Astérisque

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Astérisque, tome 237 (1996), Séminaire Bourbaki,
exp. nº 801, p. 275-293
<http://www.numdam.org/item?id=SB_1994-1995__37__275_0>

Séminaire BOURBAKI
47ème année, 1994-95, n° 801 Juin 1995

MIRROR SYMMETRY IN DIMENSION 3

by **Maxim KONTSEVICH**

1. CALABI-YAU MANIFOLDS

1.1. Definition and first properties

A *Calabi-Yau manifold* is a compact connected Riemannian manifold X of even dimension $2n$ with holonomy group contained in $SU(n) \subset O(2n, \mathbf{R})$. In other words, it is an n-dimensional complex manifold with a Kähler metric with $(1,1)$-form g and a holomorphic volume element non-vanishing everywhere $vol \in \Gamma(X, K_X)$ such that $|vol \wedge \overline{vol}|$ is equal to the Riemannian volume element arising from the metric. The volume element vol defines a holomorphic trivialization of the canonical line bundle $K_X = \wedge^n(T^*_X)$. Thus $h^{n,0}(X) = 1$. In what follows we will not fix the holomorphic volume element, i.e. we are free to multiply it by a non-zero complex number. In 1954 E. Calabi [13] conjectured and in 1976 S.-T.Yau [59] proved the following

Theorem (Yau). *If X is a compact complex manifold with $K_X = 0 \in Pic(X)$ and with a Kähler metric $g \in \Omega^{1,1}(X)$ then there exists a unique Calabi-Yau Kähler metric $g_{CY} \in \Omega^{1,1}(X)$ such that $[g_{CY}] = [g] \in H^{1,1}(X)$.*

Theorem (Bogomolov-Beauville, [12,7]). *If X is a Calabi-Yau manifold then there exists a finite covering $Y \longrightarrow X$ which is isometrically isomorphic to $A \times H \times C$ where*

1) *A is a complex torus $\mathbf{C}^k/\mathbf{Z}^{2k}$ with a flat Kähler metric (the holonomy group of A is trivial),*
2) *H is a 1-connected hyperkähler manifold (the holonomy group of H is contained in the quaternionic unitary group $Sp(l) \subset O(4l, \mathbf{R})$ and $\pi_1(H) = \{id\}$),*
3) *C is a Calabi-Yau manifold in the proper sense, i.e. $\pi_1(H) = \{id\}$ and $H^{2,0}(X) = 0$.*

Notice that the factors of type 3) in this decomposition are always projective algebraic varieties. Also, complex manifolds arising as factors of the second type (the third type respectively) can be characterized as connected simply connected smooth projective varieties with a trivial canonical class which admit a holomorphic symplectic form (for which $h^{1,0} = h^{2,0} = \ldots = h^{dim X-1,0} = 0$ respectively).

1.2. Moduli spaces

Each complex manifold X defines a deformation functor Def_X. For a germ of based analytic space (S, s_0) the set $Def_X(S, s_0)$ is the set of equivalence classes of analytic families of manifolds X_s parametrized by S, $s \in S$ with the fixed isomorphism $X_{s_0} \simeq X$.

Theorem (Bogomolov-Tian-Todorov, [11,53,55,45]). *If X is a Calabi-Yau manifold then the local deformation theory of complex structures on X is unobstructed. In other words, the deformation functor is representable by a germ of complex manifold of dimension equal to $rk\, H^1(X, T_X) = rk\, H^1(X, \wedge^{n-1}T^*_X) = h^{n-1,1}(X)$.*

The group of biholomorphic transformations of any Calabi-Yau manifold is an extension of a complex torus by a discrete group. In general, automorphism groups cause trouble in constructing moduli space. We will overcome this difficulty by considering *marked polarized* Calabi-Yau manifolds. Namely, for a given Calabi-Yau manifold X consider the set of equivalence classes of Calabi-Yau manifolds Y together with an isomorphism of graded rings $i_Y : H^*(Y; \mathbf{Z}) \simeq H^*(X; \mathbf{Z})$ inducing the identification of cohomology classes of Kähler forms. This set has a natural structure of a complex analytic space. We denote its connected component containing X by $\mathcal{M}_X^{marked}$. This space is a complex manifold of dimension equal to $h^{n-1,1}(X)$.

For algebraic X with the integral polarization we can forget marking and get an algebraic space of finite type $\mathcal{M}_X$ (moduli space) with orbifold singularities.

