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QUANTUM INTEGRABLE SYSTEMS

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1. INTRODUCTION

A quantum mechanical system is a triple consisting of an associative algebra with involution $\mathcal{A}$ called the algebra of observables, its irreducible $\star$-representation π in a Hilbert space $\mathcal{H}$ and a distinguished self-adjoint observable H called the Hamiltonian. Typical question to study is the spectrum and the eigenvectors of $\pi(H)$. Quantum mechanical systems usually appear with their classical counterparts. By definition, a classical mechanical system is again specified by its algebra of observables $\mathcal{A}_{cl}$ which is a commutative associative algebra equipped with a Poisson bracket (i.e. a Lie bracket which is a biderivation with respect to the associative algebra structure), and a Hamiltonian $H \in \mathcal{A}_{cl}$. The spectrum of $\mathcal{A}_{cl}$ is a Poisson manifold $\mathcal{M}$ (whenever its definition makes sense), and Hilbert space representations are now replaced by restriction of functions to Poisson submanifolds (in particular, to the symplectic leaves) of $\mathcal{M}$. A classical system is called *integrable* if the commutant of the Hamiltonian in $\mathcal{A}_{cl}$ contains an abelian algebra of maximal possible rank. (A technical definition is provided by the well known Liouville theorem.)

It is much less clear what should be called a quantum integrable system. To a certain extent the answer remains pragmatic: one can say it's the sort of systems which are studied by the experts in quantum integrability! Of course, numerous exactly solvable quantum mechanical models were studied in Quantum Mechanics from its early days. Starting with the works of H. Bethe, L. Hulthen, L. Onsager, E. Lieb an entire universe of exactly solvable models of Quantum Statistical Physics has been gradually discovered, these developments coming to a culminating point in the works of R. Baxter [B]. On the other hand, and quite independently of that, there was in the late 60's and in the 70's a major breakthrough in the study of classical integrable systems which led to creation of the Classical Inverse Scattering Method. (It is virtually impossible to give here a summary of this development which was started by the famous papers of M. Kruskal et al [GGKM] and P. Lax [L]. A good introduction close to the ideas of QISM may be found in [FT2].) A synthesis of these two independent trends was achieved in the Quantum Inverse Scattering Method (QISM) proposed in the late 70's by L. D. Faddeev, E. K. Sklyanin, and L. A. Takhtajan [STF] [FT1] [F1] [F2]. The underlying algebraic structures which were implicit in QISM have proved to be extremely interesting; among other things, this has led to the discovery of q-deformed affine algebras and to the general theory of quantum groups [D]. The language of Quantum Groups naturally incorporates several important aspects of QISM; however, some of its more elaborate parts (in particular, the algebraic Bethe ansatz and its generalizations which are used to study the spectra and the eigenfunctions of quantum Hamiltonians) are still not too widely known and in fact, still require much work for their proper understanding.

The origins of QISM lie in the study of concrete examples; it is designed as a working machine which produces quantum systems together with their spectra, the quantum integrals of motion, and their joint eigenvectors. In the same spirit, Classical Inverse Scattering Method (along with its ramifications) is a similar tool to produce examples of classical integrable systems together with their solutions. Despite the great diversity of these examples, the underlying construction (both in classical and in quantum cases) is fairly uniform. The first key observation is that integrable systems always have an ample hidden symmetry. (Fixing this symmetry provides some rough classification of associated examples. A nontrivial class of examples is related to loop algebras or, more generally, to their q-deformations. This is the class of examples I shall consider below.) Two other key points are the role of (classical or quantum) R-matrices and of the Casimir elements which give rise to the integrals of motion. This picture brings together all main ingredients of the theory (classical and quantum R-matrices, Lax operators, factorization problems);

it appears in several different guises which depend on the type of examples in question. The simplest (so called linear) case corresponds to classical systems which are modelled on coadjoint orbits of Lie algebras; a slightly more complicated group of examples are classical systems modelled on Poisson Lie groups or their Poisson submanifolds. In the quantum setting the difference between these two cases is deeper: while in the former case the hidden symmetry algebra remains the same, quantization of the latter requires the full machinery of Quantum Groups. Still, for loop algebras the quantum counterpart of the main construction is nontrivial even in the linear case; the point is that the universal enveloping algebra of a loop algebra has a trivial center which reappears only after central extension at the critical value of the central charge. Thus to tackle with the quantum case one needs the full machinery of representation theory of loop algebras. (By contrast, in the classical case one mainly deals with the evaluation representations which allow to reduce the solution of the equations of motion to a problem in algebraic geometry.) The corresponding construction for an important particular model (*the Gaudin model*) is a recent result of B. Feigin, E. Frenkel and N. Reshetikhin [FeiFrR]; remarkably, it allows to include into the Lie-algebraic picture the generalized Bethe ansatz and provides natural links with Conformal Field Theory. A similar treatment of quantum models related to q-deformed affine algebras is also possible, although the results in this case are still incomplete. (One should warn that much of the 'experimental material' on Quantum Integrability still resists general explanations. I'd like to mention in this respect deep results of E. K. Sklyanin [S2] relating Bethe ansatz to the separation of variables.)

To fix up the ideas I shall first discuss the classical case.

2. CLASSICAL CASE

The key idea used to study classical integrable systems is to bring them into *Lax form.* Let $(\mathcal{A}, \mathcal{M}, H)$ be a classical mechanical system. Let $\mathcal{F}_t : \mathcal{M} \to \mathcal{M}$ be the associated flow on $\mathcal{M}$ (defined at least locally). Suppose that $\mathbf{g}$ is a Lie algebra. A mapping $L : \mathcal{M} \to \mathbf{g}$ is called a *Lax representation* of $(\mathcal{A}, \mathcal{M}, H)$ if the following conditions are satisfied:

(i) The flow $\mathcal{F}_t$ factorizes over $\mathbf{g}$, i.e. there exists a (local) flow $F_t : \mathbf{g} \to \mathbf{g}$ such that the following diagram is commutative.

$$\begin{array}{ccc} & \mathcal{F}_t & \\ \mathcal{M} & \rightarrow & \mathcal{M} \\ L \downarrow & F_t & \downarrow L \\ \mathbf{g} & \rightarrow & \mathbf{g} \end{array}$$

(ii) The quotient flow F_t on $\mathbf{g}$ is isospectral, i.e. it is tangent to adjoint orbits in $\mathbf{g}$.

Clearly, $x \circ L \in \mathcal{A}$ for any affine coordinate x on $\mathbf{g}$; thus we may regard L as an element of $\mathbf{g} \otimes \mathcal{A}$; the Poisson bracket on $\mathcal{A}$ extends to $\mathbf{g} \otimes \mathcal{A}$ by linearity. Property (ii) means that there exists an element $M \in \mathbf{g} \otimes \mathcal{A}$ such that $\{H, L\} = [L, M]$.

