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SOME RECENT RIGOROUS RESULTS IN THE THEORY OF PHASE TRANSITIONS AND CRITICAL PHENOMENA

by Jürg FRÖHLICH and Thomas SPENCER

Contents

Introductory comments

The present notes have no scholarly ambition. They address a subject that has a history of more than fifty years. The number of relevant publications is truly enormous. Presumably we have missed some of the really important papers in this subject. We have only tried to review some of the main trends during the late sixties and the seventies, have emphasized their mathematical aspects and have given work in which we have been personally involved more weight than it deserves. Since these notes collect material explained in lectures that were supposed to cover our work, we do not find anything particularly wrong in that circumstance.

Many of the results mentioned in Sect. 2.1 are contained in joint work of J. Glimm, A. Jaffe and T. Spencer, of J. Fröhlich, B. Simon and T. Spencer and of J. Fröhlich, R. Israel, E. Leib and B. Simon. Moreover, various important results, due to Griffiths ; Dobrushin ; Minlos, Pirogov and Sinai ; Lebowitz, and others are mentioned or underly our presentation. The most important mathematically rigorous results in Sect. 2.2 are due to J. Glimm and A. Jaffe. The ideas and concepts in §§ 3 and 4 are part of the"conventional wisdom" of the modern form of the renormalization group, invented by Wilson ; Kadanoff ; Jona-Lasinio and al. and extended by Fisher ; Wegner ; Brézin, Le Guillou and Zinn-Justin, and many others. Our presentation has drawn inspiration from ones by Sinai, who - together with Bleher and Dobrushin - has contributed the first crucial ideas and results clarifying the mathematical status of the renormalization group. The formalism and the techniques in § 5 are inspired by work of Symanzik and were developed in joint work with D. Brydges. The main results reported in that section followed similar results by M. Aizenman. Further developments were carried out by D. Brydges, J. Fröhlich and A. Sokal.

These notes contain no proofs, and the results are often stated somewhat vaguely. They have the character of a brief status report and were written in a hurry. They are intended for light reading and may serve as a guide to the literature. It is hoped that they convey some of the beauty of the mathematical structures and problems involved in statistical mechanics (see, in particular, Sects. 2.1, 4.1, 4.2 and 5), and that they might challenge some readers to look into some of these problems.

§ 1. Introduction

1.1. General remarks

In the development of theoretical physics there have occurred several major advances during the seventies. Although it is to some extent subjective what one considers to be a major advance and although it may be too early to tell we think that many theoretical physicists would include the following ones among the most significant discoveries of the seventies :

1) *Gauge theories of the fundamental (electro-weak and strong) interactions.*

2) *Renormalizability of gauge theories, and asymptotic freedom in* QCD (i.e. the discovery of the fact that interactions mediated by *non-abelian* gauge fields in theories like quantum chromodynamics, abbreviated QCD, become weak at high energies or short distances, but strong at large distances. This latter circumstance led to the idea of quark confinement).

3) *New, productive forms of the renormalization group* (e.g. the ε-expansion [1] ; more recently the Feigenbaum theory [2]) and its applications to a quantitative theory of second order phase transitions and critical phenomena and to the study of dynamics. (The basic idea of the renormalization group is to study the behaviour of a physical system under a change of scale - in space or time - by integrating out fluctuations on successively larger length scales.)

We have not included in this list important developments in astrophysics, condensed matter physics and other fields in or related to theoretical physics. Moreover, we have not mentioned advances in mathematical physics during the past decade, yet, among which one must mention

- constructive quantum field theory ;
- fluid dynamics (e.g. dynamical systems theory, onset of turbulence..., study of shock waves, Navier-Stokes equs. ...) ;
- non-equilibrium statistical mechanics ; theory of phase transitions in equilibrium statistical mechanics ; stability of non-relativistic matter...

From the point of view of a theoretical physicist who is not concerned very much with mathematical rigour developments 1) through 3) mentioned above have reached a rather high degree of perfection and completeness, although from the point of view of rigorous mathematics the state of the art has actually remained quite rudimentary. This is a challenge to mathematical physicists and mathematicians and is why we are, in these notes, addressing the subject of phase transitions and critical phenomena, related to topic 3) above.

During the past few years there have been very important beginnings in other directions which may become major trends in the physics of the eighties and among which one might include :

a) Supersymmetry, supergravity, spontaneous (and dynamical) breaking of supersymmetries.

b) The mathematical description of complicated behaviour of (classical) macroscopic systems ; ("roads to turbulence", "transition to chaos", "theory of attractors", "stochastic resonances"...).

c) The theory of disordered systems ("localisation", "frustration"[1] in "spin-glasses", "turbulent crystals"[2] , "wave propagation in disordered media",...).

One hopes that supersymmetry will solve some of the problems left open within ordinary Yang-Mills theory and that it might show a way towards a quantum theory of gravitation. Disordered, or chaotic systems are a natural and important play ground for people previously busy with critical phenomena. While these last topics mirror perhaps the present trend of the world towards more chaos, disorder

1) a concept related to what the mathematician calls curvature
2) a notion recently proposed by Ruelle

and frustration, supersymmetry reflects our longing for order, harmony and unity. Topics b) and c) are sure to have something to do with reality, but supersymmetry may remain a dream.

