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THE ETA INVARIANT
[some recent developments]

par **Werner MÜLLER**

0. INTRODUCTION

0.1. The eta invariant was introduced and studied by Atiyah, Patodi and Singer in a series of papers on spectral asymmetry [APS1], [APS2], [APS3]. In this report we will discuss some new developments connected with the eta invariant. We will concentrate mainly on aspects related to index theory.

The eta invariant is defined for every elliptic selfadjoint differential operator A acting on sections of a vector bundle E over a closed manifold Y. Let λ_j run over the eigenvalues of A. Then the eta function of A is defined as

$$\eta_A(s) = \sum_{\lambda_j \neq 0} \frac{\operatorname{sign}\lambda_j}{|\lambda_j|^s}, \tag{0.1}$$

where $s \in \mathbf{C}$. The series is absolutely convergent in the half-plane $\operatorname{Re}(s) > \frac{\dim Y}{m}$, m being the order of A. By Mellin transform, it follows that

$$\eta_A(s) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{(s-1)/2} \operatorname{Tr}(Ae^{-tA^2})\, dt, \quad \operatorname{Re}(s) \gg 0.$$

The heat equation method implies that, as $t \to 0$, there exists an asymptotic expansion

$$\operatorname{Tr}(Ae^{-tA^2}) \sim \sum_{k=1}^{\infty} a_k t^{(-n-1+k)/m},$$

where $n = \dim Y$. Using this asymptotic expansion, it follows that the eta function admits a meromorphic continuation to the whole complex plane. It is a nontrivial result that $\eta_A(s)$ is always holomorphic at $s = 0$ [APS3], [Gi1]. This result needs global arguments because the local residue at $s = 0$ of $\eta_A(s)$ may not vanish [APS3], [Gi1]. This is in contrast to the case of the zeta function $\zeta_B(s) = \sum_{\lambda_j > 0} \lambda_j^{-s}$ of a positive elliptic operator B. The special value $\eta_A(0)$ is then called the *eta invariant* of the operator A. An important feature of the eta invariant is that it is not locally computable. However, the variation is locally computable.

0.2. The original purpose which led Atiyah, Patodi and Singer to the invention of the eta invariant was to generalize Hirzebruch's signature theorem to the case of manifolds with boundary. Recall that for a closed oriented C^∞ manifold X of dimension $4k$ Hirzebruch's signature theorem [Hi1] states that the signature $\mathrm{Sign}(X)$ of the nondegenerate quadratic form on $H^{2k}(X;\mathbf{R})$, which is defined by the cup product, is given by the formula

$$\mathrm{Sign}(X) = \langle L(p_1, ..., p_k), [X] \rangle. \tag{0.2}$$

Here $L(p_1, ..., p_k)$ is the Hirzebruch L–polynomial in the Pontrjagin classes of X and $[X] \in H_{4k}(X;\mathbf{Z})$ is the fundamental class of X. In his study of Hilbert modular surfaces [Hi2], Hirzebruch introduced certain topological invariants - called the *signature defect* - for cusp singularities of a Hilbert modular surface. Recall that a Hilbert modular surface is the compactification of a complex surface of the form $\Gamma\backslash H^2$ where H is the upper half–plane and Γ is a discrete subgroup of $\mathrm{SL}(2,\mathbf{R})^2$ which is commensurable with the Hilbert modular group of a real quadratic field. For a given cusp singularity $p \in \overline{\Gamma\backslash H^2}$ one considers a neighborhood U of p. The singularity can be resolved and in this way, one obtains a compact 4–manifold X with smooth boundary Y. The tangent bundle TX restricted to the boundary Y is parallelizable and therefore, can be pushed down to an SO(4)–bundle over X/Y. Let $\overline{p}_1 \in H^4(X,Y;\mathbf{Q})$ be the first Pontrjagin class of this bundle. Then the signature defect $\delta(p)$ of p is defined as

$$\delta(p) = \mathrm{Sign}(X) \; - \; \frac{1}{3}\langle \overline{p}_1, [X,Y] \rangle. \tag{0.3}$$

Here $\mathrm{Sign}(X)$ denotes the signature of the nondegenerate quadratic form induced by the cup product on $H_!^*(X;\mathbf{R}) = \mathrm{Im}(H^*(X,Y;\mathbf{R}) \to H^*(X;\mathbf{R}))$. The right hand side is independent of the particular choice of a resolution of p.

Using the explicit description of the resolution, Hirzebruch was able to compute the signature defect in terms of the resolution diagram. In this way he showed that $\delta(p)$ equals the value at $s = 0$ of a certain Hecke L–series $L(s)$ attached to the cusp p. Hirzebruch then conjectured [Hi2] that a similar result holds for Hilbert modular varieties of any dimension.

0.3. This conjecture was one of the primary motivations for Atiyah, Patodi and Singer to investigate this problem in the wider context of Riemannian geometry. Let X be a compact oriented $4k$–dimensional Riemannian manifold with smooth boundary Y. Suppose that the metric of X is a product near the boundary. Then the differential geometric signature defect of Y is defined as

$$\delta(Y) = \mathrm{Sign}(X) - \int_X L(p_1, ..., p_k) \tag{0.4}$$

where $L(p_1, ..., p_k)$ is now the Hirzebruch L–polynomial in the Pontrjagin forms of X and $\mathrm{Sign}(X)$ is the signature of the intersection form on $H_!^*(X; \mathbf{R})$. It follows from the Novikov additivity of the signature that the right hand side of (0.4) is independent of the manifold X that bounds Y. One of the main results of [APS1] shows that $\delta(Y)$ has an intrinsic definition: it equals the eta invariant of some particular elliptic operator A on Y. Namely, let $\Lambda^{ev}(Y)$ be the even part of the differential forms on Y and let $A : \Lambda^{ev}(Y) \to \Lambda^{ev}(Y)$ be defined by

$$A\phi = (-1)^{k+p+1}(*d - d*)\phi, \quad \phi \in \Lambda^{2p}(Y).$$

Then A is essentially selfadjoint with respect to the canonical inner product in $\Lambda^{ev}(Y)$. Let $\eta_Y(0)$ denote the eta invariant associated to A. Then $\delta(Y) = -\eta_Y(0)$ and the generalization of the Hirzebruch signature theorem (0.2) to manifolds with boundary, proved by Atiyah, Patodi and Singer in [APS1], can be stated as follows

$$\mathrm{Sign}(X) = \int_X L(p_1, ..., p_k) - \eta_Y(0). \tag{0.5}$$

The conjecture of Hirzebruch was proved independently by Atiyah, Donnely and Singer [ADS] and the author [Mü2], [Mü3].

0.4 In the same way as (0.1) is a consequence of the Atiyah–Singer index theorem, (0.5) is a consequence of an index theorem for first order elliptic operators on manifolds with boundary [APS1]. The boundary conditions used by Atiyah, Patodi and Singer are nonlocal and defined by the positive spectral projection of the tangential part of the first order operator. Instead of the spectral boundary conditions one may work with L^2 conditions on manifolds obtain by enlarging X in various ways. Cheeger [C1] was the first to study L^2–index problems on noncompact manifolds. He considered manifolds with isolated metrically conical singularities and derived the signature theorem in this context.

0.5. After the invention of the eta invariant by Atiyah, Patodi and Singer, many interesting relations of the eta invariant with other fields of mathematics have been discovered. As examples, we mention special values of certain L–series, the holonomy of determinant line bundles of families of Dirac operators, L^2-index theorems on noncompact manifolds and Witten's 3–manifold invariants.

1. THE ATIYAH–PATODI–SINGER INDEX THEOREM

1.1. Let X be a compact Riemannian manifold with C^∞ boundary Y, and suppose that on a collar neighborhood $I \times Y \subset X$ of Y, the metric equals the product metric $du^2 + g_Y$. Let E, F be Hermitian vector bundles over X and let $D : C^\infty(X, E) \to C^\infty(X, F)$ be a first order elliptic differential operator. Furthermore, suppose that on $I \times Y$, D takes the form

$$D = \gamma\Big(\frac{\partial}{\partial u} + A\Big) \tag{1.1}$$

where u is the inward normal coordinate, $\gamma : E|Y \to E|Y$ is a bundle isomorphism and $A : C^\infty(Y, E|Y) \to C^\infty(Y, E|Y)$ is an elliptic operator which is symmetric with respect to the inner product defined by the Hermitian metric of E and the metric of Y. Then A has discrete spectrum consisting of real eigenvalues λ of finite multiplicity. Let P_+ denote the orthogonal projection of $L^2(Y, E|Y)$ onto the subspace spanned by all eigenfunctions with eigenvalues $\lambda \geq 0$. Then P_+ is a pseudo-differential operator. Let $C^\infty(X, E; P_+)$ be the subspace of $C^\infty(X, E)$ consisting of all sections φ which satisfy the boundary conditions

$$P_+(\varphi|Y) = 0. \tag{1.2}$$

Then the main result of [APS1] is

THEOREM 1.1. (Atiyah-Patodi-Singer) *Let* $D_{P_+} : C^\infty(X, E; P_+) \to C^\infty(X, F)$ *be the restriction of* D. *Then* D_{P_+} *is a Fredholm operator and its index is given by*

$$\text{(1.3)} \qquad \operatorname{Ind} D_{P_+} = \int_X \omega_D(x)\, dx \; - \; \frac{1}{2}\big(\eta_A(0) + \dim \operatorname{Ker} A\big),$$

where $\omega_D(x)dx$ *is the local index density for* D *and* $\eta_A(s)$ *is the eta function (0.1).*

Note that $\omega_D(x)$ is the constant term in the local asymptotic expansion of

$$\operatorname{tr}\big(e^{-tD^*D}(x,x)\big) \; - \; \operatorname{tr}\big(e^{-tDD*}(x,x)\big)$$

as $t \to 0$. For A as above, the regularity of $\eta_A(s)$ at $s = 0$ follows from the proof of the index formula.