Let (ρ, V) be a (finite-dimensional) linear representation of $\mathbf{g}$. Then $L_V = \rho \otimes id(L) \in EndV \otimes A$ is a matrix valued function on $\mathcal{M}$; the coefficients of its characteristic polynomial $P(\lambda) = \det(L_V - \lambda)$ are integrals of the motion.

One may replace in the above definition a Lie algebra $\mathbf{g}$ with a Lie group G; in that case isospectrality means that the flow preserves conjugacy classes in G.

There is no general way to find a Lax representation for a given system (even if it is known to be completely integrable). However, there is a systematic way to produce examples of such representations.

The following basic construction (summarized in Theorem 1 below) goes back to Kostant [K] and Adler [A] in some crucial cases. Recall first of all that the symmetric algebra $S(\mathbf{g})$ of any Lie algebra $\mathbf{g}$ is equipped with a natural Poisson bracket which extends the Lie bracket on $\mathbf{g}$ (it is sometimes called the *Lie-Poisson bracket* of $\mathbf{g}$); its center coincides with the subalgebra $I = S(\mathbf{g})^{\mathbf{g}}$ of $ad\mathbf{g}$-invariants in $S(\mathbf{g})$ (called Casimir elements). The elements of I thus generate trivial dynamics in $S(\mathbf{g})$. However, one may still use Casimir elements to produce nontrivial equations of motion if the Poisson structure on $S(\mathbf{g})$ is properly modified. The formal definition is as follows. Let $r \in End\mathbf{g}$ be a linear operator (*classical r-matrix)* satisfying the *modified classical Yang-*

Baxter identity

$$[rX, rY] - r([rX, Y] + [X, rY]) = -[X, Y], X, Y \in \mathbf{g}. \tag{1}$$

Put

$$[X, Y]_r = \frac{1}{2}([rX, Y] + [X, rY]), X, Y \in \mathbf{g}, \tag{2}$$

Then (1) implies that (2) is a Lie bracket; let $\mathbf{g}_r$ be the corresponding Lie algebra (with the same underlying linear space). Put $r_\pm = \frac{1}{2}(r \pm id)$; by (1), $r_\pm : \mathbf{g}_r \to \mathbf{g}$ are Lie algebra homomorphisms. Extend them to Poisson algebra morphisms $S(\mathbf{g}_r) \to S(\mathbf{g})$ (denoted by the same letters). These morphisms also agree with the standard Hopf structure on $S(\mathbf{g})$, $S(\mathbf{g}_r)$. Define the action

$$S(\mathbf{g}_r) \otimes S(\mathbf{g}) \to S(\mathbf{g})$$

by setting

$$x \cdot y = \sum r_+(x_i^{(1)})\, y\, r_-(x_i^{(2)})', x \in S(\mathbf{g}_r), y \in S(\mathbf{g}), \tag{3}$$

where $\Delta x = \sum x_i^{(1)} \otimes x_i^{(2)}$ is the coproduct and $a \mapsto a'$ is the antipode map.

Theorem 1 *(i) $S(\mathbf{g})$ is a free graded $S(\mathbf{g}_r)$-module generated by $1 \in S(\mathbf{g})$. (ii) Let $i{:}S(\mathbf{g}) \to S(\mathbf{g}_r)$ be the induced isomorphism of graded linear spaces; its restriction to $I = S(\mathbf{g})^{\mathbf{g}}$ is a morphism of Poisson algebras. (iii) Assume, moreover, that $\mathbf{g}$ is equipped with a nondegenerate invariant bilinear form; the induced mapping $\mathbf{g}_r^* \to \mathbf{g}$ defines a Lax representation for all Hamiltonians $H = i(\hat{H})$, $\hat{H} \in S(\mathbf{g})^{\mathbf{g}}$.*

The most common examples of classical r-matrices are associated with *Manin triples.* By definition, it is a triple $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ consisting of a Lie algebra $\mathbf{c}$ equipped with a nondegenerate invariant inner product and two isotropic Lie subalgebras $\mathbf{a}, \mathbf{b} \subset \mathbf{c}$ such that $\mathbf{c} = \mathbf{a} \dot{+} \mathbf{b}$ as a linear space. Let $P_{\mathbf{a}}, P_{\mathbf{b}}$ be the projection operators associated with this decomposition; the operator $r = P_{\mathbf{a}} - P_{\mathbf{b}}$ is skew and satisfies (1); moreover, the Lie algebra $\mathbf{c}_r$ splits into two parts, $\mathbf{c}_r \simeq \mathbf{a} \oplus \mathbf{b}$. Let us choose $\mathcal{A} = S(\mathbf{a})$ as our algebra of observables. The inner product on $\mathbf{c}$ sets up the isotropic subspaces $\mathbf{a}, \mathbf{b}$ into duality; let $L \in \mathbf{b} \otimes \mathbf{a} \subset \mathbf{b} \otimes S(\mathbf{a})$ be the associated canonical element; alternatively, we may regard L as an embedding $\mathbf{a}^* \hookrightarrow \mathbf{b}$. Let $P : S(\mathbf{c}) \to S(\mathbf{a})$ be the projection onto $S(\mathbf{a})$ in the decomposition

$$S(\mathbf{c}) = S(\mathbf{a}) \oplus S(\mathbf{c})\mathbf{b}.$$

Corollary 1 *(i) The restriction of P to the subalgebra $I = S(\mathbf{c})^{\mathbf{c}}$ of Casimir elements is a Poisson algebra homomorphism. (ii) L defines a Lax representation for all Hamiltonian equations of motion defined by the Hamiltonians $\bar{H} = P(H)$, $H \in I$. (iii) The corresponding Hamiltonian flows on $\mathbf{a}^* \simeq \mathbf{b} \subset \mathbf{c}$ preserve intersections of coadjoint orbits of $\mathbf{a}$ with the adjoint orbits of $\mathbf{c}$.*

One sometimes calls L *universal Lax operator;* restricting the mapping L : $\mathbf{a}^* \hookrightarrow \mathbf{b}$ to various Poisson submanifolds in $\mathbf{a}^*$ we get Lax representations for particular systems.

Theorem 1 may be applied to finite dimensional semisimple Lie algebras [K]. However, its really important applications are connected with loop algebras, which possess sufficiently many Casimirs. As a matter of example, let us describe the so called *generalized Gaudin model* (its quantum counterpart was originally proposed in [G]); the associated Manin triple is related to the Mittag-Leffler problem on $\mathbf{C}P_1$. Later on we shall also discuss quantum Gaudin model.