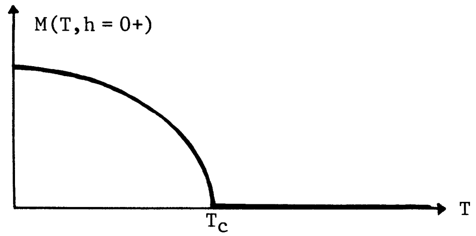
In the following we shall discuss some recent rigorous results on phase transitions and critical phenomena, topics 3) above, but we can recommend any of the other topics - 1), 2) and a) through c) - for future Bourbaki seminars. Although phase transitions and critical phenomena are perhaps not so fashionable among physicists, anymore, they do still pose serious problems challenging the mathematician and mathematical physicist. Good mathematical understanding of critical phenomena is presumably a prerequisite for further progress in quantum field theory and, quite generally, in the theory of systems with infinitely many, degrees of freedom.

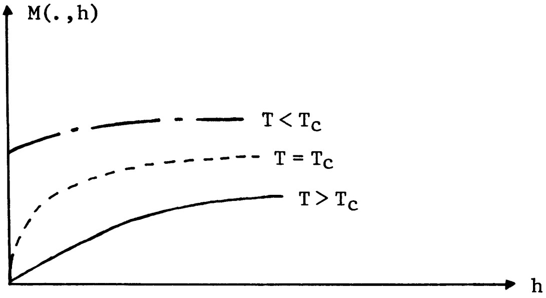
1.2 A little phenomenology of phase transitions and critical phenomena

We now try to explain, in intuitive terms, what phase transitions are and what kinds of phase transitions may occur. Our examples are chosen from *condensed matter physics*. Other examples are found in nuclear physics, astrophysics, quantum field theory... We shall study phase transitions in ferromagnets and mathematical models thereof (defined in Sect. 1.5).

A ferromagnet consists of a macroscopic (i.e. nearly infinite - with respect to a microscopic scale) piece of bulk matter, ideally arranged in a crystalline structure. At each point of the crystal lattice there is an atom or molecule with non-zero total angular momentum, (spin). There are interactions between the spins located at nearby points of the lattice which tend to align the spins. (It is argued that the dominant interactions are the so called exchange interactions which are a consequence of the Pauli principle.)

When the temperature, T, is large thermal fluctuations destroy correlations between spins located at very distant points of the lattice. If the system is placed in a magnetic field which is then slowly turned off, no magnetization remains. However, if T is sufficiently small the system remains magnetized (*spontaneous magnetization*) even after the external magnetic field has been turned off. Let h denote the strength of the magnetic field, and let $M(T,h)$ denote the magnetization as a function of temperature T and magnetic field h. The behaviour of $M(T,h)$ is shown in the following graphs :





The temperature, T_c , at which the phase transition occurs is called *critical temperature*. The so called *magnetic susceptibility*, χ , is given by

$$\chi(T,h) = \frac{\partial M(T,h)}{\partial h} .$$

It turns out that the susceptibility, $\chi(T) \equiv \chi(T,h=0)$, of a magnet in zero magnetic field diverges to $+\infty$, as T approaches T_c , as indicated in the above graph. It is an important theoretical problem to determine the way in which $\chi(T)$ diverges at T_c. This is a typical problem in the theory of critical phenomena. It is expected that, in dimension $d \neq 4$,

$$(*) \qquad \chi(T) \sim (T - T_c)^{-\gamma} , \quad T \geq T_c ,$$

for some number γ called *critical exponent*. Of course, in a laboratory, all that is available to us are three-dimensional or approximately planar pieces of ferromagnetic material. But in theory one can study d-dimensional magnets, where d is an arbitrary natural (or complex) number. It is expected that in four dimensions there are logarithmic corrections to the power law divergence of $\chi(T)$, but in five or more dimensions (*) is expected to hold with

$$\gamma = 1 .$$

This has recently been proven rigorously for some class of models ; (see §§ 2, 5). It is quite surprising that the value of γ is independent of dimension, for $d \geq 5$, and of the details of the mathematical models of ferromagnets. For $d < 4$, γ appears to depend on d , but not on the details of the mathematical model. One says, that critical exponents, like γ , are *universal*. (See §§ 4, 5.)

It should be emphasized that there are *different kinds of phase transitions* ; (see Sect. 1.5.) For example, the melting of ice is a transition which is quite different from the one in a ferromagnet : It has latent heat, and there is no quantity analogous to the susceptibility χ which would exhibit some (universal) power law divergence at the transition temperature.

In these notes we only consider the mathematical theory of the kind of phase transitions found in ferromagnets and its relation with quantum field theory.

The following two aspects will be ignored :

i) We shall study models of classical spins, i.e. quantum mechanical effects are taken into account only implicitly. (This is usually unjustified, except if the spin at each point of the lattice is enormous.) Our (naive) models of ferromagnets, *lattice spin systems*, are defined in Sect. 1.5 and analyzed in subsequent sections.

ii) We shall not discuss the connections between phase transitions and spontaneous breaking of (internal or spatial) symmetries, except in a few rather vague remarks. This topic has been considered in many excellent surveys, some of which are quoted in the bibliography.

1.3. Some physical problems mathematically related to each other

The main purpose in the following is to explain the relation between two circles of problems, namely

A) the construction of relativistic quantum field theories in the continuum limit ; and

B) higher order phase transitions and critical phenomena in lattice spin systems.