We define the period map from $\mathcal{M}_X^{marked}$ to the complex projective space $P(H^n(X; \mathbf{C}))$ by formula

$$Period((Y, i_Y)) = i_Y(H^{n,0}(Y)) \subset H^n(X; \mathbf{C})$$

It follows from the Kodaira-Spencer theory that the period map is locally an embedding. On the vector space $H^n(X; \mathbf{C})$ we have a pseudo-hermitean form

$$([\alpha], [\beta]) = \int_X \overline{\alpha} \wedge \beta$$

where α and β are smooth closed n-forms on X. This form induces a pseudo-Kähler metric on an open subset of $P(H^n(X;\mathbf{C}))$ by formulas analogous to formulas for the Fubini-Study metrics. The pullback of this pseudo-metric to $\mathcal{M}_X^{marked}$ by the period map is everywhere defined and strictly negative. Thus, after changing the sign we get a canonical Kähler metric on the moduli space called the Weil-Petersson metric.

For complex tori and for hyperkähler manifolds moduli spaces are pretty well understood, they are hermitean symmetric domains locally.

From now on we will consider only 3-dimensional Calabi-Yau manifolds in the proper sense. If X is such a manifold then it has the following diagram of Hodge numbers:

$$(h^{i,j}(X)) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & b & a & 0 \\ 0 & a & b & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad a = h^{1,1}(X),\ b = h^{2,1}(X) \quad .$$

Calabi-Yau metrics on X depend on b complex parameters (moduli of complex structures) and a real parameters (classes of Kähler forms). The cone over the image of the period map is (locally) a complex Lagrangian cone in $H^3(X;\mathbf{C}) \simeq \mathbf{C}^{2b+2}$. The symplectic form on $H^3(X;\mathbf{C})$ is the Poincaré pairing.

1.3. Constructions

A large class of Calabi-Yau manifolds can be obtained as complete intersections in Fano varieties, i.e. complex algebraic manifolds with very ample anticanonical class (see [49,tangent61]). The simplest example is a quintic 3-fold, i.e. a smooth hypersurface in $\mathbf{C}P^4$ given by a homogeneous equation of degree 5 in 5 variables. Our next example is an intersection of two cubics in $\mathbf{C}P^5$. The rule is that the sum of degrees of equations should be equal to the number of variables.

One can replace projective spaces by products of projective spaces, by weighted projective spaces, by flag varieties etc. One of most general construction covering 90% of examples is due to V. Batyrev [4,5], it is based on the consideration of toric Fano varieties.

Also one can start from a Calabi-Yau manifold admitting an action of a finite group preserving the holomorphic volume element and try to resolve singularities of the quotient space. By results of S. Roan, D. Markushevich et al. [47,40,48,39,10] for all finite subgroups $\Gamma \subset SU(3)$ the quotient space $X := \mathbf{C}^3/\Gamma$ admits a resolution of singularities X' with the trivial canonical class. Some other singularities (like the toric ones) also have such resolutions.

Earlier constructions (due to F. Hirzebruch [29]) were obtained following deformation arguments from the next section. Some other constructions were proposed by C. Voisin [56].

Playing with all this hundreds of thousands of families of 3-dimensional manifolds can be constructed, but still up to now only a *finite* number of different families of Calabi-Yau manifolds in the proper sense is known. E. Calabi conjectured that there are finitely many connected families of Calabi-Yau manifolds in any given dimension.

1.4. Rational curves

H. Clemens and R. Friedman [19,22] introduced a construction transforming the topology of 3-dimensional complex manifolds using rational curves. We define $(-1,-1)$-*curve* on X as a smooth complex rational curve $C \subset X$, $C \simeq \mathbf{C}P^1$ with the normal bundle $\mathcal{N}_C = (T_X)_{|C}/T_C$ isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. In a sense, it is typical for rational curves because the degree of the normal bundle is

$$(c_1(T_X) - c_1(T_C))[C] = 0 - c_1(T_C)[C] = -2$$

and in the space of $\overline{\partial}$-connections on a 2-dimensional C^∞-bundle $\mathcal{N}$ over $\mathbf{C}P^1$ with $c_1(\mathcal{N})[\mathbf{C}P^1] = -2$ the set of connections giving holomorphic bundles equivalent to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ is open and dense with the complement of real codimension 2.

It is easy to see that $(-1,-1)$-curves are isolated and do not disappear after small deformations of the complex structure on X. As we will see, it is reasonable to expect infinitely many $(-1,-1)$-curves on Calabi-Yau manifolds.

H. Clemens' idea was to take a finite set of non-intersecting $(-1,-1)$-curves $\{C_i\}_{i\in I}$, contract each of them to a point and try to deform the resulting analytic space X' into a smooth manifold via flat deformations.

