{4}r\omega$ by changing r.)

Thus, we conclude that for some $r_0 > 0$ the equations (16) have a reducible solution, to account for the jump in the invariant. For the reducible solution $(A, 0)$ we have $F_A^+ = -\frac{i}{4}r_0\omega$. Thus, the Chern-Weil formula implies:

$$c_1(-K) \cdot [\omega] = \int_X \frac{i}{2\pi} F_A \wedge \omega = \frac{1}{4\pi} r_0 vol > 0 \ .$$

Therefore, the analog of Theorem 3.2 for $\mathbb{C}P^2$ is:

PROPOSITION 3.6 (Taubes [19]). — *For every symplectic form ω on $\mathbb{C}P^2$ with canonical bundle K, one has $c_1(K) \cdot [\omega] < 0$.*

In the same way, we find the analog of Theorem 3.3 for $\mathbb{C}P^2$. Given a symplectic form ω on $\mathbb{C}P^2$, we choose a generator $H \in H^2(\mathbb{C}P^2, \mathbb{Z})$, such that $[\omega]$ is a positive multiple of H. Then, by Proposition 3.6, $c_1(K)$ is a negative multiple of H, which implies that it is $= -3H$.

THEOREM 3.7 (Taubes [21, 22]). — *For every symplectic form ω on $\mathbb{C}P^2$ and any compatible almost Kähler metric, the generator $H \in H^2(\mathbb{C}P^2, \mathbb{Z})$ is Poincaré dual to a pseudo-holomorphic curve.*

Proof. We consider the Seiberg-Witten invariants of $\mathbb{C}P^2$ for the canonical *Spin*^{c} structure twisted by H. Assume that for the equations in the proof of Theorem 3.3 there exists a solution for all r sufficiently large. Then letting r go to infinity, the analysis in [22] applies just as in the proof of Theorem 3.3, and one finds that H is Poincaré dual to a pseudo-holomorphic curve.

Thus, it suffices to show that for all r sufficiently large there are solutions to the Seiberg–Witten equations (16). To see this, observe that the homology class in $H_*(\mathcal{B}^*, \mathbb{Z})$ calculated from the moduli space is a differentiable invariant of $\mathbb{C}P^2$, which does not depend on the metric or on the symplectic structure ω, although here it does depend on the parameter η in (3). Thus, we can calculate the invariant using the standard Kähler form on $\mathbb{C}P^2$, and then use the result to get a statement that is valid for all symplectic forms ω.

Thus, consider first the standard Kähler form. As before, if η is small there are no solutions to the equations because of the Weitzenböck formula (4) and the invariant vanishes when η is small, i.e. when r in (16) is near 0. See Lemma 3.5.

Now increase r. For a fixed r_1 there is a unique reducible solution, and for all $r > r_1$ there is a smooth moduli space of solutions. Recall from the proof of Theorem 3.3 that, because the metric we consider is Kähler, not just almost Kähler, for large r the β component of the spinor of any solution vanishes identically, and the α component is then a holomorphic section. Using this, the moduli space of solutions for $r > r_1$ can be identified with the projective space of non–trivial holomorphic sections of the hyperplane bundle on $\mathbb{C}P^2$, cf. [28], [5]. This projective space is itself a copy of $\mathbb{C}P^2$, which is a slice in the $\mathbb{C}P^\infty \simeq \mathcal{B}^*$. Thus, for r large, the invariant is $\pm 1 \in H_4(\mathbb{C}P^\infty, \mathbb{Z}) \cong \mathbb{Z}$.

Considering now an arbitrary symplectic form ω on $\mathbb{C}P^2$, we know from Proposition 3.6 that it has the same sign as the standard Kähler form. Thus letting $r \to \infty$ in equation (16) leads into the same regime, no matter which ω we use. As the invariant for r large is non–trivial for the Kähler form, it is non–trivial for every symplectic form ω. Thus, for every ω and every r large enough, there are solutions to the equations (15), (16), (17). $\square$

Remark 3.8. — The discussion above concerning the change in the Seiberg–Witten invariant of $\mathbb{C}P^2$ as one varies η is really the same as the discussion in [14], [1] concerning the change in the invariant as one varies the metric on a manifold with $b_2^+ = 1$ and $b_2^- \neq 0$. Both are special cases of the general wall–crossing formula which applies whenever one encounters a reducible solution in a family of Seiberg–Witten equations parametrised by (g_t, η_t).