We think that the realization that A) and B) are intimately related is an important and deep idea, [1, 3, 4].

We shall then emphasize the discussion of B). In particular, we shall sketch how, mathematically, the theory of higher order phase transitions and critical phenomena is related to

- the statistical mechanics of topological defects in ordered media [5] ;
- the study of non-linear mappings on infinite dimensional spaces, of their fixed points and of the stable and unstable manifolds near those fixed points ;
- the mathematical theory of random walks and their intersection properties.

1.4. Relativistic quantum field theory

We now recall what is meant by a relativistic quantum field theory and its Euclidean description. Clearly we have to over-simplify matters.

Relativistic quantum field theory is an attempt towards combining the special theory of relativity and quantum mechanics into one mathematically consistent and physically correct theory (satisfying some causality principle). It can be characterized by various postulates, e.g. the (Gårding-) Wightman axioms[1] [6]. These axioms say that a relativistic physical system on a d-dimensional space-time can be described, in the simplest case, by the following mathematical structure :

(W0) The states of the system are the unit rays of a separable Hilbert space, $\mathcal{H}$.

(W1) With each test function f in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is associated an unbounded operator, $\phi(f)$, (the field operator) defined on and leaving invariant a dense domain $\mathcal{D} \subset \mathcal{H}$ which is independent of f , and

[1] Gauge theories require some modifications in those axioms ; see [7].

$$\phi(f)^* \supseteq \phi(\overline{f}) \ .$$

(W2) There is a continuous, unitary representation,

$$U : (a,\Lambda) \in \mathcal{P} \longrightarrow U(a,\Lambda) \ ,$$

of the Poincaré group $\mathcal{P}$ on $\mathcal{H}$, with the property that

$$U(a,\Lambda)\phi(f)U(a,\Lambda)^* = \phi(f_{(a,\Lambda)}) \ ,$$

where

$$f_{(a,\Lambda)}(x) \equiv f(\Lambda^{-1}(x-a)) \ ,$$

and

$$U(a,\Lambda)\mathcal{D} \subseteq \mathcal{D} \ .$$

(W3) The spectrum of the generators of the translation subgroup $\{U(a,\mathbb{1}) : a \in \mathbb{R}^d\}$, where d is the dimension of space-time, is contained in the forward light cone $\overline{V}_+$ ("positivity of the energy"), and 0 is an eigenvalue of those generators.

The eigenstate associated with 0 is called the *physical vacuum* and is denoted by Ω .

(W4) Field operators smeared out with test functions whose supports are space-like separated commute, (as operators defined on $\mathcal{D}$).

This is the "locality axiom" and expresses the causality principle alluded to above.

(W5) $\mathcal{D}$ is obtained by applying arbitrary polynomials in $\{\mathbb{1},\phi(f) : f \in \mathcal{S}(\mathbb{R}^d)\}$ to the physical vacuum, Ω .

From these "axioms" it follows [6] that a relativistic quantum field theory is uniquely characterized by the vacuum expectation values of products of field operators, the *Wightman distributions*,

(1.1) $$W_n(x_1,\ldots,x_n) = \langle\Omega,\phi(x_1)\ldots\phi(x_n)\Omega\rangle \ ,$$

$n = 0,1,2,\ldots$, $W_0 \equiv 1$. W_n is a tempered distribution on $\mathcal{S}(\mathbb{R}^{nd})$ which is invariant under simultaneous Poincaré transformations of its arguments and has various other properties which follow from (W0)-(W5) ; see [6].

Let

$$x = (t,\vec{x}) \quad , \quad \vec{x} \in \mathbb{R}^{d-1} \ ,$$

be the decomposition of a point in space-time into time - and space components. It can be shown that the distributions $W_n(t_1,\vec{x}_1,\ldots,t_n,\vec{x}_n)$ are the boundary values of analytic functions, the Wightman functions, whose domain of analyticity contains, in particular the points

$$\{(x_1,\ldots,x_n) : \mathrm{Im}(t_m - t_{m-1}) \neq 0 \ , \ m = 2,3,\ldots,n\} \ .$$

This permits us to introduce the functions

(1.2) $$S_n(x_1,\ldots,x_n) \equiv W_n(it_1,\vec{x}_1,\ldots,it_n,\vec{x}_n) \ ,$$

$n = 0,1,2,\ldots$, $S_0 \equiv 1$, t_m real, for $m = 1,\ldots,n$, $t_i \neq t_j$ for $i \neq j$. They are called *Euclidean Green's* or *Schwinger functions*. It has been proven by Osterwalder and Schrader [8] (see also [9, 10] for further related results) that under suitable conditions (called Osterwalder-Schrader axioms) a sequence of functions

$$\{S_n(x_1,\ldots,x_n)\}_{n=0}^{\infty}$$

uniquely determines, by analytic continuation, a sequence of Wightman distributions corresponding to a relativistic quantum field theory, in the sense of postulates (W0)-(W5). Among those conditions are

- invariance of $S_n(x_1,\ldots,x_n)$ under simultaneous Euclidean motions of all its arguments, and under arbitrary permutations, for all $n = 1,2,3,\ldots$;
- a positivity condition, called Osterwalder-Schrader - or reflection positivity, related to the positivity of the scalar product on $\mathcal{H}$ and the positivity of the energy, (W3). This condition has an analogue in statistical mechanics, (existence of a selfadjoint transfer matrix). See [8, 11].