Locally all $(-1,-1)$-curves are alike. There exists a neigbourhood of C_i in X analytically isomorphic to a neigbourhood of the zero section in the total space of the vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbf{C}P^1$. The result of the contraction of C_i into a point is an analytic space with the simple quadratic singularity looking like

$$\{(x_1, x_2, x_3, x_4) \in \mathbf{C}^4 | \sum x_j^2 = 0\} \quad .$$

In fact, this singular space can be obtained as a result of contraction of a $(-1,-1)$-curve on another 3-dimensional complex manifold $\tilde{X}$ with the trivial canonical class. This manifold $\tilde{X}$ is not necessarily a Kähler one. The passing from X

to $\tilde{X}$ is called *flop* and it is very important in Mori's theory of minimal models of algebraic varieties.

We can try to deform the complex structure on X' outside singular points and modify it near these points as

$$\{(x_1, x_2, x_3, x_4) \in \mathbf{C}^4 | \sum x_j^2 = \epsilon_i\}$$

where ϵ_i, $i \in I$ are small parameters. Topologically it is a simple surgery: we replace $(S^2 \times D^4)_i$ in X by $(D^3 \times S^3)_i$ with the common boundary $(S^2 \times S^3)_i$.

Theorem (Tian [54]). *The deformation theory of X' is unobstructed. If we choose a holomorphic volume element on X then the tangent space $\mathbf{T}$ to the local moduli space at the base point is inserted naturally into the following exact sequence:*

$$0 \longrightarrow H^1(X, T_X) \longrightarrow \mathbf{T} \longrightarrow \mathbf{C}^I \longrightarrow H_2(X; \mathbf{C})$$

where the last map is defined on the base vectors as $i \mapsto [C_i] \in H_2(X; \mathbf{C})$.

One can normalize holomorphic volume elements on deformed manifolds by the condition $\int_\gamma vol = 1$ where $\gamma \in H_3(X; \mathbf{Z})$ is a non-trivial cycle. Local parameters ϵ_i can be defined as integrals of normalized forms vol over the vanishing cycles $(S^3)_i$. One can check that the vector of ϵ_i lies in the kernel of the last arrow in Tian's theorem.

Hence if we have enough rational curves C_i such that fundamental classes of C_i span $H_2(X; \mathbf{Z})$ and there is a linear relation $\sum_i \epsilon_i [C_i] = 0 \in H_2(X; \mathbf{C})$ with all numbers ϵ_i non-zero, then there is a deformation Y of X' which is a smooth space with $\pi_1(Y) = \{id\}$ and $H_2(Y; \mathbf{Z}) = 0$. By a classification theorem of C. T. Wall [57], the manifold Y is diffeomorphic to the connected sum of several copies of $S^3 \times S^3$.

Complex manifolds appearing in such a way are not Kähler ones because the second Betti number is zero. Nevertheless there is a pure Hodge structure on their cohomology by deformation arguments. Hence we can realize moduli spaces from section 1.2 as "boundaries" of larger smooth moduli spaces with natural Kähler metrics. Also we can increase dimensions of these moduli spaces by contracting/deforming more and more rational curves. M. Reid [46] conjectured that the moduli spaces of *all* 3-dimensional Calabi-Yau manifolds (in the proper sense) can be connected in such a way. It seems to be true. Physicists proved [26], without using computers, by purely abstract arguments that all the thousands of complete intersections in products of projective spaces are on the boundary of only one connected component of moduli space of complex structures on $(S^3 \times S^3)\# \ldots \#(S^3 \times S^3)$.

2. STRINGS AND MIRROR SYMMETRY

2.1. Few words about string theory

String theory is a project for the Grand Unification of all interactions in Nature. The total Feynman integral is taken over the space of maps from all surfaces to the target pseudo-riemannian manifold M. The action functional is the Dirichlet functional plus some fermionic terms. The theory is supersymmetric only for 10-dimensional manifolds with special metrics close enough to Einstein metrics. In order to have a chance to be related to the physical world this manifold should be approximately equal to the product of the Minkowski space $\mathbf{R}^4$ and a six-dimensional manifold X of very small size. This X is essentially the Calabi-Yau 3-fold. The spectrum of particles of the physics arising in 4 dimensions is determined by the string theory on X.

It was later realized that one can replace X by any $(2,2)$-supersymmetric conformal field theory with the central charge $\hat{c} = 3$. We want to mention here that the whole notion of conformal field theory is completely rigorously defined in mathematical terms [52,23]. One of important unresolved problems is the description of the conformal theory corresponding to a given Calabi-Yau manifold.