Let $\mathbf{g}$ be a complex simple Lie algebra. Fix a finite set $D = \{z_1, ..., z_N\} \subset \mathbf{C} \subset \mathbf{C}P_1$ and let $\mathbf{g}(D)$ be the algebra of rational functions on $\mathbf{C}P_1$ with values in $\mathbf{g}$ which vanish at infinity and are regular outside D . Let $\mathbf{g}_{z_i} = \mathbf{g} \otimes \mathbf{C}((z - z_i))$ be the localization of $\mathbf{g}(D)$ at $z_i \in D$. Put $\mathbf{g}_D = \oplus_{z_i \in D} \mathbf{g}_{z_i}$. There is a natural embedding $\mathbf{g}(D) \hookrightarrow \mathbf{g}_D$ given by the Laurent expansion. Put $\mathbf{g}^+_{z_i} = \mathbf{g} \otimes \mathbf{C}\,[[z - z_i]]$, $\mathbf{g}^+_D = \oplus_{z_i \in D}\, \mathbf{g}_{z_i}$. Then

$$\mathbf{g}_D = \mathbf{g}^+_D \dot{+} \mathbf{g}(D) \tag{4}$$

as a linear space. Fix an inner product on $\mathbf{g}$ and extend it to $\mathbf{g}_D$ by setting

$$\langle X, Y \rangle = \sum_{z_i \in D} Res\,(X_i, Y_i)\, dz. \tag{5}$$

Both subspaces $\mathbf{g}(D), \mathbf{g}^+_D$ are isotropic with respect to the inner product (5) which sets them into duality. Thus $(\mathbf{g}_D, \mathbf{g}^+_D, \mathbf{g}(D))$ is a Manin triple. The linear space $\mathbf{g}(D)$ may be regarded as a $\mathbf{g}^+_D$–module with respect to the coadjoint representation. The action of $\mathbf{g}_D^+$ preserves the natural $(\mathbf{Z}_+)^N$-filtration on $\mathbf{g}(D)$ by the order of poles. In particular, rational functions with simple poles form an invariant subspace $\mathbf{g}(D)_1 \subset \mathbf{g}(D)$. Let $\mathbf{g}_{z_i}^{++} \subset \mathbf{g}_{z_i}^+$ be the subalgebra consisting of formal series without constant terms. Put $\mathbf{g}_D^{++} = \oplus_{z_i \in D}\, \mathbf{g}_{z_i}^{++}$; clearly, $\mathbf{g}_D^{++}$ is an ideal in $\mathbf{g}_D^+$ and its action in $\mathbf{g}(D)_1$ is trivial. The quotient algebra is isomorphic to $\mathbf{g}^N = \oplus_{z_i \in D}\mathbf{g}$. The inner product (5) sets the linear spaces $\mathbf{g}(D)_1, \mathbf{g}^N$ in duality. Let $L(z) \in \mathbf{g}(D)_1 \otimes \mathbf{g}^N$ be the canonical element; we shall regard $L(z)$ as a matrix-valued rational function with coefficients in $\mathbf{g}^N \subset S(\mathbf{g}^N)$. Let $\{e_a\}$ be an orthonormal basis in $\mathbf{g}$. Fix a faithful linear representation (ρ, V) of $\mathbf{g}$; it extends canonically to a representation $(\rho, V(z))$ of the Lie algebra $\mathbf{g}(z) = \mathbf{g} \otimes \mathbf{C}(z)$ in the space $V(z) = V \otimes \mathbf{C}xx(z)$. Put

$$L_V(z) = (\rho \otimes id)L(z).$$

Matrix coefficients of $L_V(z)$ generate the algebra of observables $S(\mathbf{g}^N)$. The Poisson bracket relations in this algebra have a nice expression in 'tensor form', the brackets of the matrix coefficients of L_V forming a matrix in $EndV \otimes EndV$ with coefficients in $\mathbf{C}(u,v) \otimes S(\mathbf{g}^N)$; it is explicitly given by the following formula (which was suggested for the first time by Sklyanin [S1] and was the starting point of the whole theory):

$$\{L_V(u) \otimes_, L_V(v)\} = [r_V(u,v), L_V(u) \otimes 1 + 1 \otimes L_V(v)], \tag{6}$$

where $r_V(u,v)$ is the *rational r-matrix,*

$$r_V(u,v) = \frac{t_V}{u-v},\ t_V = \sum_a \rho(e_a) \otimes \rho(e_a). \tag{7}$$

Notice that $r_V(u,v)$ is essentially the Cauchy kernel solving the Mittag-Leffler problem on $\mathbf{C}P_1$ with which we started.

Corollary 1 may now be specialized in the following way which shows that spectral invariants of the Lax operator $L(z)$ may be regarded as 'radial parts' of the Casimir elements.

Clearly, we have $\mathbf{g}_D = \mathbf{g}(D) \dot{+} \mathbf{g}^N \dot{+} \mathbf{g}_D^{++}$ and hence

$$S(\mathbf{g}_D) = S(\mathbf{g}^N) \oplus (\mathbf{g}(D)\ S(\mathbf{g}_D) + S(\mathbf{g}_D)\, \mathbf{g}_D^{++}).$$

Let $\gamma : S(\mathbf{g}_D) \to S(\mathbf{g}^N)$ be the projection map associated with this decomposition.

Proposition 1 *The restriction of γ to the subalgebra $S(\mathbf{g}_D)^{\mathbf{g}_D}$ of Casimir elements is a morphism of Poisson algebras; under the natural pairing $S(\mathbf{g}^N)\times \mathbf{g}(D)_1 \to \mathbf{C}$ induced by the inner product (5) the restricted Casimirs coincide with the spectral invariants of $L(z)$.*

One should take some caution, as Casimir elements do not lie in the symmetric algebra $S(\mathbf{g}_D)$ itself but rather in its appropriate local completion; however, their projections to $S(\mathbf{g}^N)$ are well defined. The mapping γ is an analogue of the Harish-Chandra homomorphism; its definition may be extended to the quantum case as well.

Corollary 2 *Spectral invariants of $L(z)$ are in involution with respect to the standard Lie-Poisson bracket on $\mathbf{g}^N$; $L(z)$ defines a Lax representation for any of these invariants (regarded as a Hamiltonian on $(\mathbf{g}^N)^* \simeq \mathbf{g}^N$.*

The generalized Gaudin Hamiltonians are, by definition, the quadratic Hamiltonians contained in this family; one may take, e.g.

$$H_V(u) = \frac{1}{2} tr_V \, L(u)^2. \tag{8}$$

(Physically, they describe, e.g., bilinear interaction of several 'magnetic momenta'.)