In most models of scalar relativistic quantum field theory, the Schwinger functions, S_n , turn out to be intimately related to the so called correlation functions of some lattice spin system, studied in equilibrium statistical mechanics : *Schwinger functions can be constructed as continuum limits of correlation functions of lattice spin systems, as the lattice spacing tends to* 0 , [12].

1.5. Lattice spin systems

We shall consider the simplest, classical spin systems, described by the following mathematical structure :

i) As our lattice we choose the simple (hyper-) cubic lattice, $\mathbb{Z}^d$. With each site $j \in \mathbb{Z}^d$ we associate a *classical spin*

(1.3) $$\vec{\varphi}(j) \in \mathbb{R}^N_{(j)} \cong \mathbb{R}^N ,$$

$N = 1,2,3,\ldots$. A configuration, $\vec{\varphi}$, of spins assigns to each j a fixed vector $\vec{\varphi}(j) \in \mathbb{R}^N$. For each finite subset, Λ , of the lattice, we define a space of all spin configurations on Λ

(1.4) $$\begin{cases} K_\Lambda = \bigtimes_{j\in\Lambda} \mathbb{R}^N_{(j)} , \text{ and} \\ \vec{\varphi}_\Lambda = \{\vec{\varphi}(j) : j \in \Lambda\} \in K_\Lambda \end{cases}$$

which is a configuration of spins of a finite subsystem in Λ . We set $K_\infty \equiv K_{\mathbb{Z}^d}$.

ii) *The a priori distribution of the spin* $\vec{\varphi}(j)$ at j is given by a probability measure, $d\lambda(\vec{\varphi}(j))$, (the same for all j), on the Borel sets of $\mathbb{R}^N$. The *a priori* distribution of a configuration $\vec{\varphi}_\Lambda$ of spins on Λ is given by

(1.5) $$\prod_{j\in\Lambda} d\lambda(\vec{\varphi}(j)) ,$$

which is a probability measure on K_Λ .

iii) For each configuration $\vec{\varphi}_\Lambda$ of a finite subsystem we define an *energy*, or *Hamilton function*

(1.6) $$H_\Lambda(\vec{\varphi}_\Lambda)$$

which is assumed to be a continuous function on K_Λ . Let $\{\Lambda_n\}_{n=1}^{\infty}$ be an arbitrary sequence of finite regions in $\mathbb{Z}^d$ increasing to $\mathbb{Z}^d$ (e.g. in the sense of

Fisher [13]). We assume that the Hamilton functions $\{H_\Lambda\}_{\Lambda\subset\mathbb{Z}^d}$ have the property that the thermodynamic limit of the *interaction energy* between the spins in a bounded region Λ and the ones outside Λ ,

(1.7) $$W_{\Lambda,\Lambda^c} \equiv \lim_{n\to\infty}\{H_{\Lambda_n} - (H_\Lambda + H_{\Lambda_n\sim\Lambda})\} ,$$

exists, for each finite sublattice Λ , and that for all $\beta \geq 0$, the thermodynamic limit of the *free energy per unit volume*,

(1.8) $$\beta f(\beta,\lambda) \equiv \lim_{n\to\infty} - \frac{1}{|\Lambda_n|} \log Z_\beta(\Lambda_n) ,$$

where

(1.9) $$Z_\beta(\Lambda_n) \equiv \int_{K_{\Lambda_n}} \exp[-\beta H_{\Lambda_n}(\vec{\varphi}_{\Lambda_n})] \prod_{j\in\Lambda_n} d\lambda(\vec{\varphi}(j)) ,$$

exists. Here $\beta \equiv (kT)^{-1}$ is the inverse temperature.

iv) *An equilibrium state at inverse temperature* β of the infinite lattice spin system is given by a *probability measure*, $d\mu_{\beta,\lambda}(\vec{\varphi})$, on (the σ-algebra generated by the Borel cylinder sets of) K_∞ , with the property that, for every bounded measurable function A on K_Λ , where Λ is an arbitrary finite sublattice,

(1.10) $$\langle A\rangle_{\beta,\lambda} \equiv \int A(\vec{\varphi}_\Lambda) d\mu_{\beta,\lambda}(\vec{\varphi}) = \int d\rho(\vec{\varphi}_{\Lambda^c}) \int e^{-\beta W_{\Lambda,\Lambda^c}(\vec{\varphi})} . \exp[-\beta H_\Lambda(\vec{\varphi}_\Lambda)] A(\vec{\varphi}_\Lambda) \prod_{j\in\Lambda} d\lambda(\vec{\varphi}(j)) ,$$

where $d\rho(\vec{\varphi}_{\Lambda^c})$ is a finite measure on K_{Λ^c} . These are the so called Dobrushin-Lanford-Ruelle equations [14].

Whenever reasonable we shall think of the simplest *examples of lattice systems* having properties i) through iv) above, e.g.