4. APPLICATIONS OF TAUBES'S THEOREMS

Taubes's theorems discussed in the previous section have a lot of consequences for the topology of symplectic manifolds, and for the classification of symplectic structures. At the time of writing, such applications are emerging rapidly, so we will not try to be exhaustive, but rather restrict ourselves to a few selected corollaries, most of which were pointed out by Taubes himself.

4.1. Direct applications of Theorem 3.2

As was remarked in the introduction, Theorem 3.2 implies some immediate but important constraints on the topology and geometry of symplectic 4–manifolds. Combining it with Theorem 1.11, one concludes:

COROLLARY 4.1. — *Let X be a symplectic 4–manifold with $b_2^+(X) > 1$. Then*

(1) *X does not admit any Riemannian metric of positive scalar curvature, and*

(2) *X does not admit a smooth connected sum decomposition $X \cong X_1 \# X_2$ with both $b_2^+(X_i) > 0$, $i = 1, 2$.*

Example 4.2. — Let $X_{p,q} = p\mathbb{C}P^2 \# q\overline{\mathbb{C}P^2}$. For all p and q, $X_{p,q}$ admits a metric of positive scalar curvature. It admits an almost complex structure if and only if $p \equiv 1 \pmod 2$, and a symplectic structure if and only if $p = 1$.

Theorem 3.2 also strongly constrains the possible canonical classes of symplectic structures:

COROLLARY 4.3. — *Let X be a smooth 4–manifold with $b_2^+(X) > 1$.*

(1) *If $b_1(X) = 0$, then X admits at most finitely many homotopy classes of almost complex structures compatible with symplectic forms.*

(2) *X does not admit any symplectic structure whose canonical class $K \in H^2(X, \mathbb{Z})$ is a non–zero torsion class. If X is symplectic, then the canonical classes of two symplectic structures cannot differ by a non–zero torsion class.*

(3) *If X admits a symplectic structure with $K = 0$, then all other symplectic structures on X also satisfy $K = 0$.*

Proof. (2) and (3) are clear from the second part of Theorem 3.2. Under the assumption $b_1(X) = 0$, the canonical class determines the homotopy class of an almost complex structure up to finite ambiguity, so that (1) follows from Lemma 1.10. □

Remark 4.4. — The assumption on $b_2^+(X)$ cannot be weakend in (2). The Enriques surface, a quotient of a $K3$ surface by a free holomorphic involution, is Kähler. It has $b_2^+ = 1$ and its canonical class is a non–trivial 2–torsion class. The proof of Theorem 3.2 breaks down because the Seiberg–Witten invariants of the Enriques surface depend on η, cf. the discussion of the $\mathbb{C}P^2$ case.

Example 4.5. — Part (3) of Corollary 4.3 applies to the $K3$ surfaces, to the 4-torus T^4, and to the Kodaira–Thurston manifolds [11, 26]. The latter are certain T^2–bundles over T^2 with $b_2^+ = 2$ and $b_1 = 3$. They are both complex and symplectic with trivial canonical class. These were the first known examples of symplectic non–Kähler 4-manifolds. Theorem 3.2 gives a complete calculation of the Seiberg–Witten invariants for all these examples. Recently, Biquard [2] has calculated the invariants for the Kodaira–Thurston manifolds in a different way, not using the symplectic structure.

4.2. Applications using pseudo–holomorphic curves

The existence theorems 3.3 and 3.7 for pseudo–holomorphic curves by themselves are not immediately useful, unless one has a genericity statement to ensure that the curves are not too singular or degenerate. On a fixed symplectic 4–manifold (X, ω) one has a choice when picking a calibrating almost complex structure, or, equivalently, a compatible almost Kähler metric. There is a then a genericity statement for the moduli spaces of pseudo–holomorphic curves which follows from the Sard–Smale theorem and is the analog of Fact 1.4 for the Seiberg–Witten moduli spaces. Informally, we can paraphrase this statement, Proposition 7.1 in [22], as follows, compare [15].