Corollary 2 does not mention the 'global' Lie algebra $\mathbf{g}_D$ (and may in fact be proved by elementary means). However, of the three Lie algebras involved it is probably the most important one, as it is responsible for the dynamics. It is this 'global' Lie algebra that may be called the *hidden symmetry algebra.*

The characteristic equation $\det(L_V(z) - \lambda) = 0$ associates to the Lax operator $L(z)$ an affine algebraic curve (more precisely, we get a family of curves parametrized by the values of commuting Hamiltonians); this switches on the powerful algebraic-geometric machinery and makes the complete integrability of the generalized Gaudin Hamiltonians almost immediate.

Remarks. 1. In the exposition above we kept the divisor D fixed. A more invariant way is of course to use adèlic language. Thus one may replace the algebra $\mathbf{g}_D$ with the global algebra $\mathbf{g}_A$ defined over the ring $\mathbf{A}$ of adèles of $\mathbf{C}(z)$; fixing a divisor amounts to fixing a Poisson subspace inside the Lie algebra $\mathbf{g}(z)$ of rational functions (which is canonically identified with the dual space of the subalgebra $\mathbf{g}_{\mathbf{A}}^{+}$). The use of adèlic freedom allowing to add new points to the divisor is essential in the treatment of the quantum Gaudin model.

2. Further generalizations consist in replacing rational r-matrix with more complicated ones. One is of course tempted to repeat the construction above replacing $\mathbf{C}P_1$ with an arbitrary algebraic curve. There are obvious obstructions which come from the Mittag-Leffler theorem for curves: the subalgebra $\mathbf{g}(C)$ of rational functions on the curve C does not admit a complement in $\mathbf{g}_{\mathbf{A}}$ which is again a Lie subalgebra; for elliptic curves (and $\mathbf{g} = sl(n)$) this problem may be cured [BD] by considering quasiperiodic functions on C, and in this way we recover elliptic r-matrices which were originally the first example of classical r-matrices ever considered! ([S1])

3. Another, and much more important extension of this picture, is to observe that the Poisson bracket defined on $\mathbf{g}(z)$ is a linearization of a *quadratic* Poisson bracket defined on the group $G(z)$ (we choose a group G with the Lie algebra $\mathbf{g}$ and regard $G(z)$ as an affine algebraic group defined over $\mathbf{C}(z)$). Fixing again a faithful representation of G we may define this Poisson bracket on the affine ring of $G(z)$ by the following formula [S1]

$$\{L_V(u) \otimes, L_V(v)\} = [r_V(u,v), L_V(u) \otimes L_V(v)]\,, \tag{9}$$

where we regard the matrix coefficients of $L_V(u)$ as generators of the affine ring $\mathcal{A}_{aff}$ of $G(z)$. The bracket (9) known as the *Sklyanin bracket* is multiplicative, i.e. the coproduct $\Delta : \mathcal{A}_{aff} \rightarrow \mathcal{A}_{aff} \otimes \mathcal{A}_{aff}$ is a morphism of Poisson algebras. In other words, the group $G(z)$ equipped with the bracket (9) is a Poisson-Lie group. Rational functions with prescribed divisor of poles form a finite-dimensional Poisson submanifold in $G(z)$; generic Poisson submanifolds correspond to functions with simple poles which should now be written in multiplicative form

$$L_V(z) = \overleftarrow{\prod_i}\left(I - \frac{X_i}{z - z_i}\right). \tag{10}$$

Spectral invariants of 'Lax matrices' of this type again generate completely integrable Lax equations. It may be shown (although it is less evident than before) that these spectral invariants are restrictions to $G(z)$ of formal Casimir elements associated with the global group $G_{\mathbf{A}}$. So virtually all elements of the 'linear' picture admit a generalization to the quadratic case.

Multiplicativity of the Sklyanin bracket has still another asset, for it matches perfectly with the kinematics of lattice systems in statistical mechanics. It is natural to assume that the phase space of a multiparticle system associated with a one-dimensional lattice $\Gamma = \mathbf{Z}/N\mathbf{Z}$ is the product $\mathcal{M}^N = \mathcal{M} \times ... \times \mathcal{M}$ of one-particle spaces. Let $\mathcal{M} = G(z)$ be the Poisson-Lie group of the previous example. The product $m : G(z) \times ... \times G(z) \to G(z)$ is a Poisson map (it is called *monodromy map* in the present context); it allows to pullback integrable Hamiltonians on $G(z)$ to $G(z)^N$. The pull-backed Hamiltonians still remain integrable; this may be regarded as the main content of the Inverse Scattering Method (in this slightly simplified setting). Multi-pole Lax matrices (10) naturally arise as monodromy matrices associated with single-pole matrices on the lattice. The study of Lax systems on the lattice thus breaks naturally into two parts: *(a) Solve the Lax equation for the monodromy. (b) Lift the solutions back to* $G(z)^N$. The second stage (by no means trivial) is the inverse problem *sensu strictu.*

3. QUANTUM CASE. THE GAUDIN MODEL

Speaking informally, quantization consists in replacing Poisson bracket relations with commutation relations in an associative algebra. The standard way to quantize the algebra$A_{cl} = S(\mathbf{a})$ is to replace it with the universal enveloping algebra $U(\mathbf{a})$. Whenever there is a nice correspondence between coadjoint orbits and representations, we may also quantize particular systems obtained by specialization of $S(\mathbf{a})$ to these orbits. A much more delicate problem is to construct commuting quantum Hamiltonians and to determine their spectra.

Let us consider again the Gaudin model. Here the algebra of observables is simply $U(\mathbf{g}^N)$. Let V_λ be a finite-dimensional highest weight representation of $\mathbf{g}$ with dominant integral highest weight λ. Let $\lambda = (\lambda_1, ...\lambda_N)$ be the set of such weights; put $\mathbf{V}_\lambda = \otimes_i V_{\lambda_i}$. The space $\mathbf{V}_\lambda$ is a natural Hilbert space associated with the Gaudin model; so the 'kinematical' part of quantization problem in this case is fairly simple. Let us fix also an auxiliary representation (ρ, V). By analogy with the classical case we may introduce the *quantum Lax* *operator*

$$L_V(z) \in \ EndV(z) \ \otimes \mathbf{g}^N \subset EndV(z) \otimes U(\mathbf{g}^N); \tag{11}$$

the definition remains exactly the same, but now we embed $\mathbf{g}^N$ into the universal enveloping algebra $U(\mathbf{g}^N)$ instead of symmetric algebra $S(\mathbf{g}^N)$. The commutation relations for $L_V(z)$ have the form

$$[L_V(u) \otimes_, L_V(v)] = [r_V(u,v), L_V(u) \otimes 1 + 1 \otimes L_V(v)], \tag{12}$$