In fact, there is an additional parameter in theory. It is called a B-field and can be identified with an element of $\sqrt{-1}H^2(X;\mathbf{R})$. This field multiplies the contribution of each map ϕ from a closed surface Σ to X by the topological factor $\exp\left(\int_\Sigma \phi^*(B)\right)$. The total moduli space of conformal field theories close to the theory associated with a given Calabi-Yau manifold X has the structure of the product of two Kähler complex manifolds of dimensions $a = h^{2,1}(X)$ and $b = h^{1,1}(X)$. The field B and a real parameters of the Kähler class of X together form complex coordinates in a domain of $H^2(X;\mathbf{C})$. The string theory on X is not well-defined when the size of X is too small. In a sense the Feynman integral is not convergent in this case.

There are two classes of observables in $N = 2$-theory which depend only on a part of parameters [58]. In the *B-model* correlators depend only on the complex structure of X and in the *A-model* they depend only on the symplectic structure of X together with the B-field. Moreover, one can argue that the correlators are holomorphic functions of parameters.

2.2. Discovery of Mirror Symmetry

The correspondence between Calabi-Yau manifolds and conformal field theories is locally one-to-one. Globally there is no reasons for this, and there is no natural way to reconstruct the manifold from its string theory. For example, for the target

($\mathbf{R}^{24}$/Leech lattice) the automorphism group of a version of the superstring theory is the Monster group, i.e. much larger than the symmetry group of the target.

In supersymmetric conformal field theories there is no intrinsic difference between the A and B model. Based on this, Lerche-Vafa-Werner [35] and Dixon [20] conjectured that Calabi-Yau manifolds come in pairs giving equivalent string theories. The A-model on one manifold X is equivalent to the B-model on the dual manifold Y and vice versa. Hodge diagrams of dual manifolds should be mirror reflections of each other, $h^{i,j}(X) = h^{3-i,j}(Y)$. At the same time Green and Plesser [25] proposed a first explicit pair of points in the moduli of Calabi-Yau manifolds as candidates to the Mirror symmetry and proposed very convincing arguments. Physicists [15, 27] made a table of Hodge numbers (a, b) for all 7868 families of Calabi-Yau manifolds arising as complete intersections in products of projective spaces. Very surprisingly the plot of this numbers in the plane $\mathbf{Z}^2$ was almost symmetric, with only a few ($\simeq 10$) exceptions.

The first actual calculation by Candelas et al. [17] of correlators gives an extremely beautiful prediction relating numbers of rational curves and Picard-Fuchs equations. We reproduce the results of their calculations in 3.3. Physicists realized that one can make explicit calculations in both A and B models by a standard trick in supersymmetry. One can show formally that the Feynman integral in the A-model is equivalent to the summation over all holomorphic curves on X and the B-model corresponds to the Hodge theory. The reason is that the Feynman integral over the space of all maps can be localized to the space of holomorphic maps and constant maps respectively.

3. PREDICTIONS FROM MIRROR SYMMETRY

3.1. Results from symplectic topology

Let X be a 3-dimensional Calabi-Yau manifold on which all rational curves are smooth $(-1,-1)$-curves and they do not intersect each other. Then there is a finite number n_d of these curves in each degree $d = [C] \in H_2(X;\mathbf{Z})$, $d \neq 0$. This number does not change if we vary the complex structure on X a little bit. We want to define analogous numbers for an arbitrary 3-dimensional Calabi-Yau manifold, also including curves of higher genus. The simplest way to do it, after Y. Ruan [50], is the following.

We perturb generically the almost-complex structure on X leaving it compatible in the evident way with the symplectic Kähler form. Then, by Gromov's theorem [28] there is a nice compactification of the space of smooth holomorphic curves $C \subset X$ of a given area (equivalently, of a given homology class). By theorems of D. McDuff [41] this compact space is stratified by smooth manifolds with the dimension given by indices of appropriate $\overline{\partial}$-operators. All this works for an arbitrary compact symplectic manifold. The case of $c_1(T_X) = 0$ and $dim_{\mathbf{R}}(X) = 6$ is exceptional because this index is equal to zero for all degrees d and all genera g. Thus the set of curves of given degree and genus is finite.

We associate with any such curve a sign with values in $\{+1, -1\}$. Namely, each curve is a solution of a non-linear differential equation (the Cauchy-Riemann equation). We can linearize the problem near each solution and get an invertible linear differential operator

$$\overline{\partial}' : \Gamma(C, \mathcal{N}_C) \longrightarrow \Gamma\left(C, \mathcal{N}_C \otimes \left(T^{0,1}\right)^*\right) .$$

This operator acts from a complex vector space to another complex vector space, but it is not $\mathbf{C}$-linear. Nevertheless the principal symbol of $\overline{\partial}'$ is $\mathbf{C}$-linear. We can choose an invertible $\mathbf{C}$-linear operator $\overline{\partial}$ with the same principal symbol. The quotient $\overline{\partial}' \circ (\overline{\partial})^{-1}$ is an invertible $\mathbf{R}$-linear operator of the form (Id + compact operator). The space of such operators has two connected components labeled by the sign of the determinant of a finite-dimensional approximation. This is the sign which we attach to curves.