1.2. Now suppose that $\dim X = 4k$. Let $\tau : \Lambda^*(X) \to \Lambda^*(X)$ be defined by $\tau\varphi = i^{p(p-1)+2k} * \varphi$ for $\varphi \in \Lambda^p(X)$. Then τ is an involution and $d + d^*$ anti-commutes with τ. Let $\Lambda^*_\pm(X)$ denote the ±1–eigenspaces of τ. Then $d + d^*$ interchanges $\Lambda^*_+(X)$ and $\Lambda^*_-(X)$, and hence defines by restriction an operator

$$\text{(1.4)} \qquad D_S : \Lambda^*_+(X) \to \Lambda^*_-(X)$$

which is called the *signature operator*. The signature operator satisfies (1.1) with $A : \Lambda^*(Y) \to \Lambda^*(Y)$ being the operator given by

$$\text{(1.5)} \qquad A\phi = (-1)^{k+p+1}(\varepsilon * d - d*)\phi,$$

where ϕ is either in $\Lambda^{2p}(Y)$ and $\varepsilon = 1$ or ϕ belongs to $\Lambda^{2p-1}(Y)$ and $\varepsilon = -1$. One may now apply (1.3) to compute the index of D_S with boundary conditions (1.2). On the other hand, the index of D_S is closely related to the signature of X. This implies the signature formula (0.5).

1.3. The Atiyah-Patodi-Singer index theorem and, in particular, the signature theorem (0.5) can be derived in different ways as L^2–index theorem for elliptic operators on noncompact manifolds. Let $\hat{X} = X \cup \big((-\infty, 0] \times Y\big)$ be the complete manifold obtained from X by gluing the negative half–cylinder $(-\infty, 0] \times Y$ to the boundary of X. Using (1.1), we may extend the vector bundles E, F and the differential operator D to $\hat{X}$ in the obvious way. Let $\hat{D} : C^\infty(\hat{X}, \hat{E}) \to C^\infty(\hat{X}, \hat{F})$ be the extended operator. Then $\hat{D}$ restricted to $(-\infty, 0] \times Y$ is given by (1.1). It was proved in [APS1], Proposition 3.11, that $\mathrm{Ker} D_{P_+}$ is isomorphic to the space of L^2–solutions of $\hat{D}\varphi = 0$ on $\hat{X}$ and $\mathrm{Ker} D^*_{P_+}$ is isomorphic to the space of extended L^2–solutions of $\hat{D}^*\varphi = 0$ on $\hat{X}$, where extended solution means that on $(-\infty, 0] \times Y$, φ can be written as

$$\varphi = \phi + \psi$$

with $\phi \in \mathrm{Ker}\, A$ and $\psi \in L^2$. The section ϕ is called the limiting value of the extended solution φ. Let $L(F) \subset \mathrm{Ker}\, A$ be the subspace consisting of all limiting values of $\hat{D}^*$ on $\hat{X}$. By the remark above, it follows especially that $\hat{D}$ has a well-defined L^2–index given by

$$L^2 \,\mathrm{Ind}\, \hat{D} = \dim\big(\mathrm{Ker}\, \hat{D} \cap L^2(E)\big) \;-\; \dim\big(\mathrm{Ker}\, \hat{D}^* \cap L^2(F)\big) \tag{1.5}$$

and the two indeces are related by the formula

$$\mathrm{Ind}\, D_{P_+} = L^2 \,\mathrm{Ind}\, \hat{D} \;-\; h_\infty(F), \tag{1.6}$$

where $h_\infty(F) = \dim L(F)$. Thus the nonlocal boundary conditions on X can be replaced by L^2 conditions on $\hat{X}$. The space $L(F)$ of limiting values has a different interpretation in terms of generalized eigenfunctions of $\Delta = \hat{D}\hat{D}^*$ [Mü6]. Namely, let $\Delta_0 = -\partial/\partial u^2 + A^2$, regarded as operator in $C^\infty((-\infty, 0] \times Y, \hat{F})$. If we impose Dirichlet boundary conditions, we get a selfadjoint extension $\overline{\Delta}_0$. Let $\overline{\Delta}$ be the closure of Δ in L^2. Then $\exp(-t\overline{\Delta}) - \exp(-t\overline{\Delta}_0)$ is trace class for $t > 0$. Let $J : L^2((-\infty, 0] \times Y, \hat{F}) \subset L^2(\hat{X}, \hat{F})$ be the inclusion. Then the wave operators

$$W_\pm(\overline{\Delta}, \overline{\Delta}_0) = \lim_{t \to \pm\infty} e^{it\overline{\Delta}} J e^{-it\overline{\Delta}_0}$$

exist and are complete. Hence, the scattering operator S, associated to $(\overline{\Delta}, \overline{\Delta}_0)$, and the corresponding scattering matrix $S(\lambda)$, $\lambda \in \mathbf{R}$, exist [K1]. Suppose that $\mu_1 > 0$ is the smallest positive eigenvalue of A^2. Then, for $|\lambda| < \mu_1$, $S(\lambda)$ is a linear operator in $\mathrm{Ker}\, A$ which satisfies the functional equation $S(\lambda)S(-\lambda) = \mathrm{Id}$. In particular, for $\lambda = 0$ we get an involution

$$S(0) : \mathrm{Ker}\, A \to \mathrm{Ker}\, A, \quad S(0)^2 = \mathrm{Id}. \tag{1.7}$$

Using generalized eigenfunctions, it follows that

$$L(F) = \big(\mathrm{Id} - S(0)\big)\mathrm{Ker}\, A,$$

i.e., the +1–eigenspace of $S(0)$ coincides with the space $L(F)$ of limiting values. Thus, by Theorem 1.1, we obtain the following L^2 index theorem

THEOREM 1.2. *Let the notation be as above. Then*

$$L^2 \,\mathrm{Ind}\, \hat{D} = \int_X \omega_D(x)\, dx \; - \; \frac{1}{2}\eta_A(0) \; - \; \frac{1}{2}\mathrm{Tr}\big(S(0)\big). \tag{1.8}$$

On the other hand, Theorem 1.2 can be proved directly without referring to Theorem 1.1. For example, one may use the relative index theorem of [BMS] as applied in [Mü3] in a more general setting. Another method is based on Callia's formula (cf. [St2]). In view of (1.5), this reproduces then Theorem 1.1. Melrose [Me] has studied the case of asymptotically cylindrical metrics and derived the Atiyah–Patodi–Singer index theorem within this setting.

1.4 A different approach was used by Cheeger [C1], [C2]. In fact, he was the first to study L^2 index problems on noncompact manifolds. Cheeger considered manifolds with isolated conical singularities. Let X be a compact manifold with C^∞ boundary Y and let

$$C(Y) = (0,1) \times Y, \quad ds^2 = du^2 + u^2 g_Y,$$

be the metrical cone over Y. Let $M = X \cup C(Y)$ be the manifold obtain by gluing the bottom of the cone $C(Y)$ to the boundary of X and equip M with a Riemannian metric which coincides with the given metric on $C(Y)$. This is a manifold with an isolated metrically conical singularity. The noncompact Riemannian manifold M is not complete. Therefore the Laplacian Δ on forms $\Lambda^*(M)$ may have different closed extensions. In this case, *ideal boundary conditions* must be introduced. This leads to a selfadjoint operator with pure discrete spectrum consisting of eigenvalues of finite multiplicity. The space of L^2 harmonic forms $\mathcal{H}^*_{(2)}(M)$ is naturally isomorphic to the middle intersection cohomology $IH^*(\overline{M})$ of the one point compactification $\overline{M} = M \cup \{p\}$ where p is the cone tip. Let $D_S : \Lambda^*_+(M) \to \Lambda^*_-(M)$ be the signature operator. Then the observation above implies that

$$\mathrm{Sign}(X) = \mathrm{Sign}(M) = L^2 \,\mathrm{Ind}\, D_S,$$

and application of the heat equation method gives the signature formula (0.5) (cf. [C2], §6). This approach was extended by Bismut and Cheeger [BC1] to the case of Dirac operators on manifolds with conical singularities. In this way Theorem 1.1 is reproduced from analysis on such spaces.

1.5. One may also consider other warped product metrics on $\mathbf{R}^+ \times Y$. This means that we consider $\mathbf{R}^+ \times Y$ equipped with a metric of the form $du^2 + f^2(u)g_Y$, where $f \in C^\infty(\mathbf{R}^+)$ is an appropriate warping factor. Call this Riemannian manifold $C_f(Y)$. Let $M = X \cup C_f(Y)$ and equip M with a metric which agrees on $C_f(Y)$ with the given metric. For example, we may pick $f(u) = e^{-u}$. Then M becomes a manifold with a cusp studied in [Mül]. For other examples see [Br1], [Br2], [St1]. In all cases, the signature theorem (0.5) and, more generally, Theorem 1.1 can be recovered from appropriate L^2 index formulas for first order elliptic operators.