where the l.h.s. is now a matrix of true commutators. The key point in (12) is the interplay of comm utation relations in the quantum algebra $U(\mathbf{g}^N)$ and the auxiliary matrix algebra $EndV(z)$. Formula (12) was the starting point of QISM (as applied to models with *linear* commutation relations). Put

$$S(u) = \frac{1}{2} tr_V \left(L_V(u)\right)^2; \tag{13}$$

using (12) it is easy to check that $S(u)$ form a commutative family of Hamiltonians (called Gaudin Hamiltonians) in $U(\mathbf{g}^N)$ [J]. An important property of this commuting family is that it possesses at least one 'obvious' eigenvector $\mid 0 > \in H$, the tensor product of highest weight vectors in V_{λ_i}; it is usually called *the vacuum vector* . One of the key ideas of QISM is to construct other eigenvectors by applying to the vacuum creation operators which are themselves rational functions of z. This construction is called *algebraic Bethe ansatz.* Assume that $\mathbf{g} = sl(2)$ and let $\{E, F, H\}$ be its standard basis. Put

$$F(z) = \sum_{z_i \in D} \frac{F^{(i)}}{z - z_i}, \tag{14}$$

where $F^{(i)}$ acts as F in the i-th copy of $sl(2)$ and as id on other places. The Bethe vector is, by definition,

$$\mid w_1, w_2, ...w_m >= F(w_1)F(w_2)...F(w_m) \mid 0 > \tag{15}$$

The Lax matrix (11) applied to $\mid w_1, w_2, ...w_m >$ becomes triangular; this yields after a short computation

$$S(u) \mid w_1, w_2, ...w_m >=$$

$$s_m(u) \mid w_1, w_2, ...w_m > + \textstyle\sum_{j=1}^N \frac{f_j}{u-w_j} \mid w_1, w_2, ..., w_{j-1}, u, w_{j+1}, ..., w_m >,$$

where

$$f_j = \sum_{i=1}^{N} \frac{\lambda_i}{w_j - z_i} - \sum_{s \neq j} \frac{2}{w_j - w_s},$$

and $s_m(u)$ is a rational function,

$$(16)\ s_m(u) = \frac{c_V}{2}\chi_m(u)^2 - c_V \partial_u \chi_m(u),\ \chi_m(u) = \sum_{i=1}^{N} \frac{\lambda_i}{u - z_i} - \sum_{i=1}^{m} \frac{2}{u - w_j}.$$

(The constant c_V depends on the choice of V.) If all f_j vanish, $\mid w_1, w_2, ... w_m >$ is an eigenvector of $S(u)$ with the eigenvalue $s_m(u)$; equations

$$(17) \qquad \sum_{i=1}^{N} \frac{\lambda_i}{w_j - z_i} - \sum_{s \neq j} \frac{2}{w_j - w_s} = 0$$

are called the *Bethe ansatz equations.* (Notice that (17) is precisely the condition that $s_m(u)$ is nonsingular at $u = w_i$.)

For general simple Lie algebras the study of spectra of the Gaudin Hamiltonians becomes rather complicated; one way to solve this problem is to treat it inductively by choosing in $\mathbf{g}$ a sequence of embedded Lie subalgebras of lower rank and applying the algebraic Bethe ansatz to these subalgebras. An alternative idea is to interprete the Hamiltonians as radial parts of (infinite dimensional) Casimir operator of the 'global' Lie algebra $\mathbf{g}_D$. One observes, first of all, that Theorem 1 and Corollary 1 remain valid formally in quantum case; one has just to replace in their formulation symmetric algebras with the corresponding universal enveloping algebras (cf. [K]). More precisely, let $\mathbf{g}, \mathbf{g}_r$ be as in Theorem 1, let $U(\mathbf{g}), U(\mathbf{g}_r)$ be the corresponding universal enveloping algebras. Let $Z \subset U(\mathbf{g})$ be the center of $U(\mathbf{g})$. Extend the homomorphisms $r_\pm : \mathbf{g}_r \to \mathbf{g}$ to the universal enveloping algebras and define the action $U(\mathbf{g}_r) \otimes U(\mathbf{g}) \to U(\mathbf{g})$ by the same formulae (3) as before.

Theorem 2 *(i) $U(\mathbf{g})$ is a free filtered $U(\mathbf{g}_r)$ -module generated by $1 \in U(\mathbf{g})$. (ii) Let i:$U(\mathbf{g}) \to U(\mathbf{g}_r)$ be the induced isomorphism of filtered linear spaces; its restriction to $Z \subset U(\mathbf{g})$ is an algebra homomorphism.*

Corollary 3 *Let $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ be a Manin triple, $U(\mathbf{a}), U(\mathbf{b}), U(\mathbf{c})$ the corresponding universal enveloping algebras. Let $P : U(\mathbf{c}) \to U(\mathbf{a})$ be the projection map associated with the decomposition $U(\mathbf{c}) = U(\mathbf{a}) \oplus U(\mathbf{c})\mathbf{b}$; the restriction of P to the center of $U(\mathbf{c})$ is an algebra homomorphism.*

It is impossible to apply these theorems immediately in the affine case, since the center of $U(\mathbf{g}_D)$ is trivial. However, the situation can be amended by first passing to the central extension of $U(\mathbf{g}_D)$ and then considering the quotient algebra $U_k(\mathbf{g}_D) = U(\hat{\mathbf{g}}_D)/(c-k)$. It is known that for the *critical value* of the central charge $k = -h^\vee$ (here $h^\vee$ is the dual Coxeter number of $\mathbf{g}$) the (appropriately completed) algebra $U_{-h^\vee}(\mathbf{g}_D)$ possesses an ample center (for each 'local' factor in $U_{-h^\vee}(\mathbf{g}_D) = \otimes_{z_i\in D}U_{-h^\vee}(\mathbf{g}_{z_i})$ it is isomorphic to the classical W-algebra associated with the Langlands dual of $\mathbf{g}$ [FeiFr]). Moreover, the Gaudin Hamiltonians are radial parts of the appropriate Casimir operators. Let us describe this construction (due to B.Feigin, E.Frenkel and N.Reshetikhin [FeiFrR]) more precisely.