We define the "number of curves" $n_{d,g} \in \mathbf{Z}$ as the sum of signs over all curves of degree d and genus g.

Theorem (Ruan). *The number $n_{d,g}$ is independent of the choice of generic almost-complex structure. It is invariant under continuous deformations of the symplectic form on X.*

In fact, Y. Ruan made this statement only for genus zero curves, but his argument works for higher genera too. If $n_{d,0} < 0$ for some d then there are unavoidably whole continuous families of rational curves for arbitrary integrable complex structure compatible with the symplectic form. It follows from the positivity of multiplicities of complete intersections in algebraic/analytic geometry.

The number of curves $n_{d,g}^{phys}$ used in string theory is not the same as the number defined above. It is not an integer number in general. The reason is that in string theory one wants to count in a sense the number of equivalence classes of holomorphic maps from curves to X. Any such a map is a composition of a (ramified) covering map $C \longrightarrow C'$ and an embedding $C' \hookrightarrow X$. The space of ramified coverings has a positive dimension, and one needs an additional perturbation argument in an auxiliary space which is an infinite-dimensional orbifold with finite isotropy groups. The intersection theory on orbifolds relevant to physics contains non-trivial denominators.

For example, ramified coverings of degree k of a $(-1,-1)$-curve give the contribution equal to $1/k^3$. This formula was proposed by P. Aspinwall and D. Morrison [3] and recently checked by Yu. Manin [38] using a new definition [34] of numbers $n_{d,g}^{phys}$ (so called Gromov-Witten invariants) which we will not reproduce here. Hence

$$n_{d,0}^{phys} = \sum_{k|d} \frac{1}{k^3} n_{d/k,0} \ .$$

3.2. Two Lagrangian cones

For a 3-dimensional Calabi-Yau manifold X with $h^{1,0}(X) = 0$ we define two complex lagrangian cones $\mathcal{L}_A(X)$ and $\mathcal{L}_B(X)$. The second cone $\mathcal{L}_B(X)$ is simply the cone over the image of the period map. It lies in the symplectic vector space $H^{odd}(X; \mathbf{C}) = H^3(X; \mathbf{C})$.

In order to describe the first cone we need to construct a certain analytic function (prepotential) in an open domain of the complex vector space $H^2(X; \mathbf{C})$:

$$F([\omega]) := \frac{1}{3!}\int_X \omega^3 + \sum_d n_{d,0}\mathrm{Li}_3\left(\exp\left(\int_d \omega\right)\right) = \frac{1}{3!}\int_X \omega^3 + \sum_d n_{d,0}^{phys}\exp\left(\int_d \omega\right)$$

Here $\omega \in \Omega^2(X)$ is a closed 2-form and $\mathrm{Li}_3(z) = \sum_{k=1}^{\infty} z^k/k^3, \ \ |z| < 1$ is the usual 3-logarithm function. One expects that the series defining F converges absolutely in some domain "$\mathrm{Re}[\omega] \longrightarrow -\infty$". The cubic term in the formula for F represents the contribution of constant holomorphic maps from rational curves to X.

We define an analytic function $F^{(2)}$ of homogeneity degree 2 in a conical domain of the vector space $V := H^2(X; \mathbf{C}) \oplus \mathbf{C}$ by formula $F^{(2)}(x,t) := t^2 F(\frac{x}{t})$ where $x \in H^2(X; \mathbf{C})$ and $t \in \mathbf{C} \setminus \{0\}$. Let us consider the graph of the differential of $F^{(2)}$. It is clear that it is a Lagrangian cone in the symplectic vector space $V \oplus V^* \simeq H^{even}(X)$. We define $\mathcal{L}_A(X)$ to be the analytic continuation of this graph.

Mirror Conjecture. *For dual varieties X, Y, the associated cones are equivalent after linear symplectic transformations:*

$$\mathcal{L}_A(X) \simeq \mathcal{L}_B(Y), \ \ \mathcal{L}_B(X) \simeq \mathcal{L}_A(Y) \ .$$

Some open domains in projectivizations of cones $\mathcal{L}_A(X)$, $\mathcal{L}_A(Y)$ can be identified with some open domains in $H^2(X;\mathbf{C})$, $H^2(Y;\mathbf{C})$ respectively. Mirror symmetry give rise to a certain biholomorphic map between these domains in affine spaces and domains in the moduli spaces of dual manifolds. This map is called the *mirror map*. Affine coordinates on the second cohomology correspond to so-called *flat coordinates* on moduli spaces.