2. ETA INVARIANTS FOR MANIFOLDS WITH BOUNDARY

2.1. Let $M = X \cup_Y C(Y)$ be a manifold with an isolated conical singularity of dimension $4k-1$. Consider the Laplacian Δ_{2k-1} on $2k-1$–forms. If $H^{2k-1}(Y;\mathbf{R}) \neq 0$ one must introduce $*$–invariant ideal boundary conditions (cf. [C2]) to get a self-adjoint extension which we also denote by Δ_{2k-1}. This operator has pure point spectrum and the heat operator is trace class. The eta function can then be defined by

$$\eta_M(s) = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{(s-1)/2} \operatorname{Tr}(*de^{-t\Delta_{2k-1}^2})\, dt, \quad \operatorname{Re}(s) > 4k-1.$$

Cheeger [C2] has proved that $\eta_M(s)$ has a meromorphic continuation to $\mathbf{C}$ which is holomorphic at $s=0$. Therefore, the eta invariant $\eta(M)$ of M can be defined as $\eta_M(0)$. As stressed by Cheeger [C2], p.612, this may be regarded as a definition for the eta invariant of the manifold with boundary X. In place of a manifold with isolated conical singularities one may consider more general pseudomanifolds and define eta invariants for such spaces in the same way.

2.2. Using eta invariants for piecewise flat manifolds, Cheeger obtained combinatorial formulas for the Pontrjagin classes. For example, let X be a closed oriented $4k$–dimensional Riemannian pseudomanifold with a piecewise flat metric. Then the signature of X is given by

$$\operatorname{Sign}(X) = \sum_{\sigma^0} \eta(L(\sigma^0)),$$

where the sum runs over the vertices of X and $L(\sigma^0)$ denotes the link of the vertex σ^0 [C2].

2.3. Another way to define eta invariants for manifolds with boundary is to use the spectral boundary conditions of Atiyah, Patodi and Singer (cf. [DW]). Let X, Y and E be as in section 1.1 and let $D : C^\infty(X, E) \to C^\infty(X, E)$ be an elliptic first order operator which is formally selfadjoint with respect to the Hermitian metric in E. Suppose that near the boundary, D takes the form (1.1). This means that

$$\gamma^2 = -\mathrm{Id}, \quad A\gamma = -\gamma A \quad \text{and} \quad A = A^*. \tag{2.1}$$

Let P_+ be the positive spectral projection with respect to A. By (2.1), γ induces a map $\gamma : \operatorname{Ker} A \to \operatorname{Ker} A$. Therefore, $\operatorname{Ker} A$ has a natural symplectic structure defined by $\Phi(x, y) = \langle x, y\rangle$, $x, y \in \operatorname{Ker} A$, where $\langle x, y\rangle$ denotes the L^2 inner product. If $\operatorname{Ker} A \neq 0$, let $L \subset \operatorname{Ker} A$ be a Lagrangian subspace, that is, L satisfies $L \oplus \gamma L = \operatorname{Ker} A$ and $\Phi(L, L) = 0$. Let P_L be the orthogonal projection onto L. Put

$$\Pi_L = P_+ + P_L.$$

Then we introduce boundary conditions for D by setting $\Pi_L(\phi|Y) = 0$ for smooth sections ϕ of E. This gives rise to a selfadjoint extension D_L. It has pure discrete spectrum consisting of real eigenvalues of finite multiplicity. The eta function of D_L can then be defined by the same formula (0.1) and it follows that $\eta_{D_L}(s)$ is a meromorphic function on $\mathbf{C}$ [DW], [Mü5]. For Dirac type operators, the eta function is regular at $s = 0$ and the eta invariant of D_L can therefore be defined as $\eta_{D_L}(0)$. It this clear that such eta invariants should be related to index formulas for Dirac type operators on manifolds with corners.

3. THE HIRZEBRUCH CONJECTURE

3.1. As explained in the introduction, one of the main motivations for Atiyah, Patodi and Singer to prove the signature theorem (0.5) for manifolds with smooth boundary was a conjecture by Hirzebruch concerning the equality of the topological signature defect of a cusp singularity of a Hilbert modular variety and the value at zero of a certain Hecke L–function attached to the cusp.

Let F be a totally real number field of degree n and let $\mathcal{O}_F$ be the ring of integers of F. Then $\mathrm{SL}(2,\mathcal{O}_F)$ is called *Hilbert modular group*. It can be identified with a discrete subgroup Γ_0 of $\mathrm{SL}(2,\mathbf{R})^n$. In particular, $\mathrm{SL}(2,\mathcal{O}_F)$ acts properly discontinuously on the product H^n of n copies of the upper half–plane. If $x \in F \mapsto x^{(i)} \in \mathbf{R}$, $i = 1,...,n$, denote the n different embeddings of F into $\mathbf{R}$, then the action of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,\mathcal{O}_F)$ on $(z_1,...,z_n) \in H^n$ is given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(z_1,...,z_n) = \left(\frac{\alpha^{(1)}z_1+\beta^{(1)}}{\gamma^{(1)}z_1+\delta^{(1)}},...,\frac{\alpha^{(n)}z_n+\beta^{(n)}}{\gamma^{(n)}z_n+\delta^{(n)}}\right).$$

The quotient $\Gamma_0\backslash H^n$ of H^n by this action of Γ_0 is called a Hilbert modular variety. The group Γ_0 may have isolated fixed points which give rise to quotient singularities. To avoid this problem, we may pass to a torsion free normal subgroup $\Gamma \subset \Gamma_0$ of finite index. In fact, we may take Γ to be a congruence subgroup. Then $\Gamma\backslash H^n$ is a complex manifold which is noncompact. It can be compactified as analytic space by adding a finite number of points called "cusps". Each cusp is a singular point of $\overline{\Gamma\backslash H^n}$. The manifold has a finite number of ends Z_i, $i = 1,...,m$, which are in one-to-one correspondence with the cusps p_i. Each end Z_i is diffeomorphic to a half–cylinder $[1,\infty) \times Y_i$ and Y_i is a fibre bundle

$$T^n \to Y_i \xrightarrow{\pi} T^{n-1}, \tag{3.1}$$

where T^m denotes a torus of dimension m. The torus bundle (3.1) can be described explicitly in terms of the number field F [Hi2]. There is an abelian subgroup $\mathbf{M}_i$ of F of rank n and a group of totally positive units $\mathbf{V}_i$ acting on $\mathbf{M}_i$ such that $(\mathbf{M}_i,\mathbf{V}_i)$ determines the stabilizer Γ_i of the cusp p_i in Γ (cf. [Hi2]). If $(\mathbf{M},\mathbf{V})$ is any pair as above, then the L–function attached to $(\mathbf{M},\mathbf{V})$ is defined as

$$L(s,\mathbf{M},\mathbf{V}) = \sum_{\mu\in(\mathbf{M}-0)/\mathbf{V}} \frac{\operatorname{sign} N(\mu)}{|N(\mu)|^s}, \quad \mathrm{Re}(s) > 1, \tag{3.2}$$

where $N(\mu) = \mu^{(1)} \cdots \mu^{(n)}$ denotes the norm of $\mu \in \mathbf{M}$. The series (3.2) has an analytic extension to the whole complex plane. There is some similarity with the eta–function (0.1).

3.2. The conjecture of Hirzebruch [Hi2] can now be stated as follows: Let p be a cusp singularity of $\overline{\Gamma \backslash H^n}$ and let $(\mathbf{M}, \mathbf{V})$ be the datas attached to p as above. Then the following equality holds

$$\delta(p) = L(0, \mathbf{M}^*, \mathbf{V}), \tag{3.3}$$

where $\mathbf{M}^*$ is the lattice dual to $\mathbf{M}$ and $\delta(p)$ is the signature defect of p which is defined analogously to (0.3).

The proof of Hirzebruch's conjecture given in [ADS] proceeds in two steps. Using Fourier analysis along the fibres of the torus bundle $Y \xrightarrow{\pi} T^{n-1}$, Atiyah, Donnelly and Singer identify the eta invariant $\eta_Y(0)$ attached to the operator (1.5) and $L(0, \mathbf{M}, \mathbf{V})$. The second step is to prove that $\eta_Y(0)$ equals the signature defect $\delta(p)$. This approach is closely related to the computation of the adiabatic limit of the eta invariant $\eta_Y(0)$ if the metric of Y is shrinked along the fibres. We shall explain this connection in more detail in §4.

3.3. The proof of Hirzebruch's conjecture given in [Mü2], [Mü3] is based on an L^2 index theorem for the signature operator $D_S : \Lambda^*_+(\Gamma \backslash H^n) \to \Lambda^*_-(\Gamma \backslash H^n)$. The L^2 index of D_S is defined in the same way as in (1.5). Using the Selberg trace formula, the following theorem was proved in [Mü2]:

THEOREM 3.1. *Let $n = 2k$ and suppose that $L(s, \mathbf{M}_i, \mathbf{V}_i)$, $i = 1, ..., m$, are the L–series attached to the cusp singularities of $\Gamma \backslash H^n$ as above. Then*

$$L^2 \operatorname{Ind} D_S = \int_{\Gamma \backslash H^n} L(p_1, ..., p_k) + \sum_{i=1}^{m} L(0, \mathbf{M}_i^*, \mathbf{V}_i).$$

Using Hirzebruch's proportionality theorem, it follows that $L(p_1, ..., p_k) \equiv 0$ in the present case. Let $\mathcal{H}^*_{(2)}(\Gamma \backslash H^n)$ be the space of L^2 harmonic forms of $\Gamma \backslash H^n$ and let $\mathcal{H}^{2k}_{(2),\pm}(\Gamma \backslash H^n)$ be the ± 1–eigenspaces of the involution $\tau = (-1)^k *$. Then

$$L^2 \operatorname{Ind} D_S = \dim \mathcal{H}^{2k}_{(2),+}(\Gamma \backslash H^n) - \dim \mathcal{H}^{2k}_{(2),-}(\Gamma \backslash H^n).$$

On the other hand, $\mathcal{H}^*_{(2)}(\Gamma\backslash H^n)$ is isomorphic to the middle intersection cohomology $IH^*(\overline{\Gamma\backslash H^n})$ of the compactification $\overline{\Gamma\backslash H^n}$. This implies that

$$L^2 \operatorname{Ind} D_S = \operatorname{Sign}(\Gamma\backslash H^n)$$

and we get the following signature formula

$$\operatorname{Sign}(\Gamma\backslash H^n) = \sum_{i=1}^{m} L(0, \mathbf{M}_i^*, \mathbf{V}_i). \tag{3.4}$$

For a Hilbert modular variety with a single cusp, this gives a proof of (3.3). In general, one has to isolate cusps using more general manifolds with cusps of Hilbert modular type [Mü3]. As a consequence one gets $\eta_Y(0) = L(0, \mathbf{M}^*, \mathbf{V})$ for all Hilbert modular cusps.