(1.11) $$d\lambda(\vec{\varphi}) = \text{const.}\exp[-\lambda|\vec{\varphi}|^4 + \frac{\mu^2}{2}|\vec{\varphi}|^2 + \beta h\varphi^1]d^N\varphi ,$$

$\lambda > 0$, μ^2 and h real numbers,

(1.12) $$H_\Lambda(\vec{\varphi}) = - \sum_{\substack{j,j'\in\Lambda \\ |j-j'|=1}} \vec{\varphi}(j).\vec{\varphi}(j') .$$

Note that, for $\mu^2 = \lambda$, $N = 1$, this model approaches the usual *Ising model*, as $\lambda \to \infty$.

For $N = 1,2$ and $1 \leq d \leq 5$, this example exhibits all kinds of phase transitions and ciritcal behaviour, as β ranges over $(0,\infty)$ and h over $(-1,1)$. The parameter h has the physical interpretation of a *magnetic field*. In the following, β and h will usually be the only parameters that we shall vary. We therefore write $f(\beta,h)$ instead of $f(\beta,\lambda)$, $\langle(.)\rangle_{\beta,h}$ instead of $\langle(.)\rangle_{\beta,\lambda}$, etc. Moreover

(1.13) $$f(\beta) \equiv f(\beta,h=0) , \quad \langle(.)\rangle_\beta \equiv \langle(.)\rangle_{\beta,h=0} .$$

Next, we introduce some basic quantities in terms of which phase transitions and critical phenomena can be discussed. (For simplicity, we shall often consider one - component spins, i.e. $N = 1$.)

The basic objects in terms of which lattice spin systems are analyzed are the *correlation functions*

(1.14) $$< \varphi(x_1)\dots\varphi(x_n)>_{\beta,h} \equiv \int \prod_{k=1}^{n} \varphi(x_k)d\mu_{\beta,h}(\varphi) \ .$$

It is these correlation functions which often turn out to be directly related to the Euclidean Green's functions, $S_n(x_1,\dots,x_n)$, defined in (1.2), of a relativistic quantum field theory. Of particular importance are

a) *the magnetization*

$$M(\beta,h) \equiv <\varphi(x)>_{\beta,h} = \frac{\partial f(\beta,h)}{\partial h} \ ;$$

b) *the susceptibility*

$$\chi(\beta,h) = \beta^{-1}\frac{\partial M(\beta,h)}{\partial h} = \sum_{x\in\mathbb{Z}^d} <\varphi(0)\varphi(x)>^c_{\beta,h} \ ,$$

where

$$<\varphi(0)\varphi(x)>^c_{\beta,h} = <\varphi(0)\varphi(x)>_{\beta,h} - <\varphi(0)>^2_{\beta,h} \ ;$$

c) *the internal energy density*

$$u(\beta,h) = -\frac{\partial(\beta f(\beta,h))}{\partial\beta} \ ,$$

and

d) *the specific heat*

$$c(\beta,h) = -k\beta^2\frac{\partial u(\beta,h)}{\partial\beta} \ .$$

We are also interested in the asymptotic behaviour of the two-spin correlation, $<\varphi(0)\varphi(x)>^c_{\beta,h}$, as $|x|\to\infty$. One measure of that behaviour is

e) *the inverse correlation length (mass)*

$$m(\beta,h) = -\lim_{|x|\to\infty}\frac{1}{|x|}\log<\varphi(0)\varphi(x)>^c_{\beta,h}$$

which measures the exponential decay rate of $<\varphi(0)\varphi(x)>^c_{\beta,h}$.

We now come to the description of various types of *phase transitions* and introduce the notion of *critical exponents*.

We all have some intuitive understanding of what is meant by a phase transition : If some thermodynamic parameter is varied there may occur a sudden change in the behaviour of the system, as described in Sect. 1.2. Let us imagine that we vary the inverse temperature β . It is convenient to distinguish between the following two kinds of phase transitions :

I) "*Phase transitions with local order parameter*" : For β small the equilibrium state is *unique*, while for large β there are *several, mutually singular solutions* of the DLR equations (1.10). In the example specified by (1.11), (1.12) this kind of phase transition occurs in zero magnetic field $(h=0)$ in two or more dimensions, provided $N = 1$ (i.e. in the Ising model) and in three or more dimensions, provided $N \geq 2$.

Remark.— It may happen that the equilibrium state is degenerate (i.e. that there are several solutions of the DLR equations) only at the phase transition point.

II) "*Phase transitions without local order parameter*" : The equilibrium state $<(.)>_\beta$ is *unique for all values of* β *, but does not depend analytically on* β . This kind of phase transitions has been established in the example introduced in (1.11) and (1.12), for $N = d = 2$: For $h = 0$ and small β , correlations in

$<(.)>_\beta$ have exponential fall-off, i.e. $m(\beta) > 0$, while for large β they have only power law fall-off and $m(\beta) = 0$. Mathematically, this is a rather subtle problem ; see [15].

In both cases, I) and II), there will be *at least* one value, β_o , of the inverse temperature which separates two different regimes, i.e. at which the transition occurs. One can distinguish two kinds of transition points :

(1) $\beta_o = \beta_c$ *is a critical point* :

We say that β_o is a critical point if

$$m(\beta) \searrow 0 \quad , \quad \text{as} \quad \beta \nearrow \beta_c \quad , \quad \text{or} \quad \beta \searrow \beta_c \; .$$

A phase transition with a critical point is traditionally called a "higher order phase transition" (although Ehrenfest's definition of the order of a transition is actually different and is not very useful).