3.3. Example of quintics

The most popular example is a quintic 3-fold. Hodge numbers here are $a = 1$ and $b = 101$. The dual family of varieties consists of resolutions of singularities of quotient spaces

$$\{(x_1 : \ldots : x_5) \in \mathbf{C}P^4 | \sum_j x_j^5 = z^{-1/5} \prod_j x_j\} / (\mathbf{Z}/5\mathbf{Z})^3 \ , \quad z \in \mathbf{C} \text{ is a parameter}$$

where the group $(\mathbf{Z}/5\mathbf{Z})^3 \subset PGL(5,\mathbf{C})$ acts by diagonal transformations $x_j \mapsto \xi_j x_j$ preserving the equation from above and the volume element $dx_1 \wedge \ldots \wedge dx_5$:

$$(\mathbf{Z}/5\mathbf{Z})^3 \simeq \{(\xi_j) | \xi_j^5 = 1, \ \prod_j \xi_j = 1\} / \{(\xi_j) | \xi_1 = \ldots = \xi_5 = \xi, \ \xi^5 = 1\} \ .$$

The cone $\mathcal{L}_A(X)$ comes from the function in one variable

$$F(t) = \frac{5}{6} t^3 + \sum_{d \geq 1} n_d \mathrm{Li}_3 \left(e^{dt}\right)$$

which is defined for $\mathrm{Re}(t) < t_0 = -7.590\ldots$.

The variation of Hodge structures on 1-parameter family of dual manifolds Y can be described by a fourth-order linear differential equation:

$$\left(\left(z \frac{d}{dz} \right)^4 - 5z(5z\frac{d}{dz} + 1)(5z\frac{d}{dz} + 2)(5z\frac{d}{dz} + 3)(5z\frac{d}{dz} + 4) \right) \psi(z) = 0 \ .$$

It has four linearly independent solutions in domain $|z| \ll 1, \ |\mathrm{Arg}\ z| \ll 1$:

$$\psi_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n$$

$$\psi_1(z) = \log z \cdot \psi_0(z) + 5 \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} \left(\sum_{k=n+1}^{5n} \frac{1}{k} \right) z^n$$

$$\psi_2(z) = \frac{1}{2}(\log z)^2 \cdot \psi_0(z) + \dots$$

$$\psi_3(z) = \frac{1}{6}(\log z)^3 \cdot \psi_0(z) + \dots$$

The Mirror prediction [17, 42] is the following identity:

$$F\left(\frac{\psi_1}{\psi_0}\right) = \frac{5}{2} \cdot \frac{\psi_1\psi_2 - \psi_0\psi_3}{\psi_0^2} \quad .$$

One can write easily a computer program and get numbers

$$n_1 = 2875, \; n_2 = 609250, \; n_3 = 317206375, \; n_4 = 242467530000, \dots$$

Miraculously, all numbers n_d coming from the mirror prediction are integers. It is still not proven. It is known that all rational curves up to degree 4 on generic quintics are smooth $(-1,-1)$-curves. Results of the mirror prediction for generic quintics were confirmed by more or less direct algebro-geometric calculations up to degree 4 [21,34]. Another remarkable virtue is that the exponent of the ratio of periods appearing in the mirror map expands into a series with integral coefficients in appropriate algebraic coordinates on the moduli space of complex structures of Y. In the case of quintics one has

$$\exp\left(\psi_1(z)/\psi_0(z)\right) \in z + z\mathbf{Z}[[z]] \quad .$$

This fact was recently proved by B. Lian and S.-T. Yau [36] for complete intersections in projective spaces using p-adic results of B. Dwork.

3.4. Other examples

There are hundreds of other manifolds for which numbers of rational curves of small degrees were computed and mirror predictions checked (see [21,33,37,6,56,30, 31,16,18]).

An evident trouble is that there are rigid Calabi-Yau manifolds X with $h^{2,1}(X) = 0$ which cannot have any mirror manifold Y because $h^{1,1}(Y) > 0$ for any Kähler Y. Still one can count curves on X and make generating functions. Physicists conjectured that the dual variation of Hodge structures comes from certain higher dimensional Fano varieties [14]. For example, the H^7 of cubics in $\mathbf{C}P^8$ looks like the H^3 of a non-existing mirror to one of rigid manifolds. Mathematically more natural possibility of Hodge structures on non-Kähler 3-dimensional manifolds, as in 1.4, has not been explored yet.