3.4. Satake [Sa] (see also [SO]) has considered generalizations of Hirzebruch's conjecture to cusps of other $\mathbf{Q}$–rank one locally symmetric spaces. This case can also be treated by the method described above [Mü4]. Ogata has proved some of the generalizations by different methods [Og].

3.5. Instead of the signature operator one may consider other geometric operators on $\Gamma\backslash H^n$ and compute their L^2 index. For example, this applies to the Dolbeault operator. Using the corresponding index formula, one obtains formulas for the dimension of the space $\mathcal{H}^{p,q}_{(2)}(\Gamma\backslash H^n)$ of L^2 harmonic forms of type (p,q). This generalizes results of Shimizu [Sh]. In these cases, special values of L–functions also occur as contributions of the cusps.

4. ADIABATIC LIMITS OF ETA INVARIANTS

4.1. The computation of the eta invariant for torus bundles of the form (3.1) and the identification of this eta invariant with the value at $s = 0$ of some L–series is closely related the study of "adiabatic limits" of eta invariants [BF2], [C3]. These investigations were initiated by a formula, derived by Witten [W1], [W2], for the eta invariant of a mapping torus $Z \to M \xrightarrow{\pi} S^1$ associated to a diffeomorphism $\psi : Z \to Z$. The formula of Witten is connected with the study of global anomalies in general relativity. The word adiabatic refers to the so called adiabatic approximation used by Witten to solve the Dirac equation. This method is applied frequently in quantum mechanics, but is not rigorous.

4.2. We shall now describe the results of [BF2]. Let $Z \to M \xrightarrow{\pi} B$ be a fibration of compact manifolds with compact connected fibre Z of even dimension $n = 2l$. Suppose that g, g_B are Riemannian metrics on M, B respectively. Then π is called a Riemannian submersion, if

$$g = \pi^* g_B + g_Z, \tag{4.1}$$

where g_Z vanishes on the normal space of the fibres.

Let $TZ \subset TM$ be the subbundle consisting of all vectors tangent to the fibres. Suppose that TZ is oriented and has a spin structure. Let $F = F^+ \oplus F^-$ be the associated spinor bundle on M. Let E be a Hermitian bundle on M, equipped with a unitary connection ∇^E. For each $b \in B$, we get a twisted Dirac operator $D_b = D_b^+ \oplus D_b^-$ along the fibre $Z_b = \pi^{-1}(b)$ where

$$D_b^\pm : C^\infty(Z_b, F^\pm \otimes E) \to C^\infty(Z_b, F^\mp \otimes E).$$

This defines a family $\mathcal{D} = \{D_b\}_{b \in B}$ of twisted Dirac operators. Associated to the family $\mathcal{D}$ is the determinant line bundle λ over B whose fibre λ_b at $b \in B$ is canonically isomorphic to

$$\lambda_b \cong (\det \operatorname{Ker} D_b^+)^* \otimes \det(\operatorname{Ker} D_b^-).$$

Recall that the determinant of a finite–dimensional vector space is by definition the top exterior power of this vector space. Generalizing a construction by Quillen [Q], the line bundle λ can be equipped with a canonical metric, called Quillen metric, and a unitary connection ${}^1\nabla$. The curvature of this connection is the degree 2 term of the following differential form on B:

$$2\pi i \int_Z \hat{A}\Big(\frac{R^Z}{2\pi}\Big) \operatorname{Tr}\Big[\exp\Big(-\frac{L}{2\pi i}\Big)\Big],$$

where R^Z is the curvature tensor of the connection on TZ which is canonically induced by the Levi-Civita connection of TM and L is the curvature tensor of E.

The integral over Z denotes integration along fibres [BF2]. Witten's formula for global anomalies is related to the computation of the holonomy τ of the determinant line bundle λ over a closed loop c in B. Witten showed that in certain cases the holonomy of the loop c could be calculated using the eta invariant of a Dirac operator on $N = \pi^{-1}(c)$. After changing the parametrization of c and rescaling the metric of B, one can assume that c is isometric to S^1. Thus $N \xrightarrow{\pi} S^1$ is a Riemannian submersion and the metric of N takes the form

$$g = \pi^*(du^2) + g_Z.$$

Let $\varepsilon \neq 0$ and put

$$g_\varepsilon = \varepsilon^{-2}\pi^*(du^2) + g_Z. \tag{4.2}$$

Let D_ε be the twisted Dirac operator on N associated with the metric g_ε. Let $\eta^\varepsilon(0)$ be the eta invariant of D_ε. Put

$$\overline{\eta}^\varepsilon = \frac{1}{2}(\eta^\varepsilon(0) + \dim \operatorname{Ker} D_\varepsilon)$$

and let $[\overline{\eta}^\varepsilon]$ be the reduction mod $\mathbf{Z}$. Then we have the following holonomy theorem of Bismut and Fried:

THEOREM 4.1. ([BF2]) *As $\varepsilon \downarrow 0$, $[\overline{\eta}^\varepsilon]$ has a limit $[\overline{\eta}]$ in $\mathbf{R}/\mathbf{Z}$ and the holonomy τ of λ over the loop c for the connection ${}^1\nabla$ is given by*

$$\tau = (-1)^{\operatorname{Ind} D^+} \exp(-2\pi i[\overline{\eta}]).$$

The proof is based on the superconnection formalism.

4.3. Another proof of the holonomy theorem was given by Cheeger for the case of the signature operator [C3]. Let N be a closed oriented Riemannian manifold of dimension $4k-1$ and let $Z \to N \xrightarrow{\pi} S^1$ be a Riemannian submersion. Let $\tilde{d}_u, *_u$ denote exterior differentiation and the Hodge $*$–operator, on the fibre $Z_u = \pi^{-1}(u)$. Put

$$\beta_u = \begin{cases} (-1)^p \tilde{*}_u, & \Lambda^{2p}(Z_u); \\ \tilde{*}_u, & \Lambda^{2p-1}(Z_u); \end{cases}$$

and

$$\mathcal{A}(u) = \begin{cases} (-1)^{p+1}(\tilde{d}_u\beta_u + \beta_u\tilde{d}_u), & \Lambda^{2p}(Z_u); \\ \tilde{d}_u\beta_u - \beta_u\tilde{d}_u, & \Lambda^{2p-1}(Z_u). \end{cases}$$

Then $\mathcal{A}(u)$ is selfadjoint, elliptic, and the following relations hold

$$\beta_u^2 = -1, \quad \mathcal{A}(u)\beta_u = -\beta_u\mathcal{A}(u).$$

As above, let $A : \Lambda^{ev}(N) \to \Lambda^{ev}(N)$ be defined by

$$A = d * +(-1)^p * d \quad \text{on } \Lambda^{2p}(N). \tag{4.3}$$

Let u be the arc length on S^1 and let $U \subset S^1$ be an interval. Then, over $\pi^{-1}(U) \subset N$ a differential form $\psi \in \Lambda^{2p}(N)$ can be written as $\psi = \phi + du \wedge \omega$. Thus

$$\Lambda^{2p}(\pi^{-1}(U)) \cong \Lambda^{2p}(Z) \oplus \Lambda^{2p-1}(Z)$$

and, with respect to this identification, we have

$$A = \begin{pmatrix} \beta_u\partial_u & \mathcal{A}(u) \\ \mathcal{A}(u) & \partial_u\beta_u \end{pmatrix}.$$

Now let $\varepsilon \neq 0$, and consider the metric g_ε on N defined by (4.1). Let A_ε be the operator (4.3) with respect to g_ε. Let $\eta(N, g_\varepsilon) = \eta_{A_\varepsilon}(0)$ be the eta invariant of (N, g_ε).

THEOREM 4.2. ([C3]) *Suppose that* $\operatorname{Ker}\mathcal{A}(u) = 0$ *for all* $u \in S^1$. *Then*

$$\lim_{\varepsilon\to 0}\eta(N, g_\varepsilon) = \frac{1}{\pi}\int_{S^1}\lim_{t\to 0}\operatorname{Tr}\Big(\frac{\beta}{2}\mathcal{A}(u)^{-1}\dot{\mathcal{A}}(u)e^{-t\mathcal{A}(u)^2}\Big). \tag{4.4}$$

We remark that $\operatorname{Tr}\Big(\frac{\beta}{2}\mathcal{A}(u)^{-1}\dot{\mathcal{A}}(u)e^{-t\mathcal{A}(u)^2}\Big)$ equals i times the imaginary part of the connection form of λ.