FACT 4.6. — *For a generic almost Kähler metric on (X, ω) compatible with ω, pseudo-holomorphic curves in a given homology class are transverse zeroes of the Cauchy–Riemann equations. In particular, they consist of disjoint smooth components, the only rational curves of negative self–intersection are smooth (-1)–curves, and the only non–reduced components (if any) are tori of self–intersection zero and rational (-1)–curves.*

In Theorems 3.3 and 3.7, we can always use an almost Kähler metric which is generic in this sense to make the pseudo–holomorphic curve Poincaré dual to E (respectively H) smooth. Each component Σ of such a curve is then itself a smooth pseudo–holomorphic, in particular symplectic, submanifold, whose genus is given by the adjunction formula

$$g(\Sigma) = 1 + \frac{1}{2}(\Sigma^2 + K \cdot \Sigma) \ . \tag{26}$$

With this understood, Theorem 3.7 implies that for a generic almost Kähler metric compatible with an arbitrary symplectic form on $\mathbb{C}P^2$, the positive generator $H \in H^2(\mathbb{C}P^2, \mathbb{Z})$ is represented by a smooth pseudo–holomorphic curve of genus

$$g(H) = 1 + \frac{1}{2}(H^2 + K \cdot H) = 1 + \frac{1}{2}(1 - 3) = 0 \ ,$$

where we have used Proposition 3.6 to conclude $K = -3H$. Then a result of Gromov [10] implies that the symplectic structure is the standard one (up to symplectomorphism):

THEOREM 4.7 ([21, 22]). — *The symplectic structure on $\mathbb{C}P^2$ is unique.*

Example 4.8. — It is instructive to try to apply the above argument to other manifolds with the rational homology of $\mathbb{C}P^2$. If one forms the connected sum of $\mathbb{C}P^2$ with a non–trivial rational homology sphere, the resulting manifold usually has no symplectic structure [13]. If it does admit a symplectic structure, then the generator of the homology cannot be represented by a pseudo–holomorphic rational curve [10].

Mumford [16] found a Kähler surface X uniformised by the complex ball which has the Betti numbers of $\mathbb{C}P^2$. For this X Theorem 3.7 applies, showing that the positive generator of the homology is represented by a pseudo–holomorphic curve. However, Proposition 3.6 fails for X. Indeed, the Seiberg–Witten invariants defined by equations (2) and (3) with $\eta = 0$ and with $\eta = -\frac{i}{4} r\omega$ with r large coincide, as one sees easily using the natural Kähler–Einstein metric on X. Thus, there is no reducible solution with $r > 0$, the wall in the parameter space appears for some $r_0 < 0$. For the Mumford surface one has $K \cdot [\omega] > 0$, which forces $K = 3H$. Thus, the pseudo–holomorphic curve Poincaré dual to H has genus 3 by the adjunction formula (26).

Combining the first part of Theorem 3.2 with Theorem 3.3, one concludes that if X is a symplectic 4–manifold with $b_2^+(X) > 1$ and $K \neq 0$, then K is Poincaré dual to a pseudo–holomorphic curve. From this one deduces:

COROLLARY 4.9 (Taubes [21, 22]). — *If X is a minimal symplectic 4–manifold with $b_2^+(X) > 1$, then $K^2 \geq 0$, equivalently $2\chi(X) + 3\sigma(X) \geq 0$.*

Recall that a symplectic 4–manifold is called *minimal* if it contains no symplectically embedded 2–sphere of self–intersection -1.

Proof. If $K = 0$ there is nothing to prove. If $K \neq 0$, consider the pseudo–holomorphic curve Poincaré dual to K. Because of genericity and minimality, this curve has no genus zero component of negative self–intersection. For the components Σ of positive genus, the adjunction formula (26) implies $1 \leq g(\Sigma) = 1 + \frac{1}{2}(\Sigma^2 + K \cdot \Sigma) = 1 + \Sigma^2$. Thus K^2, which is the sum of the Σ^2 over all components, is non–negative. □

Minimality of symplectic 4–manifolds admits a purely differentiable characterisation. This follows from the next result.