The most general construction of dual manifolds for complete intersections in toric varieties was proposed by V. Batyrev and L. Borisov [5] in terms of the usual duality between convex polyhedra.

P. Aspinwall, B. Green and D. Morrison [2] studied the behavior of A and B models under birational transformations. It seems that both cones are invariant under such transformations. For the B-model it is clear because the moduli space and the variation of Hodge structure of birationally equivalent manifolds are essentially the same. If we apply the simplest birational transformation (the flop) then we get new numbers of curves but the whole analytic continuation of the cone of the A-model will be the same.

We do not discuss higher dimensional generalizations here (see [43,44,24,1,32]).

4. HOLOMORPHIC ANOMALY EQUATIONS

We describe in this section predictions for numbers of curves of positive genus from remarkable papers by Bershadsky, Cecotti, Ooguri and Vafa [8,9]. It is impossible to explain here all the arguments of physicists. We just formulate final results in mathematical terms.

4.1. Flat coordinates on Kähler manifolds

The moduli space of Calabi-Yau manifolds carries a natural Kähler metric (Weil-Petersson metric). We describe now a general construction applicable to arbitrary manifold M with a real-analytic Kähler form ω.

Denote by $\overline{M}$ the same manifold M endowed with the complex structure conjugate to the original one. The diagonal submanifold M^{diag} of $M \times \overline{M}$ is totally real. Hence the differential form ω on M^{diag} has the analytic continuation to the holomorphic form $\omega^{\mathbf{C}}$ in a neighbourhood U of M^{diag}. Thus U is a complex symplectic manifold. By the Kähler property submanifolds $M \times \{\overline{m}\} \cap U$ where $\overline{m} \in \overline{M}$ are Lagrangian. It means that we have a Lagrangian foliation of U. It is well known that leaves of such a foliation carry a natural flat affine structure. Hence we construct holomorphic affine structures on open subsets of M depending antiholomorphically on points of M. For functions on affine space there are canonically defined higher derivatives which are symmetric covariant tensors.

Résumé. *For any smooth function f on a Kähler manifold M we defined its canonical higher derivatives $\partial^k(f) \in \Gamma\left(M, S^k\left(T_M^{1,0}\right)^*\right)$.*

Analogously, if L is a hermitean line bundle over M such that the curvature form of L is proportional to ω and s is a smooth section of L then higher derivatives of s are defined. The reason is that the pullback of L to $M \times \overline{M}$ carries a flat connection along the Lagrangian foliation as above.

4.2. Equations

Let X be a 3-dimensional Calabi-Yau manifold and $\mathcal{M}_X$ be its moduli space. We consider now $M = \mathcal{M}_X$ not as an algebraic space but as an orbifold. It carries the Weil-Petersson metric ω_{WP} and a hermitean line bundle L with fiber over each point (X) equal to $\left(H^{3,0}(X)\right)^*$.

Physicists claim that there are canonical global objects

$$\partial^k(F_g) \in \Gamma\left(M, S^k\left(T^{1,0}\right)^* \otimes L^{2-2g}\right)$$

for all integers $g, k \geq 0$ obeying inequality $2 - 2g - k < 0$. This inequality is exactly the condition of hyperbolicity for surfaces of genus g with k punctures. The first constraint on these objects is that locally there exist sections F_g of L^{2g-2} such that global sections $\partial^k(F_g)$ are its derivatives. Thus everything can be obtained from $\partial^3(F_0)$, $\partial^1(F_1)$ and F_g for $g \geq 2$.

The second constraint is an explicit formula for $\partial^3(F_0)$. Namely, the Kodaira-Spencer theory defines a map $T_{(X)}M \longrightarrow H^1(X, T_X)$. Its third tensor power gives a map of vector bundles $S^3(T_{(X)}M) \longrightarrow H^3(X, \wedge^3 T_X) = L^2_{|(X)}$. This tensor field is equal to $\partial^3(F_0)$ after obvious identifications. Because this construction is complex-analytic we have

0) $\overline{\partial}(\partial^3(F_0)) = 0$. In general one can write the formula for $\overline{\partial}(\partial^k(F_0))$, $k \geq 4$ with the r.h.s. equal to an expression quadratic in $\partial^l(F_0)$, $l < k$ and linear in $\overline{\partial^3(F_0)}$.

Holomorphic anomaly equations:

1) $\overline{\partial}(\partial^1(F_1)) = \omega_0 - \frac{\chi(X)}{24}\omega_{WP}$.