If $\mathcal{A}(u)$ is not invertible for all $u \in S^1$, the theorem can be modified by deforming the family $\mathcal{A}(u)$ to a family $\mathcal{A}_1(u)$ which is invertible for all $u \in S^1$. The adiabatic limit of the eta invariants is then expressed by the same formula with $\mathcal{A}(u)$ replaced by $\mathcal{A}_1(u)$, except that equality holds only mod $\mathbf{Z}$.

4.4. In particular, (4.4) may be applied to the case of a 2–torus bundle over the circle, $T^2 \to N \xrightarrow{\pi} S^1$. Such bundles occur as the boundary of a neighborhood of a cusp singularity in a Hilbert modular surface. The holonomy of the 2–torus bundle $N \xrightarrow{\pi} S^1$ is given by a hyperbolic matrix

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z}), \quad |a+d| > 2.$$

The matrix B defines an isomorphism of the standard torus $T^2 = \mathbf{R}/\mathbf{Z}$ and $N = ([0,1] \times T^2)/\sim$ where $(0,x) \sim (1, B(x))$, $x \in T^2$. As a linear fractional transformation of the upper half–plane, B has two real fixed points $w > w'$ given by

$$w = \frac{1}{2c}(a - d + \sqrt{(a+d)^2 - 4}), \quad w' = \frac{1}{2c}(a - d - \sqrt{(a+d)^2 - 4}).$$

Let

$$\mathbf{M} = \mathbf{Z}w \oplus \mathbf{Z}.$$

Then $\mathbf{M}$ is an abelian subgroup of rank two in the real quadratic field $F = \mathbf{Q}(\sqrt{D})$, $D = \sqrt{(a+d)^2 - 4}$. Let $\varepsilon > 1 > \varepsilon' > 0$ be the eigenvalues of B and set

$$\mathbf{V} = \{\varepsilon^n \mid n \in \mathbf{Z}\}.$$

Then $\mathbf{V}$ is the group of totally positive units in the ring of integers $\mathcal{O}_F$ which satisfy $\varepsilon \mathbf{M} = \mathbf{M}$. Let σ be the semi–circle in the upper half–plane with endpoints w, w' and let γ be the arc on σ which connects $z_0 = (w + iw')/(1+i)$ and $B(z_0)$. Furthermore, let

$$E(z,s) = \sum_{(m,n) \in \mathbf{Z}^2 - 0} \frac{y^s}{|mz + n|^{2s}}.$$

Then Theorem 4.2, applied to the torus bundle $N \xrightarrow{\pi} S^1$, gives the following formula

$$4\pi i \lim_{s \to 0} \frac{1}{s} \int_\gamma \frac{\partial}{\partial z} E(z,s)\, dz = L(0, \mathbf{M}^*, \mathbf{V})$$

(cf. [Mü4] for more details). This is a formula which was first proved by Hecke [He, p.415].

4.5. The case of a fibration $M \xrightarrow{\pi} B$ with arbitrary compact base B was treated by Bismut and Cheeger in [BC1]. Let $Z \to M \xrightarrow{\pi} B$ be a fibration of closed oriented manifolds with even–dimensional fibres and odd–dimensional base B. Assume that

$M \xrightarrow{\pi} B$ is a Riemannian submersion. Then the metric g of M satisfies (4.1). Let E be a Hermitian vector bundle over M with unitary connection ∇^E and curvature L^E. Let R^B be the curvature of B and let $\hat{A}(R^B/(2\pi))$ be the differential form representing the $\hat{A}$–genus of B, which is defined by Chern–Weil theory.

Let D^Z be the Dirac operator along fibres with coefficients in E. This means that D^Z acts fibrewise and its restriction to $Z_b = \pi^{-1}(b)$, $b \in B$, is the Dirac operator on Z_b twisted by $E|Z_b$.

To state the main result of [BC1], one has to introduce the Levi–Civita superconnection. Let $F = F^+ \oplus F^-$ be the spinor bundle associated to the vertical tangent bundle $TZ \subset TM$ and g_Z. Let H^∞, $H^\infty_\pm$ denote the space of C^∞ sections of $F \otimes E$, $F^\pm \otimes E$ respectively. Then H^∞ and $H^\infty_\pm$ may be regarded as spaces of C^∞ sections of infinite–dimensional bundles $\mathbf{H}^\infty$ and $\mathbf{H}^\infty_\pm$ over B where the fibres $\mathbf{H}^\infty_b$, $\mathbf{H}^\infty_{\pm,b}$ over $b \in B$ are the C^∞ sections of $F \otimes E$, $F^\pm \otimes E$ over Z_b. The fibre $\mathbf{H}^\infty_b$ has a natural Hermitian inner product given by

$$\langle h, h' \rangle = \int_{Z_b} \langle h(x), h'(x) \rangle \, dx$$

and the connection ∇ on $F \otimes E$ induces a connection $\tilde{\nabla}$ on $\mathbf{H}^\infty$ by the prescription

$$\tilde{\nabla}_Y h = \nabla_{\tilde{Y}} h, \quad Y \in T_b B,$$

where $\tilde{Y}$ is the horizontal lift of Y. This connection does, in general, not preserve the Hermitian structure, but an elementary modification gives a unitary connection $\tilde{\nabla}^u$. Furthermore, let T be the torsion of the connection $\pi^*(\nabla^B) \oplus \nabla^Z$. The tensor determines a 2–form $c(T) \in \Lambda^2(T^*B) \otimes \mathrm{End}(TZ)$ which assigns to $U, V \in T^*_b B$ the Clifford multiplication by $T(\tilde{U}, \tilde{V}) = -p^{TZ}([\tilde{U}, \tilde{V}])$, $\tilde{U}, \tilde{V}$ being horizontal lifts. Then the Levi–Civita superconnection on $\mathbf{H}^\infty$ attached to the datas $(T^H M, \frac{1}{t} D^Z, \nabla^E)$ is defined as

$$A_t = \tilde{\nabla}^u + t^{1/2} D^Z - c(T)/(4t^{1/2}).$$

The curvature A_t^2 is a second–order elliptic differential operator acting fibrewise. It may be regarded as element in $\mathrm{End}(\mathbf{H}^\infty) \hat{\otimes} \Lambda^*(T^*B)$. Moreover, the fibrewise trace of $(D^Z + c(T)/(4t))e^{-A_t^2}$ exists. Let Tr_s denote the fibrewise supertrace. Then the $\hat{\eta}$–form is defined as

$$\hat{\eta} = \int_0^\infty \mathrm{Tr}_s\big((D^Z + c(T)/(4t))e^{-A_t^2}\big) \frac{dt}{2t^{1/2}}.$$

The convergence of this integral is proved in [BC1]. Note that $\hat{\eta}$ is an odd form on B which, by definition, depends only on global information along the fibres, on the metric g_Z, the connection ∇^E and on the splitting of TM into its horizontal and vertical subbundles. Moreover, one has

$$d\hat{\eta} = \frac{1}{(2\pi i)^l} \int_Z \hat{A}(iR^Z)\,\mathrm{tr}\big(\exp(-L^E)\big),$$

where $l = \frac{1}{2}\dim Z$ and R^Z is the curvature of the horizontal subbundle $TZ \subset TM$.

For $\varepsilon > 0$, let

$$D_\varepsilon : C^\infty(M, S \otimes E) \to C^\infty(M, S \otimes E)$$

be the twisted Dirac operator for the metric

$$g_\varepsilon = \varepsilon^{-1}\pi^*(g_B) + g_Z$$

on M. Let

$$\overline{\eta}(D_\varepsilon) = \frac{1}{2}\big(\dim \operatorname{Ker} D_\varepsilon + \eta_{D_\varepsilon}(0)\big)$$

be the reduced eta invariant. Then one has

THEOREM 4.3. ([BC1]) *As $\varepsilon \downarrow 0$, $\overline{\eta}(D_\varepsilon)$ has a limit in $\mathbf{R}$, which is given by*

$$\lim_{\varepsilon \to 0} \overline{\eta}(D_\varepsilon) = \frac{1}{(2\pi i)^k} \int_B \hat{A}(iR^B)\hat{\eta}$$

where $k = (1 + \dim B)/2$.

There are modifications of this result for $\dim B$ even and for the case where D^Z is not invertible [Da].

4.6. In [BC2], this result was used to study adiabatic limits of eta invariants for torus bundles. Applying their result to torus bundles over tori, Bismut and Cheeger obtained a new proof of Hirzebruch's conjecture. Furthermore, the adiabatic limit technique can also be used to derive the signature formula (3.4).

5. FAMILIES INDEX THEOREM FOR MANIFOLDS WITH BOUNDARY

5.1. The $\hat{\eta}$–form occurs also in the families version of the Atiyah-Patodi-Singer index theorem proved by Bismut and Cheeger [BC3], [BC4]. Let $M \xrightarrow{\pi} B$ be a Riemannian submersion of a compact manifold with boundary with even dimensional fibres diffeomorphic to a fixed manifold with boundary Z. Again suppose that the vertical tangent bundle TZ has a spin structure. Let E be a Hermitian bundle with unitary connection ∇^E. Furthermore, suppose that the metrics are products near the boundary and that the connection ∇^E is also of product type near the boundary. Then on each fibre Z_b, $b \in B$, we can consider the twisted Dirac operator $D_b^+ : C^\infty(Z_b, F^+ \otimes E) \to C^\infty(Z_b, F^- \otimes E)$ where $F^\pm$ are the $\frac{1}{2}$–spinor bundles. By assumption, D_b^+ takes the form (1.1) near the boundary and we may impose boundary conditions with respect to the spectral projection P_b^+. In this way we get a family D^+ of Dirac operators satisfying boundary conditions of Atiyah–Patodi–Singer type. In order to define the index bundle for this family, one has to take into account the spectral flow of the family of Dirac operators $D^{\partial Z}$ on the boundaries. In [BC3], Bismut and Cheeger make the following assumption:

$$\operatorname{Ker} D_b^{\partial Z} = \{0\} \quad \text{for all } b \in B.$$

Then there is a well–defined virtual index bundle

$$\operatorname{Ind} D^+ = \operatorname{Ker} D^+ \; - \; \operatorname{Ker} D^- \in K^0(B)$$

in the sense of Atiyah and Singer [AS].