THEOREM 4.10 (Taubes [21, 22]). — *Let X be a symplectic 4–manifold with $b_2^+(X) > 1$. A class $E \in H^2(X, \mathbb{Z})$ with $E^2 = -1$ is Poincaré dual to a smoothly embedded 2–sphere if and only if E or $-E$ is Poincaré dual to a symplectically embedded 2–sphere. In this case $K \cdot E = \pm 1$.*

Proof. As K is a characteristic element for the intersection form of X, we can write $K = K^\perp + aE$, with $K^\perp \cdot E = 0$ and a an odd integer. If E is Poincaré dual to a smoothly embedded 2–sphere, then the reflection in the hyperplane orthogonal to E is realised by a self–diffeomorphism f of X, supported in a tubular neighbourhood of the 2–sphere. This diffeomorphism pulls back the canonical *Spin*^{c} structure to the one twisted by aE. By Theorem 3.3, aE is then Poincaré dual to a pseudo–holomorphic curve C.

Suppose C has several components, then by Fact 4.6 the components are disjoint. As C has negative selfintersection number, at least one of its components, C_1 say, has a negative selfintersection number. It is not hard to see that C_1 is a (-1)–sphere. But C_1 is not orthogonal to E, so that the reflections in E and in C_1 generate a group of diffeomorphisms of X for which the orbit of the canonical *Spin*^{c} structure is infinite. This contradicts Lemma 1.10.

It follows that C is connected. By the adjunction formula (26) it must have genus zero, and we must have $a = \pm 1$. □

Corollary 4.9, Theorem 4.10, and the other consequences of Taubes's theorems constitute substantial steps towards a classification of symplectic 4–manifolds in the spirit of the Kodaira classification of compact complex surfaces. Elaborating on the work of Gromov [10], initial steps in this direction had already been taken, notably by McDuff. However, no results approaching the strength of the above were in sight before the work of Taubes.

Here is a consequence of Theorem 3.3 constraining further the $Spin^c$ structures with non–trivial Seiberg–Witten invariants on symplectic 4–manifolds with $b_2^+ > 1$:

THEOREM 4.11 (Taubes [21, 22]). — *If X is a symplectic 4–manifold with $b_2^+(X) > 1$, then the only non–trivial Seiberg–Witten invariants of X arise from zero–dimensional moduli spaces, i.e., from $Spin^c$ structures with auxiliary line bundles L with $L^2 = 2\chi(X) + 3\sigma(X)$.*

Proof. A non–minimal symplectic 4–manifold smoothly splits as $Y \# \overline{\mathbb{C}P^2}$, and it is known that the claimed property holds for X if and only if it holds for Y. This follows from the blowup–formula for the Seiberg–Witten invariants, see [6] or Proposition 2 of [13].

Thus it suffices to consider the case when X is minimal. By Theorem 3.3, if $E \neq 0$ is a class for which the canonical $Spin^c$ structure twisted by E has a non–trivial Seiberg–Witten invariant, then there is a pseudo–holomorphic curve Poincaré dual to E. Consider a component Σ of this curve. It can not be of genus zero and negative self–intersection because X is assumed minimal. If $g(\Sigma) > 0$, then $\Sigma^2 + K \cdot \Sigma \geq 0$. If the self–intersection of Σ is negative, this implies $E \cdot \Sigma = \Sigma^2 < -\Sigma^2 \leq K \cdot \Sigma$.

If Σ has non–negative self–intersection, the following adjunction inequality holds [14, 5, 1]:

$$g(\Sigma) \geq 1 + \frac{1}{2}(\Sigma^2 + L \cdot \Sigma) \ ,$$

for all L which are determinant bundles of $Spin^c$ structures with non–trivial Seiberg–Witten invariants. Applying this with $L = 2E - K$ and calculating $g(\Sigma)$ from the adjunction formula (26), we conclude $E \cdot \Sigma \leq K \cdot \Sigma$.