Let the Lie algebras $\mathbf{g}_D$, $\mathbf{g}_D^+$, $\mathbf{g}_D^{++}$, $\mathbf{g}(D)$ be as above. It will be convenient to add one more point $\{u\}$ to the divisor D and to attach to it the trivial representation V_0 of $\mathbf{g}$. Thus we write $D' = D \cup \{u\}$, etc. Let ω be the 2-cocycle on $\mathbf{g}_{D'}$ defined by

$$\omega(X,Y) = \sum_{z_i\in D'} Res(X_i, dY_i). \tag{18}$$

Let $\hat{\mathbf{g}}_{D'} = \mathbf{g}_{D'} \oplus \mathbf{C}c$ be the central extension of $\mathbf{g}_{D'}$ defined by this cocycle. Note that since the restriction of ω to the subalgebra $\mathbf{g}(D') \subset \mathbf{g}_{D'}$ is zero, the algebra $\mathbf{g}(D')$ is canonically embedded into $\hat{\mathbf{g}}_{D'}$. Put $\hat{\mathbf{g}}_{D'}^+ = \mathbf{g}_{D'}^+ \oplus \mathbf{C}c$. Fix a highest weight representation $\mathrm{V}_{(\lambda,0)} = \otimes_{z_i\in D}V_{\lambda_i} \otimes V_0$ of the Lie algebra $\mathbf{g}^{N+1} = \mathbf{g}_{D'}^+/\mathbf{g}_{D'}^{++}$ as above and let $V_{(\lambda,0)}^k$ be the associated representation of $\hat{\mathbf{g}}_{D'}^+$ on which the center $\mathbf{C}c$ acts by multiplication by $k \in \mathbf{Z}$. Let $\mathbf{V}_{(\lambda,0)}^k$ be the induced representation of $\hat{\mathbf{g}}_{D'}$,

$$\mathbf{V}_\lambda^k = U(\hat{\mathbf{g}}'_D) \otimes_{U(\hat{\mathbf{g}}_{D'}^+)} V_\lambda^k. \tag{19}$$

There is a canonical embedding $V_\lambda^k \hookrightarrow \mathbf{V}_\lambda^k : v \mapsto 1 \otimes v$. Let $\left(\mathbf{V}_\lambda^k\right)^*$ be the dual of $\mathbf{V}_\lambda^k$, $\mathbf{H}_\lambda \subset \left(\mathbf{V}_\lambda^k\right)^*$ the subspace of $\mathbf{g}(D')$–invariants. Decomposition (4) immediately implies that $\mathbf{H}_\lambda$ is canonically isomorphic to V_λ^*. Let $\Omega \in \mathbf{H}_\lambda \otimes V_\lambda$ be the canonical element; it defines a natural mapping $\mathbf{V}_\lambda^k \to V_\lambda : \varphi \mapsto \langle\varphi\rangle_\Omega$. (In Conformal Field Theory $\langle\varphi\rangle_\Omega$ are usually called *correlation functions.*) For any $x \in U(\hat{\mathbf{g}}_{D'})$ let $\gamma_\lambda(x) \in End V_\lambda$ be the linear operator defined by the composition mapping $v \mapsto \langle x(1\otimes v)\rangle_\Omega, v \in V_\lambda$.

Now suppose that $k = -h^\vee$; let T be the canonical element which corresponds to the inner product (5) in $\mathbf{g}((z))$. Fix an auxiliary linear representation (ρ, V) of $\mathbf{g}$; it extends naturally to a representation of $\mathbf{g}((z))$ in $V((z)) = V \otimes C((z))$. Put formally $T_V(z) = (id \otimes \rho)T$ (although $T_V(z)$ does not belong to $U(\mathbf{g}_z \otimes EndV((z))$, it still makes sense as a formal series in z infinite in both directions). Let $J_V(z) = tr_V : T_V(z)^2 :$ be the corresponding Sugawara current (here :: means usual normal ordering with respect to decomposition into positive and negative powers of z). For the critical value of the central charge all coefficients $J_V(n)$ of $J_V(z)$ are central in $U_{-h^\vee}(\mathbf{g}((z)))$. Let us embed $U_{-h^\vee}(\mathbf{g}(z)))$ into $U_{-h^\vee}(\mathbf{g}_{D'})$ sending it to the extra place $\{u\} \subset D'$ to which we attached the trivial representation of $\mathbf{g}$.

Proposition 2 $S(u) = \gamma_\lambda(J_V(-2))$ *coincides with the Gaudin Hamiltonian.*

Remark. Besides quadratic Hamiltonians associated with the Sugawara current there are also higher commuting Hamiltonians which may be obtained using the methods of [FeiFr].

This construction (if a bit tedious) presents several advantages. One is conceptual, as we could unravel the role of Casimir elements for the study of (one class of) quantum integrable models. Moreover, we identified the Hilbert space of the models in question with the space of CFT correlators at the critical value of the central charge (this value corresponds to the semiclassical limit in CFT). The technical advantage is that now the study of spectra and construction of the Bethe creation operators may be lifted to the 'big' space. This is the main content of the recent paper of Feigin, E.Frenkel, and Reshetikhin [FeiFrR]. An important ingredient of their construction is the use of 'adèlic' freedom: one may attach extra factors to the big space (these factors correspond to the poles of the creation operators) without affecting the reduced space. It appears to be more convenient technically to deal with Wakimoto modules instead of Verma modules for each 'local' factor in $U_{-h^\vee}(\mathbf{g}_\mathbf{A})$ and to replace the finite dimensional representation $V_\lambda = \otimes_i V_{\lambda_i}$ with the corresponding Verma module M_λ. (The generalized Bethe equations appear to be universal and Bethe vectors in V_λ are projections of Bethe vectors in M_λ.) To parametrize a Bethe vector choose first a set $\{w_1, ..., w_m\}$, $w_j \in \mathbf{C}$, and assign to each w_j a set of simple roots $\{\alpha_{i_j}\}_{i=1}^N$, one for each copy of $\mathbf{g}$ in $\mathbf{g}^N$. Let $F^i_{i_j}$

be the corresponding Chevalley generator of $\mathbf{g}$ acting nontrivially in the i-th copy of $\mathbf{g}$. The straightforward generalization of the Bethe creation operator (14) to the higher rank case is ([BF])

$$F(w_j) = \sum_{i=1}^{N} \frac{F^i_{i_j}}{w_j - z_i}. \tag{20}$$

The problem with (20) is that these operators no longer commute with each other. However, their counterparts in the big representation space (acting each in its own local factor attached to $w_j \in \mathbf{C}$) are already decoupled; acting on the vacuum they create singular vectors of imaginary weight. The conditions which assure it are precisely the generalized Bethe equations which read

$$\sum_{i=1}^{N} \frac{(\lambda_i, \alpha_{i_j})}{w_j - z_i} - \sum_{s \neq j} \frac{(\alpha_{i_s}, \alpha_{i_j})}{w_j - w_s} = 0, j = 1, ..., m. \tag{21}$$

Bethe vectors in the reduced space are computed as correlation functions which correspond to the tensor product of these singular vectors.