THEOREM 5.1. ([BC4]) *Let R^Z be the curvature tensor of the connection on the vertical tangent bundle $TZ \subset TM$ which is induced by the Levi–Civita connection of TM. Let L^E be the curvature of E. Then one has*

$$\operatorname{ch}(\operatorname{Ind} D^+) = \int_Z \hat{A}\Big(\frac{R^Z}{2\pi}\Big)\operatorname{Tr}\Big(\exp\Big(-\frac{L^E}{2\pi i}\Big)\Big) \; - \; \hat{\eta}.$$

Here $\int_Z$ means integration along fibres.

This is the family version of Theorem 1.1. If M is closed, this is precisely the local families index theorem proved by Bismut [Bi].

5.2. To prove Theorem 4.3, Bismut and Cheeger replace the family of manifolds with boundary $Z \to M \xrightarrow{\pi} B$ by the family of manifolds with conical singularities $Z' \to M' \xrightarrow{\pi'} B$ which is obtained by attaching cones fibrewise. The typical fibre is then $Z' = Z \cup_{\partial Z} C(\partial Z)$ and, for $\varepsilon > 0$, $C(\partial Z)$ is endowed with the metric

$$\frac{dr^2}{\varepsilon} + r^2 g^{\partial Z}. \tag{4.4}$$

Let D_ε^+ be the corresponding family of twisted Dirac operators associated with $M' \xrightarrow{\pi'} B$ with respect to the metric (4.4). It turns out that, for sufficiently small $\varepsilon > 0$, the index bundle $\operatorname{Ind} D_\varepsilon^+ \in K^0(B)$ coincides with the index bundle $\operatorname{Ind} D^+$. In a sense, this is the extension to the case of families of the results explained in §1.

5.3. Another proof of the families index theorem for Dirac operators on manifolds with boundary has been given by Melrose and Piazza [MP]. The general framework for this proof is the so-called *b*–geometry developed by Melrose [Me]. This means that the family of manifolds with boundary is replaced by a corresponding family of complete manifolds with cylindrical ends. In this setting, traces have to be regularized which leads to the *b*–trace formalism of Melrose. The conditions on the boundary family can be relaxed so that only the existence of a spectral section is required.

6. L^2 INDEX THEOREMS FOR LOCALLY SYMMETRIC SPACES OF FINITE VOLUME

6.1. In the previous sections we have seen that there is a close connection between index theory for elliptic operators on Hilbert modular varieties, eta invariants and special values of certain L–series. This is true in a much more wider sense if we study locally symmetric spaces $\Gamma\backslash G/K$ of finite volume and arbitrary rank. For example, Stern [St2] has proved an L^2 index theorem for the signature operator on Hermitian locally symmetric manifolds $M = \Gamma\backslash G/K$ of finite volume. The manifold M has a natural compactification as manifold with corners $\overline{M}$. This is the Borel–Serre compactification of M [BS]. The boundary of $\overline{M}$ has a stratification and only the faces of maximal dimension provide a nontrivial contribution to the index formula. It is very conceivable that these terms can be reinterpreted as adiabatic limits of eta invariants. This will provide the link with §4 and will also put Stern's results into a different perspective. In subsequent papers [St3], [St4], Stern has related the boundary contributions in his index formula to special values of Sato–Shintani L–functions. It is, of course, very likely that a similar index formula can be proved for twisted Dirac operators. As observed by Moscovici [Mo], the index of special twisted Dirac operators is related to the dimension of certain spaces of automorphic forms (see below). This explains the significance of such an index formula.

In a subsequent paper [St3], Stern has expressed some of the contributions of the "cusps" to the index in terms of special values of Sato–Shintani L–functions.

6.2. Let G be a real semisimple Lie group, K a maximal compact subgroup, and Γ a discrete torsion–free subgroup of G such that $\mathrm{Vol}(\Gamma\backslash G) < \infty$. Then $X = G/K$ is a Riemannian symmetric space and $\Gamma\backslash X$ is a locally symmetric manifold of finite volume. Let E, F be finite–dimensional unitary K–modules and $\tilde{\mathcal{E}}, \tilde{F}$ the induced homogeneous vector bundles over X. These bundles can be pushed down to bundles $\mathcal{E} = \Gamma\backslash\tilde{\mathcal{E}}$, $\mathcal{F} = \Gamma\backslash\tilde{\mathcal{F}}$ over X. Let

$$\tilde{D} : C^\infty(X, \tilde{\mathcal{E}}) \to C^\infty(X, \tilde{\mathcal{F}})$$

be a G–invariant elliptic differential operator, that is, $\tilde{D}$ commutes with the action of G. Then $\tilde{D}$ descends to an elliptic operator

$$D : C^\infty(\Gamma\backslash X, \mathcal{E}) \to C^\infty(\Gamma\backslash X, \mathcal{F}).$$

The operator D is called "locally invariant". Let $L^2(\mathcal{E})$, $L^2(\mathcal{F})$ be the spaces of L^2 sections of $\mathcal{E}$, $\mathcal{F}$ respectively.

Proposition 6.1. ([Mo]) *Let* $D : C^\infty(\Gamma\backslash X, \mathcal{E}) \to C^\infty(\Gamma\backslash X, \mathcal{F})$ *be a locally invariant elliptic differential operator. Then we have*

$$\dim(\mathrm{Ker}\, D \cap L^2(\mathcal{E})) < \infty.$$

Let D^* denote the formal adjoint operator to D. Then D^* is again locally invariant and, by Proposition 6.1, we may define the L^2 index of D by

$$L^2\,\mathrm{Ind}\, D = \dim(\mathrm{Ker}\, D \cap L^2(\mathcal{E})) \;-\; \dim(\mathrm{Ker}\, D^* \cap L^2(\mathcal{F})). \tag{6.1}$$

The basic problem is now to prove an index formula. If $\Gamma\backslash X$ has real rank-one, Barbasch and Moscovici [BM] derived an index formula for twisted Dirac operators. Stern [St2] has studied the case where G is the group of real points of an algebraic group defined over $\mathbf{Q}$ and X is Hermitian and $\mathbf{Q}$–irreducible. In this context, he derives an index formula for the signature operator with coefficients in a flat bundle. We shall only consider the case of the trivial flat bundle. Then we are dealing with the usual signature operator $D_S : \Lambda_+^*(\Gamma\backslash X) \to \Lambda_-^*(\Gamma\backslash X)$. To state the index theorem, we have to recall some facts about the structure of $\Gamma\backslash X$. The structure at infinity of $\Gamma\backslash X$ is described by the Γ–conjugacy classes of rational parabolic subgroups of G, which replace the cusps in the case of a Hilbert modular variety. Let P be a rational parabolic subgroup of G. Then P has a Langlands decomposition

$$P = M_P\, A_P\, N_P \tag{6.2}$$

where N_P is the unipotent radical of P, $L_P = A_P M_P$ is the unique Levi subgroup of P which is stable under the Cartan involution corresponding to K, A_P is the identity component of the maximal $\mathbf{Q}$–split torus of the radical of L_P and M_P is reductive. Then $K \cap P = K \cap M_P$ and $\Gamma \cap P/\Gamma \cap N_P = \Gamma_M$ is a discrete subgroup of M_P with finite covolume. Set

$$X_M = M/K \cap M, \quad Y_P = \Gamma \cap P\backslash(X_M \times N_P).$$

Then $\Gamma_M\backslash X_M$ is again a locally symmetric space of finite volume and there is a fibration

$$\Gamma \cap N\backslash N \to Y_P \xrightarrow{\pi} \Gamma_M\backslash X_M \tag{6.3}$$

with typical fibre the compact nilmanifold $\Gamma \cap N\backslash N$. Furthermore, each Y_P can be identified with a face of the Borel–Serre compactification $\Gamma\backslash\overline{X}$ [BS] ($Y_P = e'(P)$ in the notation of [BS]). The faces of maximal dimension, i.e. the faces of codimension one in $\Gamma\backslash\overline{X}$ correspond to the Γ–conjugacy classes of the maximal rational parabolic subgroups. A parabolic subgroup P is maximal if the torus A_P in (6.2) has dimension 1. The formula for the L^2 index of the signature operator derived by Stern [St2] is of the following nature

$$L^2 \operatorname{Ind} D_S = \int_{\Gamma\backslash X} L \;+ \sum_{\{P\}_{max}} \delta_P \tag{6.4}$$

where L is the Hirzebruch L–polynomial and the sum is running over the Γ–conjugacy classes of maximal rational parabolic subgroups. The contribution δ_P of a given maximal rational parabolic subgroup to the index formula (6.4) is associated with the fibration (6.3). The explicit description of δ_P is too technical to be recalled here. For more details we refer to [St2]. Instead we would like to stress our point of view that in the light of §4, the term δ_P should be interpreted as adiabatic limit of eta invariants for Y_P. This needs, of course, further explanation, because Y_P is noncompact if the $\mathbf{Q}$–rank of $\Gamma\backslash X$ is greater than 1. The one–parameter family of metrics on Y_P which has to be considered for the adiabatic limit is defined as follows. Since P is maximal parabolic, the torus A_P in the Langlands decomposition (6.1) is of dimension 1. Furthermore $X = P/K \cap P$, and we get diffeomorphisms

$$X \cong \mathbf{R}^+ \times X_M \times N_P \quad \text{and} \quad \Gamma \cap P\backslash X \cong \mathbf{R}^+ \times Y_P.$$