The transitions in the example (1.11), (1.12) with $h = 0$ and $N = 1,2, d \geq 2$, are transitions passing through a critical point, β_c , as β is varied. This is typical of transitions in a ferromagnet ; (see Sect. 1.2).

(2) β_o *is not a critical point* :

β_o is not a critical point if $m(\beta)$ is strictly positive in an open interval containing β_o .

If in example (1.11) one fixes $\beta > \beta_c$ and varies h then a phase transition occurs at $h = 0$, and $h = 0$ is *not* a critical point. Moreover (for $N = 1,2,3$) the equilibrium state is unique, except at $h = 0$. A more interesting example of this kind of transition (traditionally called first order phase transition) is discussed in [16]. The melting of ice is such a transition ; (see Sect. 1.2).

For the construction of relativistic quantum field theories only transitions with critical points are relevant.

With "critical phenomena" is meant the behaviour of a physical system in thermal equilibrium near the critical point of a (higher order) phase transition.

Among the first theoretical attempts towards understanding higher order transitions and critical phenomena were the Landau theory of second order phase transitions and mean field theory. These theories are quantitatively wrong in dimension two or three and do not describe experiments accurately.

It is the purpose of the following to pin point some of the mathematical questions arising in the modern theory of critical phenomena, as developed by Wilson, Kadanoff, Jona-Lasinio and collaborators, and many others ; see [1, 3, 4, 17].

Furthermore, we shall try to explain how *the construction of the Schwinger functions of a relativistic quantum field theory can be reduced, in principle, to the study of the behaviour of lattice spin systems in the vicinity of some critical point.*

The approach to the critical point in a lattice spin system is described in terms of *critical exponents* which we define next. For the sake of concreteness we consider the examples introduced in (1.11), (1.12). The only critical points of these systems lie on the line $h = 0$, (for $N = 1,2,3$. This is a consequence of the Lee-Yang theorem [13, 18] and refs. given there).

We assume, temporarily, that $d \neq 4$. Let β_c be some critical point. It has been expected for a long time (originally on the basis of scaling arguments, more recently as a consequence of the renormalization group) that the quantities $M(\beta)$, $\chi(\beta)$, $c(\beta)$, $m(\beta)$, ... introduced above have a *power law behaviour* in $t \equiv \frac{\beta_c - \beta}{\beta_c}$, as $\beta \to \beta_c$:

$$(1.15) \qquad \begin{array}{lll} M(t) \sim |t|^{\beta'} & , & \text{for } \beta > \beta_c \\ \chi(t) \sim t^{-\gamma} & & \\ c(t) \sim t^{-\alpha} & & \text{for } \beta < \beta_c , \\ m(t) \sim t^{\nu} & & \end{array}$$

where β', γ, α and ν are some positive numbers which are called *critical exponents*. (We hasten to add that the law $c(t) \sim t^{-\alpha}$ is violated in two dimensions.) The mathematical meaning of $f(x) \sim x^{\mu}$ is

$$\mu = \lim_{x \searrow 0} \frac{\log f(x)}{\log x} .$$

One also introduces a critical exponent η (the "anomalous dimension") for the two-spin correlation $\langle \varphi(0)\varphi(x) \rangle_\beta^c$. To simplify matters, suppose that $\beta = \beta_c$, so that $m(\beta) = m(\beta_c) = 0$. Then η is defined by

$$(1.16) \qquad \langle \varphi(0)\varphi(x) \rangle_\beta^c \underset{|x| \to \infty}{\sim} |x|^{-(d-2+\eta)} ,$$

in the sense that

$$\eta = 2 - d - \lim_{|x| \to \infty} \log \langle \varphi(0)\varphi(x) \rangle_\beta^c / \log |x| .$$

It is expected that in *four dimensions* there are *logarithmic corrections* to the scaling law [19], e.g.

$$(1.17) \qquad m(t) \sim t^{1/2} (\log(\tfrac{1}{t}))^{-\tilde{\nu}} , \quad \text{etc.}$$

One of the main problems in the theory of phase transitions with critical points is a proof of the scaling laws (1.15)-(1.17) and the calculation of the critical exponents. Of help in this task are the so-called scaling relations and critical exponent inequalities, e.g.

$$(1.18) \qquad (2 - \eta)\nu - \gamma = 0$$

(for a proper definition of η), due to Fisher, or

$$(1.19) \qquad d\nu \geq 2 - \alpha ,$$

the Josephson inequality, etc. For a survey of recent, rigorous results concerning such inequalities see [20, 21, 22, 23].

One of the main achievements of the renormalization group is just precisely that it predicts values for the exponents which fit the experimental data extremely well. (Those predictions are obviously non-rigorous and obviously correct.)

The main idea of the renormalization group is to study the behaviour of a system under a change of scale, given by a transformation acting on an appropriately chosen space of states, or Hamilton functions. (It appears that it is not always possible to let those scale transformations act on a space of Hamilton functions, so defining them on some convex manifold of states is a better starting point.) In particular, one tries to find the *fixed points* of these scale transformations, corresponding to scale-invariant systems. Critical exponents are then related to real eigenvalues > 1 of the linearization of the scale transformation at a hyperbolic fixed point.