Another important point is the connection with the Knizhnik-Zamolodchikov equations. Recall that this is a system of equations satisfied by the correlation functions for an arbitrary value of the central charge; the critical value $c = -h^\vee$ corresponds to the semiclassical limit for the KZ system (small parameter before the derivatives). The outcome of this is two-fold: first, the Bethe vectors for the Gaudin model appear naturally in the semiclassical asymptotics of the solutions of the KZ system [RV]. Moreover, the exact integral representation of the solutions (for any value of the central charge) also involves the Bethe vectors [FeiFrR].

4. QUANTUM INTEGRABLE SYSTEMS AND QUANTUM GROUPS

For an expert in Quantum Integrability the Gaudin model is certainly sort of a limiting special case. The real thing starts with the quantization of *quadratic* Poisson bracket relations (9). This is a much more complicated problem which eventually requires the whole machinery of Quantum Groups (and has led to their discovery). The substitute for the Poisson bracket relations is the famous relation

$$R(u\ v^{-1})L_1(u)L_2(v)R(u\ v^{-1})^{-1} = L_2(v)L_1(u), \tag{22}$$

where $R(u)$ is the quantum R-matrix satisfying the quantum Yang-Baxter identity

$$R_{12}(u)R_{13}(u\ v)R_{23}(v) = R_{23}(v)R_{13}(u\ v)R_{12}(u). \tag{23}$$

To bring a quantum mechanical system into Lax form one has to arrange quantum observables into a Lax matrix $L(u)$ (which is a rational function of u) and to find an appropriate R-matrix satisfying (22), (23). First examples of quantum Lax operators were constructed by error and trial method; in combination with the Bethe ansatz technique this has led to explicit solution of important problems [STF] [FT1] [F1] [F2]. The algebraic concept which brings order to the subject is that of *quasitriangular Hopf algebra* [D]. The main examples of quasirtiangular Hopf algebras arise as q-deformations of universal enveloping algebras associated with Manin triples. Remarkably, the general pattern represented by Theorems 1, 2 survives q-deformation.

Remark. In the context of Quantum Integrability the term 'Quantum Groups' now in current use is slightly misleading; as a matter of fact, the deformation parameter q has nothing to do with the Planck's constant h which distinguishes quantum systems from the classical ones; we have seen already that certain quantum integrable systems (e.g the Gaudin magnets) are related to ordinary Lie algebras. The real reason to deal with the q-deformed case is that the Hopf structure which is inherent to the quasitriangular Hopf algebras is adapted to the kinematics of multiparticle systems (see below).

Definition 1 *Let A be a Hopf algebra with coproduct Δ and antipode S; let Δ' be the opposite coproduct in A. A is called quasitriangular if*

$$\Delta'(x) = R\Delta(x)R^{-1} \tag{24}$$

for all $x \in A$ and for some distinguished invertible element $R \in A \otimes A$ (universal R-matrix) and, moreover,

$$(\Delta \otimes id)\, R = R_{13}R_{23}, (id \otimes \Delta)\, R = R_{13}R_{12}. \tag{25}$$

(I use standard tensor notation to denote different copies of spaces concerned.) Identities (25) imply that R satisfies the Yang-Baxter identity

$$(26) \qquad R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

Let A^0 be the dual Hopf algebra equipped with the opposite coproduct (in other words, its coproduct is dual to the opposite product in A). Put $R^+ = R, R^- = \sigma(R^{-1})$ (here σ is the permutation map in $A \otimes A$, $\sigma(x \otimes y) = y \otimes x$) and define the mappings $R_\pm : A^0 \to A : f \mapsto \langle f \otimes id, R^\pm \rangle$; by (25) $R^\pm$ are Hopf algebra homomorphisms. Define the action $A^0 \otimes A \to A$ by

$$(27) \quad f \circ x = \sum_i R^+(f_i^{(1)})\, x\, S(R^-(f_i^{(2)}), \; where\; \Delta^0 f = \sum_i f_i^{(1)} \otimes f_i^{(2)}.$$

A is called *factorizable* if A is a free A^0-module generated by $1 \in A$. (Let us denote the corresponding linear isomorphism $A^0 \to A$ by F for future reference.) It is easy to see that Theorem 2 remains valid for factorizable Hopf algebras. There is a more interesting way to state this theorem which brings to light the role of quantum Lax operators and the parallels with the classical case. Main applications of this theorem are connected with quantized loop algebras; the two important classes are quantum affine Lie algebras and the Yangians [D]. For concreteness I shall consider only the first class; the case of the Yangians requires more effort, as there is still no good realization of their doubles (see, however, [Sm]). Accurate formulation will require some extra notation.

Let $\mathbf{g}$ be a simple Lie algebra. Let $\hat{\mathbf{g}}$ be the corresponding extended affine Lie algebra containing both the central element c and the conjugate scaling element d. Let $A = U_q(\hat{\mathbf{g}})$ be the corresponding quantum affine algebra.Fix a finite dimensional representation (V, ρ_V) of $U_q(\hat{\mathbf{g}})$;for $z \in \mathbf{C}^*$ let $\rho_{V(z)}$ be the representation of A given by $\rho_{V(z)}(a) = \rho(z^d a z^{-d}$. Let A^0 be the dual of A with the opposite coproduct. Let $\mathcal{R}$ be the universal R-matrix of A.

Take A^0 as the algebra of observables. Let $L \in A \otimes A^0$ be the canonical element. Let $L^V(z) = (\rho_{V(z)} \otimes id)L, L^W(z) = (\rho_{W(z)} \otimes id)L, R^{VW}(z) = (\rho_{V(z)} \otimes \rho_W(z))\mathcal{R}$. We may call $L^V(z) \in EndV \; \otimes A^0$ the *universal quantum Lax operator* (with auxiliary space V). Fix a finite-dimensional representation π of A^0; one can show that $(id \otimes \pi)\, L_V(z)$ is rational in z. (Moreover, Tarasov

[T] showed how to use this dependence on z to completely classify finite-dimensional representations of A^0.) Property (24) immediately implies that

$$(28) \qquad L_2^W(w)L_1^V(v) = R^{VW}(vw^{-1})L_1^V(v)L_2^W(w)\left(R^{VW}(vw^{-1})\right)^{-1}.$$

Moreover, $R^{VW}(z)$ satisfies the Yang-Baxter identity (23).