Now consider the metric on $\mathbf{R}^+ \times X_M \times N_P$ which is the pull back of the invariant metric on X. This metric can be described explicitly in terms of the invariant metric on X_M and the root space decomposition of Lie(N) (cf. [Bo]). Let g_t be the metric on Y_P which is induced on the slice $\{t\} \times Y_P$. Let $*_t$ be the Hodge $*$–operator with respect to g_t and define $A_t : \Lambda^{ev}(Y_P) \to \Lambda^{ev}(Y_P)$ by

$$A_t = (d *_t + (-1)^p *_t d) \quad \text{on} \quad \Lambda^{2p}(Y_P).$$

Since A_t may have continuous spectrum, the eta invariant of A_t can not be defined by (0.1). Instead one has to use a regularized version. One possible way to regularize the eta invariant is to use the eta density defined by

$$\eta_{A_t}(0, y) = \int_0^\infty u^{-1/2} \operatorname{tr}(A_t e^{-uA_t^2}(y, y))\, du.$$

The convergence of this integral needs justification. It can be handled using techniques similar to [Mü5]. Next one has to verify that $\eta_{A_t}(0, y)$ is absolutely convergent on Y_P. If this is so, we may define

$$\eta(Y_P, g_t) = \int_{Y_P} \eta_{A_t}(0, y)\, dy. \tag{6.5}$$

If Y_P is compact, this number coincides with the usual eta invariant defined in §1. If this problem is settled, one may proceed and study $\eta(Y_P, g_t)$ as $t \to \infty$. We claim that the limit exists and equals δ_P. The answer can be expressed in terms of the Levi–Civita superconnection in the same way as in Theorem 4.3. In [St3] and [St4], Stern has expressed δ_P in terms of special values of Sato–Shintani L–functions. Following our approach, this should be a consequence of an extension of [BC3] to the noncompact case. This is under consideration in joint work with Bismut and Cheeger.

6.3. The index formula (6.4) can, in principle, be extended to twisted Dirac operators. The method of Stern, however, relies on the fact that the signature operator defines a Fredholm operator in L^2. This is were the hypothesis that X is Hermitian is needed. If one considers twisted Dirac operators it may occur that the corresponding operator in L^2 is not Fredholm. In this case, the index formula will contain additional contributions from the continuous spectrum which are similar to the trace of the scattering matrix $S(0)$ occuring in (1.8). To determine these contributions, one has to use the spectral resolution of D^-D^+. The continuous part of the spectral resolution is described by Eisenstein series and the intertwining operators correspond to the scattering matrix in (1.8). For special arithmetic groups like principal congruence subgroups of $SL(n, \mathbf{Z})$ the intertwining operators can be expressed in terms of automorphic L–functions and therefore, the contribution of the continuous spectrum to the index formula is given in terms of special values of ratios of automorphic L–functions.

6.4. Let R^Γ be the right regular representation of G in $L^2(\Gamma\backslash G)$. Then $L^2(\Gamma\backslash G)$ decomposes into the direct sum of two invariant subspaces

$$L^2(\Gamma\backslash G) = L^2_d(\Gamma\backslash G) \oplus L^2_c(\Gamma\backslash G)$$

where $L^2_d(\Gamma\backslash G)$ is the smallest invariant subspace containing all irreducible unitary subrepresentations of R^Γ. Thus

$$L^2_d(\Gamma\backslash G) = \oplus_{\pi\in\hat{G}} m_\Gamma(\pi)\mathcal{H}(\pi),$$

where $m_\Gamma(\pi) < \infty$ is the multiplicity of a given representation $\pi \in \hat{G}$ in R^Γ and $\mathcal{H}(\pi)$ is the representation space of π. Assume that rank G=rank K. Let $H \subset K$

be a compact Cartan subgroup of G. Let $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{h}$ be the Lie algebras of G, K and H, respectively. Let Φ be the root system of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$, $\Phi_c \subset \Phi$ the root system of $(\mathfrak{k}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ and $\Phi_n = \Phi - \Phi_c$. Fix a positive root system $\Psi \subset \Phi$, and let ρ be the half–sum of positive roots and ρ_c the half–sum of positive compact roots. Let $\Lambda \subset i\mathfrak{h}^*$ be the lattice which, by exponentiation, correponds to $\hat{H}$. Recall that $\Lambda + \rho$ parametrizes the discrete series representations $\hat{G}_d \subset \hat{G}$. Let $\mu \in \Lambda$ be the highest weight of an irreducible K–module E_μ and let $\mathcal{E}_\mu$ be the corresponding locally homogeneous vector bundle over $\Gamma\backslash X$. Let

$$D_\mu^{\pm} : C^\infty(\Gamma\backslash X, \mathcal{S}^{\pm} \otimes \mathcal{E}_\mu) \to C^\infty(\Gamma\backslash X, \mathcal{S}^{\mp} \otimes \mathcal{E}_\mu)$$

be the Dirac operators with coefficients in $\mathcal{E}_\mu$. Moscovici [Mo] has proved that for sufficiently regular μ, $\operatorname{Ker} D_\mu^- = 0$ and

$$L^2 \operatorname{Ind} D_\mu^+ = \dim \operatorname{Ker} D_\mu^+ = m_\Gamma(\pi_{\mu+\rho_c}), \tag{6.6}$$

where π_λ denotes the discrete series representation of G parametrized by $\lambda \in \Lambda + \rho$. This is one reason why it is interesting to prove an L^2 index theorem for twisted Dirac operators.

6.5. What has been said about the L^2 index applies to the Lefschetz fixed point theorem as well. Stern [St4] has derived a Lefschetz formula for Hecke operators on Hermitian locally symmetric manifolds $\Gamma\backslash X$ with respect to the signature operator. More precisely, let $C(\Gamma)$ be the commensurator of Γ which consists of all $\alpha \in G$ such that $\Gamma \cap \alpha\Gamma\alpha^{-1}$ is of finite index in both Γ and $\alpha\Gamma\alpha^{-1}$. Fix $\alpha \in C(\Gamma)$ and let $\{\alpha_1, ..., \alpha_r\}$ denote a set of representatives of $\Gamma\alpha\Gamma$. Thus $\Gamma\alpha\Gamma$ is the disjoint union of the cosets $\alpha_i\Gamma$. Then the Hecke operator $T_G(\alpha)$ on $L^2(\Gamma\backslash G)$ corresponding to α, is defined by

$$(T_G(\alpha)f)(\Gamma g) = \sum_{i=1}^{r} f(\Gamma\alpha_i^{-1}g), \quad f \in L^2(\Gamma\backslash G).$$

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. Using the identification of $L^2\Lambda^*(\Gamma\backslash X)$ with $(L^2(\Gamma\backslash G) \otimes \Lambda^*\mathfrak{p}^*)^K$, one can extend $T_G(\alpha)$ as $T_G(\alpha) \otimes \mathrm{Id}$ to an endomorphism of $L^2\Lambda^*(\Gamma\backslash X)$, which we also denote by $T_G(\alpha)$. Then $T_G(\alpha)$ commutes with $d + d^*$ and the involution τ on $\Lambda^*(\Gamma\backslash X)$ which is defined by the Hodge $*$–operator. Let

$D_S : \Lambda^*_+(\Gamma\backslash X) \to \Lambda^*_-(\Gamma\backslash X)$ be the signature operator. Then $T_G(\alpha)$ preserves $\operatorname{Ker} D_S = \mathcal{H}^*_{(2),+}(\Gamma\backslash X)$ and $\operatorname{Ker} D_S^* = \mathcal{H}^*_{(2),-}(\Gamma\backslash X)$. Then the Lefschetz number of $T_G(\alpha)$ acting on the signature complex is defined to be

$$L(T_G(\alpha), D_S) = \operatorname{Tr}(T_G(\alpha)|\mathcal{H}^*_{(2),+}(\Gamma\backslash X)) - \operatorname{Tr}(T_G(\alpha)|\mathcal{H}^*_{(2),-}(\Gamma\backslash X)).$$

The Lefschetz formula proved by Stern contains a term which is the same as in the compact case and a sum of contributions attached to the maximal rational parabolic subgroups of G. These terms are the equivariant versions of the numbers δ_P in (6.4) which, according to our interpretation, should correspond to adiabatic limits of regularized equivariant eta invariants associated to Y_P.

If one considers the Gauss–Bonnet complex in place of the signature complex, then the Lefschetz fixed point formula for Hecke operators is simplified considerably; the eta invariants do not show up. In this case, one can also use the Arthur trace formula to derive the Lefschetz fixed point formula [Ar]. There is also a pure topological Lefschetz fixed point formula developed by Goresky and MacPherson [GM]. On the other hand, as (6.6) shows, more information is gained if one considers twisted Dirac operators. A corresponding Lefschetz fixed point formula will then determine the trace of Hecke operators acting in certain spaces of automorphic forms.