We have proved $E \cdot \Sigma \leq K \cdot \Sigma$ for all components Σ, regardless of their self–intersection numbers. Summing over all Σ, we have $E^2 \leq K \cdot E$, which is equivalent to $L^2 = (2E - K)^2 \leq K^2 = 2\chi(X) + 3\sigma(X)$. On the other hand, the dimension of the Seiberg–Witten moduli space for L has to be non–negative, so $L^2 \geq 2\chi(X) + 3\sigma(X)$ by Fact 1.5 and we conclude $L^2 = 2\chi(X) + 3\sigma(X)$. □

Remark 4.12. — The conclusion of Theorem 4.11 can be interpreted as saying that symplectic 4–manifolds have (Seiberg–Witten) *simple type*.

5. TOWARDS THE DIFFEOMORPHISM CLASSIFICATION OF FOUR–MANIFOLDS

5.1. The symplectic uniformisation conjecture

A closed smooth manifold is said to be *irreducible* if it admits no smooth connected sum decomposition in which neither summand is a homotopy sphere. A natural approach to the classification of manifolds is the search for and classification of their irreducible summands. After Taubes's theorems, for a while the following attractive conjecture seemed plausible:

CONJECTURE 5.1. — (1) *Every simply connected smooth 4–manifold is a connected sum of symplectic manifolds, with both the symplectic and the opposite orientations allowed*[3].

(2) *Minimal symplectic 4–manifolds are irreducible.*

(3) *The symplectic structure of a minimal symplectic 4–manifold is unique up to deformation equivalence and symplectomorphism.*

The first part of the conjecture was motivated by an older conjecture[4] to the effect that all simply connected 4–manifolds should be connected sums of complex surfaces, equivalently, of complex projective algebraic surfaces. This was disproved by Gompf–Mrowka [9]. Their counterexamples, and many later ones, have turned out to be symplectic [8], thus suggesting the modified conjecture (1). Six months after the lecture in the Séminaire Bourbaki, as I was revising the text for Astérisque, Z. Szabo announced counterexamples to (1). The manifolds he constructs are irreducible with non–trivial Seiberg–Witten invariants which are incompatible with Theorem 3.2. Immediately after Szabo, Fintushel and Stern announced the existence of large families of related examples.

Part (2) was conjectured by Gompf in [8]. We shall prove in Corollary 5.5 below that as a consequence of Taubes's results it is true (for simply connected manifolds) under the additional assumption $b_2^+ > 1$. Of course, non–minimal symplectic manifolds are reducible because they smoothly split off $\overline{\mathbb{C}P^2}$, and the other summand cannot be a homotopy sphere for cohomological reasons.

Part (3) is true for $\mathbb{C}P^2$ by Theorem 4.7. The other consequences of Taubes's work discussed in the previous section provide evidence that (3) may hold more generally. In higher dimensions, counterexamples to (3) are known [17], and are detected by Gromov invariants. Taubes has shown that the Gromov invariants in dimension 4 are equivalent to the Seiberg–Witten invariants, which are diffeomorphism invariants. One can take this as further evidence for (3).

Remark 5.2. — Making connected sums of symplectic manifolds with non–simply connected manifolds with negative definite intersection forms, one can build manifolds with non–trivial Seiberg–Witten invariants which do not admit any symplectic structures [13].

[3]The 4–sphere is the empty connected sum.

[4]Sometimes attributed to Thom, although he has disassociated himself from it.

This is because the property of being symplectic is preserved under passage to finite coverings, whereas having non–trivial Seiberg–Witten invariants is not.

Some of the examples in [13] have the same Seiberg–Witten invariants as the symplectic manifolds they were built from, whereas for others the Seiberg–Witten invariants are not compatible with the constraints on the invariants of symplectic manifolds proved in subsection 4.1. Recently Biquard [2] has shown that the Seiberg–Witten invariants of properly elliptic surfaces with $b_1 \equiv 1 \pmod 2$ are (non–zero and) incompatible with the constraints.