It will now be our task to make these remarks more precise and to summarize some of the progress that has been made in understanding phase transitions with critical points, critical exponents, scale transformations and the renormalization group.

§ 2. Recent results on phase transitions with critical points

In this section we describe some recent results on phase transitions with critical points and we briefly outline some general ideas that go into the proofs of those results.

2.1. Existence of phase transitions

Presently there are basically three general methods to rigorously establish the existence of phase transitions in lattice systems of statistical mechanics.

(a) *Exact solutions.* This technique applies only to a limited class of models such as one-dimensional systems with finite range interactions, the two-dimensional Ising model, the eight-vertex models,...[1]. In recent years, the interest in exact solutions has been revived through the work of Jimbo, Miwa and Sato [24], Faddeev and collaborators [25] and Thacker and collaborators [26]. Exact solutions tend to provide a fairly detailed description of the phase transition, including quantitative information, but often somewhat obscure the physical mechanisms leading to the transition. We shall not discuss any exact solutions in the following.

(b) *Energy-entropy (Peierls-type) arguments.* In its most general form this method can be viewed as a way of reinterpreting spin systems as gases of ("topologically stable") defects in an ordered medium [5] (Bloch walls = Peierls contours, vortices, magnetic monopole lines...) and of analyzing transitions in defect gases by estimating defect - energies and - entropies.

This method can be applied to study thermodynamic phases in which the defect gas is dilute. The original Peierls argument [28] was invented to analyze the Ising model. It was reconsidered and extended by Griffiths and Dobrushin, in the

[1] See e.g. E.H. Lieb's survey, [27].

sixties [29]. Subsequently, Minlos, Pirogov and Sinai developed a very general, constructive form of the Peierls argument [30]. Glimm, Jaffe and Spencer first applied it to quantum field models, introducing a new technique to analyze "contour probabilities" [31]. Furthermore they combined a Peierls argument with expansion methods permitting to estimate small fluctuations around defect configurations [32]. Some of their ideas were systematized and extended in [11, 33, 34].

The observation that the basic elements of the Peierls argument, energy-entropy considerations, can be applied to rigorously analyze a much wider class of model systems equivalent to gases of defects, including ones with long-range-interactions and massless phases, is contained in work by the authors, [15, 35, 36]. In particular, we have succeeded to set up Peierls-type arguments in systems with continuous (but *abelian*) symmetry groups. Our techniques combine entropy - (i.e. combinatorial) estimates for suitably constructed blocks of defects with some kind of "block spin integration", borrowed from the renormalization group, which serves to exhibit self-energies of defects.

We now briefly describe some general elements of the simplest kind of Peierls argument somewhat more precisely : Consider a physical system whose configurations can be described by a classical spin field, $\vec{\varphi}$. We suppose for the moment that $\vec{\varphi}$ is defined on $\mathbb{R}^d$ (rather than $\mathbb{Z}^d$), continuous except on surfaces of co-dimension ≥ 1 and with values in a compact manifold M (e.g. S^N , $N=0,1,2,\dots$).

Consider, as an example, a configuration $\vec{\varphi}$ which is continuous except on a hyperplane H_k of dimension $k \leq d-1$. The space of all configurations $\vec{\varphi} : \mathbb{R}^d \sim H_k \longrightarrow M$ can be decomposed into homotopy classes labelled by the elements of the homotopy groups

$$\pi_{d-k-1}(M) \tag{2.1}$$

A configuration $\vec{\varphi}$ labelled by a non-trivial element of $\pi_{d-k-1}(M)$ is called a topological defect of dimension k .

The idea is now to interpret the equilibrium configurations of the spin field $\vec{\varphi}$ (distributed according to an equilibrium state $d\mu_\beta(\vec{\varphi})$) as equilibrium configurations of a *gas of interacting, topological defects*. The locus of a defect, δ_k , in this gas, corresponding to a non-trivial element $g_k \in \pi_{d-k-1}(M)$, is a closed, bounded surface, Σ_k , of dimension k . In the following we assume that all homotopy groups of M are discrete.

It turns out that the main features of the statistical mechanics of defect gases can often be described by an energy-entropy argument of the following type : One calculates a self-energy density, $\varepsilon(g_k)$, of a defect δ_k corresponding to a non-trivial element $g_k \in \pi_{d-k-1}(M)$. The energy of δ_k is then estimated by

$$E(\delta_k) \gtrsim \varepsilon(g_k)\,|\Sigma_k| \ , \tag{2.2}$$

where $|\Sigma_k|$ is the k-dimensional area of Σ_k .

After introducing some coarse graining (e.g. replacing continuum models by *lattice models*) one can argue that the entropy $S(g_k,n)$ of the class of all defects labelled by g_k whose loci contain a given point, e.g. the origin, and have area

$$|\Sigma_k| = \text{const. } n \quad , \quad n = 1,2,3,\ldots ,$$

is estimated by

$$S(g_k,n) \leq c(g_k).n \ , \tag{2.3}$$

where $c(g_k)$ is a geometrical constant. The density, $\rho(g_k,n)$, of such defects, δ_k , is then proportional to

$$\rho(g_k,n) \propto e^{-\beta E(\delta_k)+S(g_k,n)} \lesssim e^{(-\beta\varepsilon(g_k)+c(g_k))n} \tag{2.4}$$

provided the interactions between different defects are, in some sense, weak. Formula (2.4) suggests that when the inverse temperature β decreases below the point

$$\beta(g_k) \approx c(g_k)/\varepsilon(g_k) \ , \tag{2.5}$$

defects labelled by g_k condense, and there are, with high probability, infinitely extended defects of type g_k . One expects, therefore, that there is a phase transition, as β is varied through $\beta(g_k)$.