Let H be the Cartan subgroup of G. Fix $h \in H$ and put $l_V^h(v) = tr_V \rho_V(h)L^V(v)$, then $l_V^h(v)l_W^h(w) = l_W^h(w)l_V^h(v)$. Elements $l_V^h(v)$ are usually called (twisted) *transfer matrices.*

Remark. Twisting the transfer matrix by $h \in H$ is of course possible in the classical case or for the Gaudin model as well; for $U_q(\mathbf{g})$, however, there is a natural twist (which comes from the square of the antipode), and so it is worth introducing the generic twist from the very beginning. For $h = 1$ $l_V^h(v)$ is the simplest 'spectral invariant' of the quantum Lax operator $L^V(v)$.

It is convenient to extend A by adjoining to it group-like elements which correspond to the extended Cartan subalgebra $\hat{\mathbf{h}} = \mathbf{h} \oplus \mathbf{C}d$ in $\hat{\mathbf{g}}$. Let $\hat{H} = H \times \mathbf{C}^*$ be the corresponding "extended Cartan subgroup" in A generated by elements $h = q^\lambda, \lambda \in \mathbf{h}, t = q^{kd}$. Let $\hat{H} \times A \to A$ be its natural action on A by right translations, $\hat{H} \times A^0 \to A^0$ the contragredient action. The algebra $A = U_q(\hat{\mathbf{g}})$ is factorizable; let $F : A^0 \to A$ be the isomorphism induced by the action (27); for $s \in \hat{H}$ put $F^s = F \circ s$. It is natural to compute the image of the universal Lax operator $L_V(z) \in EndV(z) \otimes A^0$ under the 'factorization mapping' $(id \otimes F^s)$; this image is a formal series in z which is infinite in both directions, but its coefficients are well defined elements in $EndV \otimes A$. Put

$$(29) \qquad \begin{array}{c} T_V^s(z)^\pm = (\,id \otimes R^\pm \circ s)\,L_V(z) \in EndV\left[\left[z^{\pm 1}\right]\right] \otimes A, \\ T_V^s(z) = T_V^s(z)^+\ (id \otimes S)\,T_V^s(z)^-, h \in \hat{H}; \end{array}$$

it is easy to see that

$$T_V^s(z)^\pm = \left(id \otimes s^{\pm 1}\right) T_V^1(z)^\pm$$

and hence

$$T_V^s(z) = (id \otimes s)\,T_V^1(z)\,(id \otimes s)\,;$$

moreover, if $s = h^{-1}t$, where $h \in H$ and $t = q^{kd}$, we have also

$$T_V^s(z) = (\rho_V(h) \otimes id)\,T_V^t(z)\,(\rho_V(h) \otimes id)\,.$$

Put $t_V^s = tr_V T_V^s(z) = tr_V \rho_V(h)^2 T_V^t(z)$; clearly, we have $t_V^s(z) = F^t(l_V^{h^2}(z))$. Let U be any highest weight A-module of the critical level $k = -h^\vee$. Let 2ρ be the sum of positive roots of $\mathbf{g}$.

Theorem 3 *(i) [RS] Suppose that $s = q^{-\rho} q^{-h^\vee d}$. Then all coefficients of $t_V^s(z)$ are central in U. (ii) For any $h \in H$ we have $l_V^{h^2}(z) = F^{hs^{-1}}(t_V^s(z))$.*

Thus the duality between Hamiltonians and Casimir operators holds for quantum affine algebras as well. This allows to anticipate connections between the generalized Bethe ansatz, representation theory of quantum affine algebras at the critical level and the q-KZ equation [FrR], [Sm]. The results bearing on these connections are already abundant [DE], [TV], although they are still not in their final form.

As already noticed in the classical context, the Hopf structure on A^0 is perfectly suited to the study of lattice systems. Let $\Delta^{(N)} : A^0 \to \otimes^N A^0$ be the iterated coproduct map. $\otimes^N A^0$ may be interpreted as the algebra of observables associated with a multiparticle system. Put $\hat{t}_V^h(z) = \Delta^{(N)} t_V^h(z)$; Laurent coefficients of $\hat{t}_V^h(z)$ provide a commutative family of Hamiltonians in $\otimes^N A^0$. Let $i_n : A^0 \to \otimes^N A^0$ be the natural embedding, $i_n : x \mapsto 1 \otimes \ldots \otimes x \otimes \ldots \otimes 1$; put $L_V^n = (id \otimes i_n) L_V$. Then

$$\hat{t}_V^h(z) = tr_V \left(\rho_V(h) \overleftarrow{\prod_n} L_V^n \right). \tag{30}$$

Formula (30) has a natural interpretation in terms of lattice systems: L_V^n may be regarded as 'local' Lax operators attached to the points of a periodic lattice $\Gamma = \mathbf{Z}/N\mathbf{Z}$; commuting Hamiltonians for the big system arise from the monodromy matrix $M_V = \overleftarrow{\prod} L_V^n$ associated with the lattice. Finally, the twist $h \in H$ defines a quasiperiodic boundary condition on the lattice. The study of the lattice system again breaks into two parts: (a) *Find the joint spectrum of $\hat{l}_V(z)$.* (b) *Reconstruct the Heisenberg operators corresponding to 'local' observables and compute their correlation functions.* This is the *Quantum Inverse Problem* (profound results on it are due to F.Smirnov [Sm].)

Remarks. 1. The realization of the Casimirs described in Theorem 3 is a result of a relatively long development started by Faddeev, Reshetikhin and

Takhtajan [FRT]; they introduced the concept of 'universal Lax operator' and used the commutation relations for (29) to give an alternative definition of quasitriangular Hopf algebras corresponding to classical simple Lie algebras and to describe their center.

2. One may have the impression from the discussion above (and especially after reading Drinfeld's report [D]) that Hopf structure is a *sine qua non* condition for the study of integrable systems. This is not exactly the case. The properties of $\Delta^{(N)} : A^0 \to \otimes^N A^0$ imply that quantum observables corresponding to different copies of A^0 commute with each other. There are many physically interesting examples when it is not true. An extension of the outlined formalism which drops out this restriction is possible. Interesting nontrivial examples were found recently by Faddeev and Volkov [FV].

Let us finally return once again to the construction of quantum eigenstates described in Section 3 for the sl(2) Gaudin model. One important point of this construction is the existence of a vacuum, or a reference state (in the realization using Casimir operators at the critical level this vacuum is the image of the highest weight vector). There are many physically interesting models where there is no such vacuum vector, and the Bethe ansatz technique does not apply immediately. These problems certainly represent a major challenge from the point of view of representation theory. Remarkably, the direct methods of QISM still work in this situation. Sklyanin [S2] proposed an advanced version of the Bethe ansatz (*the functional Bethe ansatz*) which is equivalent to a separation of variables for the commuting quantum Hamiltonians; this technique is still mainly confined to the rank one case, but even so it presents striking parallels with the classical separation of variables based on the Jacobi inversion problem for abelian integrals (cf. also [Kuz], [HW]).

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