Remark 5.3. — It is clear that part (1) of Conjecture 5.1 fails in the non–simply connected case, for example because of the existence of non–trivial homology spheres. If one tries to modify the conjecture by allowing arbitrary summands with definite intersection forms it is still false. Counterexamples include irreducible 4–manifolds with fundamental group $\mathbb{Z}_2$ and indefinite intersection forms not admitting any almost complex structure [12], and various $K(\pi, 1)$s, e. g. certain hyperbolic 4–manifolds and the non–Kähler minimal properly elliptic surfaces considered in [2].

5.2. Irreducibility of minimal symplectic 4–manifolds

THEOREM 5.4 ([12]). — *Let X be a minimal symplectic 4–manifold with $b_2^+(X) > 1$. If $X \cong X_1 \# X_2$ is a smooth connected sum decomposition, then one of the X_i is an integral homology sphere whose fundamental group has no non–trivial finite quotient.*

COROLLARY 5.5 ([12]). — *Minimal symplectic 4–manifolds with $b_2^+ > 1$ and with residually finite fundamental groups are irreducible*[5].

Proof of Theorem 5.4. Let X be a closed symplectic 4–manifold with $b_2^+(X) > 1$. If X splits as a connected sum $X \cong M \# N$, then by Proposition 1 of [13] we may assume that N has a negative definite intersection form and that its fundamental group has no non–trivial finite quotient. In particular $H_1(N, \mathbb{Z}) = 0$. This implies that the homology and cohomology of N are torsion–free.

Donaldson's theorem about (non–simply connected) definite manifolds [3] implies that the intersection form of N is diagonalizable over $\mathbb{Z}$. If N is not an integral homology sphere, let $e_1, \ldots, e_n \in H^2(N, \mathbb{Z})$ be a basis with respect to which the cup product form is the standard diagonal form. This basis is unique up to permutations and sign changes.

By Theorem 3.2 the Seiberg–Witten invariants of X are non–trivial for the natural *Spin*^{c} structures with auxiliary line bundles $\pm K_X$. Note that we can write

$$K_X = K_M + \sum_{i=1}^{n} a_i e_i \ ,$$

[5] An assumption about the fundamental group should have been added on page 65 of [5], where this is mentioned for Kähler manifolds.

where $K_M \in H^2(M, \mathbb{Z})$ and the a_i are odd integers because $e_i^2 = -1$ and K_X is characteristic. Considering $-K_X$ and using a family of Riemannian metrics which pinches the neck connecting M and N down to a point, we conclude that M has a non-trivial Seiberg–Witten invariant for a $Spin^c$ structure with auxiliary line bundle $-K_M$.

Now we can reverse the process and glue together solutions to the Seiberg–Witten equation for $-K_M$ on M and reducible solutions on N for the unique $Spin^c$ structure with auxiliary line bundle $e_1 - \sum_{i \neq 1} e_i$, as in the proof of Proposition 2 in [13], cf. also [5], section 5. This gives a Seiberg–Witten invariant of X which is equal (up to sign) to the Seiberg–Witten invariant of M for $-K_M$, which is non-zero.

This implies that $L = -K_M + e_1 - \sum_{i \neq 1} e_i$ has self-intersection number $= K_X^2$ because for X all the non-trivial Seiberg–Witten invariants come from zero-dimensional moduli spaces by Theorem 4.11. Thus, $a_i = \pm 1$ for all $i \in \{1, \dots, n\}$. Without loss of generality we may assume $a_i = 1$ for all i.

The line bundle L is obtained from $-K_X$ by twisting with e_1. Thus, by Theorem 3.3 the non-triviality of the Seiberg–Witten invariant of X with respect to L implies that e_1 can be represented by a symplectically embedded 2–sphere in X. Thus X is not minimal.