Here $\chi(X)$ is the Euler characteristic of X and ω_0 is certain canonical $(1,1)$-form on M. Namely, there are holomorphic vector bundles $H^{i,j}$ over M with fibers at (X) equal to $H^j(X, \wedge^i T_X^*)$). By Hodge theory these bundles are endowed with natural hermitean forms. We define closed $(1,1)$-form ω_0 on M by formula

$$\omega_0 = -\frac{1}{2}\sum_{j=0}^{3} j\, c_1(H^{j,3-j}, \text{hermitean form}) \ .$$

2) For $g \geq 2$ we have

$$\overline{\partial}(F_g) = 1/2 \times Contraction\left(\overline{\partial^3(F_0)} \otimes \left(\partial^2(F_{g-1}) + \sum_{r=1}^{g-1} \partial^1(F_r)\partial^1(F_{g-r})\right)\right)$$

The operator *Contraction* maps sections of $S^3\left(\left(T_M^{0,1}\right)^*\right)\otimes S^2\left(\left(T_M^{1,0}\right)^*\right)\otimes \overline{L}^2\otimes L^{4-2g}$ to sections of $(T_M^{0,1})^*\otimes L^{2-2g}$. It is essentially the contraction with natural hermitean forms on T_M^* and L.

One can deduce formulas for $\overline{\partial}(\partial^k(F_g))$ and get expressions quadratic in previous objects and linear in $\overline{\partial^3(F_0)}$. Terms in this formula correspond to the boundary divisors of the moduli space of stable complex curves of genus g with k punctures.

It is not possible to compute F_g inductively using these equations. In each step one has certain indeterminacy. Namely, one can add to F_g any global holomorphic section of L^{2-2g} satisfying certain growth conditions at infinity (see 4.3). Thus the number of unknown parameters is finite. Up to now it is unknown how to fix this indeterminacy and what is the Hodge-theoretic meaning of holomorphic anomaly equations.

The generating function $\mathcal{F}$ of all F_g is (locally) a function on the total space of bundle L^*. It satisfies a nice system of differential equations of type $\overline{\partial}(\exp(\mathcal{F}))=\partial^2(\exp(\mathcal{F}))$.

4.3. Predictions

Flat structure in an open domain $\mathcal{U}$ of the moduli space $\mathcal{M}_Y$ arising from the Mirror symmetry is the limit of canonical affine structures (from 4.1) when the base point $\overline{m}\in\mathcal{M}_Y$ tends to a point "$-\infty$" on some compactification of the moduli space, which is usually an intersection of normally crossing compactification divisors. Also the line bundle L is trivialized in $\mathcal{U}$. Thus all objects $\overline{\partial}(\partial^g(F_k))$ can be considered as real-analytic functions on a domain of $H^2(X;\mathbf{C})$ after applying the mirror map. We can consider them as analytic functions in two groups of variables $\left([\omega],\overline{[\omega]}\right)$, where again ω is a closed 2-form on X. The next step is to evaluate these functions at the limit $\overline{[\omega]}\longrightarrow-\infty$ using the analytic continuation. We get as a result holomorphic symmetric tensor fields on a domain of $H^2(X;\mathbf{C})$.

The predictions of Mirror symmetry are as follows:

genus=0: $\partial^3(F_0)_{|\overline{[\omega]})=-\infty}=\partial^3(F):=\partial^3\left(\frac{1}{3!}\int_X\omega^3+\sum_d n_{d,0}^{phys}\exp\left(\int_d\omega\right)\right)$.

This symmetric 3-tensor is called the *Yukawa coupling*.

genus=1: $\partial^1(F_1)_{|\overline{[\omega]})=-\infty}=\partial^1\left(\frac{1}{24}\int_X\omega\wedge c_1(T_X)+\sum_d n_{d,1}^{phys}\exp\left(\int_d\omega\right)\right)$

genus$\geq$ 2: $F_g([\omega])_{|\overline{[\omega]})=-\infty}=constant_g\chi(X)+\sum_d n_{d,g}^{phys}\exp\left(\int_d\omega\right)$. The constant term comes from the contribution of constant maps to X.

Rational numbers $n_{d,g}^{phys}$ are not equal to integers $n_{d,g}$. The difference comes again from multiple coverings. In genus 1 we have the following relations:

$$n_{d,1}^{phys} = \sum_{a,b:\, ab|d} \frac{1}{a} n_{d/ab,1} + \frac{1}{12} \sum_{a|d} \frac{1}{a} n_{d/a,0} \quad .$$

Again, a few predictions for elliptic curves were checked by algebraic geometers.

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