REFERENCES

[Ar] J. Arthur - *Characters, harmonic analysis, and an l^2-Lefschetz formula*, Proc. Sympos. Pure Math., Vol. **48**, Amer. Math. Soc., Providence, R.I. (1987), 167–179.

[At] M.F. Atiyah - *The logarithm of the Dedekind η-function*, Math. Annalen **278**, 335–380 (1987).

[APS1] M.F. Atiyah, V.K. Patodi, and I.M. Singer - *Spectral asymmetry and Riemannian geometry*, I, Math. Proc. Camb. Phil. Soc. **77**, 43–69 (1975).

[APS2] M.F. Atiyah, V.K. Patodi, and I.M. Singer - *Spectral asymmetry and Riemannian geometry*, II, Math. Proc. Camb. Phil. Soc. **78**, 405–432 (1975).

[APS3] M.F. Atiyah, V.K. Patodi, and I.M. Singer - *Spectral asymmetry and Riemannian geometry*, III, Math. Proc. Camb. Phil. Soc. **79**, 71–99 (1976).

[AS] M.F. Atiyah and I.M. Singer - *The index of elliptic operators*, IV, Ann. of Math. **93**, 119–138 (1971).

[Ba] H. Baumgärtel and M. Wollenberg - *Mathematical scattering theory*, Akademie–Verlag, Berlin 1983.

[Bi] J.–M. Bismut - *The Atiyah-singer index theorem for families of Dirac operators: Two heat equation proofs*, Invent. math. **83**, 91–151 (1986).

[BC1] J.–M. Bismut and J. Cheeger - *Eta invariants and their adiabatic limits*, J. Amer. Math. Soc. **2**, 33–70 (1989).

[BC2] J.–M. Bismut and J. Cheeger - *Transgressed Euler classes of SL(n,Z) vector bundles, adiabatic limits of eta invariants and special values of L-functions*, Ann. scient. Éc. Norm. Sup., 4^e série, t. **25**, 335–391 (1992).

[BC3] J.–M. Bismut and J. Cheeger - *Families index for manifolds with boundary, superconnections and cones*, I, Journal Funct. Anal. **89**, 313–363 (1990).

[BC4] J.–M. Bismut and J. Cheeger - *Families index for manifolds with boundary, superconnections and cones*, II, Journal Funct. Anal. **90**, 306–354 (1990).

[BC5] J.–M. Bismut and J. Cheeger - *Remarks on the index theorem for families of Dirac operators on manifolds with boundary*, in: *Differential Geometry*, 59–83, A symposium in honor of Manfredo do Carmo. ed. B. Lawson and K. Tenenblat, Longman Scientific & Technical, 1991.

[BF1] J.–M. Bismut and D.S. Freed - *The analysis of elliptic families*, I, Commun. Math. Phys. **106**, 159–176 (1986).

[BF2] J.–M. Bismut and D.S. Freed - *The analysis of elliptic families*, II, **107**, 103–163 (1986).

[Bo] A. Borel - *Stable real cohomology of arithmetic groups*, Ann. Sci. Éc. Norm. Super., 4^e série, **7**, 235–272 (1974).

[BS] A. Borel and J.-P. Serre - *Corners and arithmetic groups*, Comment. Math. Helv. **48**, 244–297 (1974).

[BMS] N.V. Borisov, W. Müller and R. Schrader - *Relative index theorems and supersymmetric scattering theory*, Commun. Math. Phys. **114**, 475–513 (1988).

[BG] T. Branson and P. Gilkey - *Residues of the η–function for operators of Dirac type*, J. Funct. Anal. **108**, 47–87 (1992).

[Br1] J. Brüning - *L^2–index theorems on certain complete manifolds*, J. Diff. Geom. **32**, 491–532 (1990).

[Br2] J. Brüning - *On L^2-index theorems for complete manifolds of rank-one type*, Duke Math. J. **66**, 257–309.

[C1] J. Cheeger - *On the spectral geometry of spaces with cone-like singularities*, Proc. Nat. Acad. Sci. U.S.A. **76**, 2103–2106 (1979).

[C2] J. Cheeger - *Spectral geometry of singular Riemannian spaces*, J. Diff. Geom. **18**, 575–657 (1983).

[C3] J. Cheeger - *η-invariants, the adiabatic approximation and conical singularities*, J. Diff. Geom. **26**, 175–221 (1987).

[Da] X. Dai - *Adiabatic limits, the non-multiplicativity of signature and Leray spectral sequence*, J. Amer. Math. Soc. **4**, 265 - 321 (1991).

[DW] R.D. Douglas and K.P. Wojciechowski - *Adiabatic limits of the η-invariants. The odd-dimensional Atiyah-Patodi-Singer problem*, Commun. Math. Phys. **142**, 139–168 (1991).

[Gi1] P.B. Gilkey - *The residue of the global eta function at the origin*, Advances in Math. **40**, 290–307 (1981).

[Gi2] P.B. Gilkey - *The boundary integrand in the formula for the signature and the Euler characteristic of a Riemannian manifold with boundary*, Advances in Math. **15**, 334–360 (1975).

[GS] P.B. Gilkey and L. Smith - *The eta invariant for a class of elliptic boundary value problems*, Commun. Pure Appl. Math. **36**, 85–132 (1983).

[GM] M. Goresky, R. MacPherson - *Topological trace formula* (To appear).

[He] E. Hecke - *Darstellung von Klassenzahlen als Perioden von Integralen 3. Gattung aus dem Gebiet der elliptischen Modulfunktionen*, Mathematische Werke, Göttingen, Vandenhoeck & Ruprecht, 1959, 405 – 417.

[Hi1] F. Hirzebruch - *Topological methods in algebraic geometry*, Springer–Verlag, Berlin, 1966.

[Hi2] F. Hirzebruch - *Hilbert modular surfaces*, Enseignement Math. **19**, 183–281 (1973).

[K1] T. Kato - *Perturbation theory for linear operators*, Berlin, Heidelberg, New York, Springer 1980.

[K2] T. Kato - *Wave operators and unitary equivalence*, Pacific J Math. **15**, 171–180 (1965).

[La] J.–P. Labesse - *Signature des variétés modulaires de Hilbert et représentations diédrales*, Lect. Notes in Math. **1447**, 249–260, Springer-Verlag, Berlin 1990.

[Me] R. Melrose - *The Atiyah-Patodi-Singer index theorem*, A.K. Peters, Boston, 1993.

[MP] R. Melrose and P. Piazza - *Families of Dirac operators, boundaries and the b-calculus*, Preprint, MIT, 1994.

[Mo] H. Moscovici - *L^2-index of elliptic operators on locally symmetric spaces of finite volume*, Contemporary Math. **10**, 129–137 (1982).

[Mü1] W. Müller - *Spectral theory for noncompact Riemannian manifolds with cusps and a related trace formula*, I.H.E.S.-Preprint, Nr. 46, 1980.

[Mü2] W. Müller - *Signature defects of cusps of Hilbert modular varieties and values of L-series at $s = 1$*, J. Diff. Geom. **20**, 55–119 (1984).

[Mü3] W. Müller - *Manifolds with cusps of rank one*, Lect. Notes in Math. **1244**, Springer–Verlag, Berlin, 1987.

[Mü4] W. Müller - *L^2-index theory, eta invariants, and values of L-functions*, Contemporary Math. **105**, 145–189 (1990).

[Mü5] W. Müller. *Eta invariants and manifolds with boundary.* MPI–Preprint, 93-50, Bonn, 1993.

[Mü6] W. Müller - *L^2-index and resonances*, in: *Analysis on manifolds (Katata 1987)*, pp. 203-211, Lect. Notes in Math. **1339**, Springer, Berlin-New York, 1988.

[O] S. Ogata - *Hirzebruch conjecture on cusp singularities*, Math. Annalen **296**, 69–86 (1993).

[S] I. Satake - *On numerical invariants of arithmetic varieties of* **Q**–*rank one*, in: "Automorphic forms of several variables", Taniguchi Symposium, Katata, 1983 (I. Satake and Y. Morita eds.), Adv. Studies in Pure Math. **46**, Birkhäuser, Basel, 1984, 353–369.

[SO] I. Satake and S. Ogata - *Zeta functions associated to cones and their special values*, in: "Automorphic forms and geometry of arithmetic varieties" (Y. Namikawa and K. Hashimodo eds), Adv. Studies in Pure Math. **15**, Kinokuniya, Tokyo and North–Holland, Amsterdam, New York, Oxford, 1989, 1–27.

[Sh] H. Shimizu - *On discontinuous groups operating on the product of upper half–planes*, Ann. of Math. **77**, 33–71 (1963).

[St1] M. Stern - *L^2-index theorems on warped products*, Princeton University Ph.D. thesis, October 1984.

[St2] M. Stern - *L^2-index theorems on locally symmetric spaces*, Invent. math. **96**, 231–282 (1989).

[St3] M.Stern - *Eta invariants and hermitian locally symmetric spaces*, J. Diff. Geom. **31**, 771–789 (1990).

[St4] M. Stern - *Lefschetz formulae for arithmetic varieties*, Invent. math. **115**, 241–296 (1994).

[W1] E. Witten - *Global gravitational anomalies*, Commun. Math. Phys. **100**, 197–229 (1985).

[W2] E. Witten - *Global anomalies in string theory*, in: Proc. of the Symposium on Anomalies, Geometry, Topology of Chicago (1985), Bardeen, W.A., White, A.R. (eds.), Singapore, World Scientific, 1985, pp. 61–99.

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