We conclude that if X is minimal, then N must be an integral homology sphere. This completes the proof of Theorem 5.4. □

Remark 5.6. — Gompf [8] has shown that all finitely presentable groups occur as fundamental groups of minimal symplectic 4–manifolds, and conjecturally all these manifolds are irreducible. As was the case in [13], our arguments do not give an optimal result because we cannot deal with fundamental groups without non-trivial finite quotients. With regard to Theorem 5.4, note that there are such groups which occur as fundamental groups of integral homology 4–spheres. Let G be the Higman 4–group, an infinite group without non-trivial finite quotients, which has a presentation with 4 generators and 4 relations. Doing surgery on $4(S^1 \times S^3)$ according to the relations produces an integral homology sphere with fundamental group G.

Remark 5.7. — In another direction, the assumption $b_2^+(X) > 1$ can probably be removed from Theorem 5.4 and Corollary 5.5. To do this one needs to understand how the neck–pinching in the proof of Theorem 5.4 and the perturbations in the proofs of Theorems 3.2 and 3.3 interact with the chamber structure of the Seiberg–Witten invariants for manifolds with $b_2^+ = 1$.

However, some results about the case when $b_2^+ = 1$ can be deduced from the proof of Theorem 5.4. For example, all manifolds with non-trivial finite fundamental groups are dealt with by the following:

COROLLARY 5.8 ([12]). — *Let X be a minimal symplectic 4–manifold with $b_2^+(X) = 1$ and $b_1(X) = 0$. If $\pi_1(X)$ is a non-trivial residually finite group, then X is irreducible.*

Proof. Suppose $X \cong M\#N$. We may assume that N has negative definite intersection form and its fundamental group has no non–trivial finite quotient. Residual finiteness then implies that N is simply connected, and $\pi_1(M) \cong \pi_1(X)$. By assumption, X has a finite cover $\overline{X}$ of degree $d > 1$ which is diffeomorphic to $\overline{M}\#dN$, where $\overline{M}$ is a d–fold cover of M. The multiplicativity of the Euler characteristic and of the signature imply $b_2^+(\overline{X}) \geq 3$.

Suppose that N is not a homotopy sphere. Then, as in the proof of Theorem 5.4, Theorem 3.3 shows that in $\overline{X}$ the generators of the second cohomology of the d copies of N are represented by pseudo–holomorphic embedded spheres. Two such spheres have algebraic intersection number zero with each other, and must therefore be disjoint, because intersections of pseudo–holomorphic curves always count positively. The spheres in the different copies are permuted by the covering group, and project to pseudo–holomorphic embedded (-1)–spheres in X. This contradicts the minimality of X. Thus N must be a homotopy sphere, and X is irreducible. □

6. FINAL COMMENTS

As mentioned in the introduction, Theorem 3.3 has a converse [24] which constructs solutions to the Seiberg–Witten equations (2) and (3) (for a suitable η) starting from sufficiently generic pseudo–holomorphic curves. One takes standard solutions to the vortex equation on the curve (the 2–dimensional reduction of the Seiberg–Witten equations), and glues these into trivial solutions in the complement of a tubular neighbourhood of the curve. This produces approximate solutions to the Seiberg–Witten equations which are then perturbed to actual solutions using the implicit function theorem.

If one counts the pseudo–holomorphic curves appropriatly [23], and keeps track of the associated signs [25], one finds that the Gromov invariant they define coincides with the value of the Seiberg–Witten invariant on the *Spin*^{c} structure obtained from the canonical one by twisting with the complex line bundle Poincaré dual to the fundamental homology class of the curves. Taubes has explained in [23] how one has to count the curves for this statement to be true; this counting is as announced in [21] except in the case of tori of self–intersection zero, when additional complications arise from an obstruction to the gluing procedure mentioned above. One has to use multi–vortices in order to overcome the obstructions.

Finally, let us remark that a lot of the analysis in [22] is valid for smooth 4–manifolds equipped with a non–trivial self–dual harmonic form ω (with respect to a generic metric) which does not have to be non–degenerate. Work of Taubes which is now in progress suggests that in this case one obtains pseudo–holomorphic curves in the complement of the vanishing locus of ω which have sufficient regularity near this vanishing locus to be useful in topological arguments. If one is very optimistic, one may hope that this will lead to an "intrinsic" characterisation of symplectic manifolds.

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