The argument sketched in (2.2)-(2.5) is called an energy-entropy argument. The art is then to apply such arguments to specific spin systems to actually *prove* that a transition occurs. This has been done for a large class of lattice spin systems with abelian symmetry groups[1]. This may sound confusing, because the notion of a "topological defect" does not make sense when one considers spin configurations on a lattice. It turns out, however, that in models with abelian symmetry groups one can use a duality transformation (Fourier transformation on the group) to exhibit what in the continuum limit corresponds to topological defects. Since this will presumably sound rather vague, we now briefly describe two examples.

(1) *The Ising model* (see (1.11) and (1.12)). In this example : $M = \{-1,1\}$, $\varphi(x) = \pm 1$ with probability $1/2$, for all $x \in \mathbb{Z}^d$, and

$$H_\Lambda(\varphi) = \sum_{\substack{j,j'\in\Lambda \\ |j-j'|=1}} \{1 - \varphi(j).\varphi(j')\} \ . \tag{2.6}$$

The defects are the Peierls contours, i.e. $(d-1)$-dimensional, closed connected surfaces in the dual lattice separating a domain where φ takes the value $+1$ from a domain where it takes the value -1 . By (2.6), the energy of a contour is equal to its $(d-1)$-dimensional area. It is a simple, combinatorial exercise to show that in $d \geq 2$ dimensions the number of contours of area n enclosing the origin is bounded above by c^n , where c is a geometrical constant. The inter-

[1] or non -abelian, but *discrete* symmetry groups.

actions between contours are given by an exclusion principle.

Suppose now that, for all x outside an arbitrarily large, finite set $\Lambda \subset \mathbb{Z}^d$, $d \geq 2$, $\varphi(x) = +1$. Let $p_+(\beta)$ and $p_-(\beta)$ be the probabilities that $\varphi(0) = +1, -1$, respectively, in an equilibrium state at inverse temperature β, with the above boundary conditions outside Λ. Clearly every configuration φ for which $\varphi(0) = -1$ must contain at least one Peierls contour enclosing the origin. Hence

(2.7) $$p_-(\beta) \leq \sum_{n=2d}^{\infty} e^{-\beta n} c^n \ll \frac{1}{2},$$

if β is large enough, and thus

(2.8) $$\langle\varphi(0)\rangle_\beta = p_+(\beta) - p_-(\beta) = 1 - 2p_-(\beta) > 0,$$

for large β. This shows that in zero magnetic field $(h=0)$ and for large β there is a spontaneous magnetization in the direction imposed by the boundary conditions. It is not hard to show that for small β there is no spontaneous magnetization, (the equilibrium state in the thermodynamic limit is unique for small β). Thus there is a phase transition.

(2) *The two-component rotor (classical XY) model.* In this model : $M = S^1$, $d\lambda(\vec{\varphi})$ is the Lebesgue measure on S^1, the Hamilton function is given by

(2.9) $$H_\Lambda(\vec{\varphi}) = \sum_{\substack{j,j'\in\Lambda \\ |j-j'|=1}} \{1 - \vec{\varphi}(j).\vec{\varphi}(j')\} = \sum_{\substack{j,j'\in\Lambda \\ |j-j'|=1}} \{1 - \cos(\vartheta(j) - \vartheta(j'))\},$$

where $\vartheta(j)$ is the angle parametrizing the unit vector $\vec{\varphi}(j)$.

Since $\pi_1(S^1) = \mathbb{Z}$, $\pi_i(S^1) = 0$, $i \neq 1$, the defects of this model are labelled by an integer and their loci have co-dimension 2. They are called *vortices*. In order to study the transitions in this model, the idea is to invent a rigorous version of the energy-entropy argument (2.2)-(2.5) for the gas of vortices equivalent to the rotator model. The equivalence between the rotator model and a vortex gas can be seen by Fourier series expansion of the equilibrium state, $d\mu_\beta(\vec{\varphi})$, in the angular variables $\{\vartheta(j)\}$ and subsequent application of the Poisson summation formula ; see e.g. [37, 15, 35]. The problem that one meets when one tries to analyze the vortex gas is that there are interactions of extremely long range between individual vortices. In *three or more dimensions*, these interactions turn out to be quite irrelevant, and the arguments (2.2)-(2.5) can be made rigorous. One concludes from (2.4) that, for large β, the density of vortices is small, i.e. the number of defects per unit volume in each equilibrium configuration $\vec{\varphi}$ is very small. Therefore one expects that, in the average, $\vec{\varphi}$ has a fixed direction, i.e.

(2.10) $$\langle\vec{\varphi}(x)\rangle_\beta = \vec{M}(\beta) \neq 0,$$

for large β ; $\vec{M}(\beta)$ is determined by the boundary conditions. These arguments are made rigorous in [35] (a slightly non-trivial task). It is well known that for small β , or for arbitrary β and $d = 1,2$ [38],

(2.11) $$<\vec{\varphi}(x)>_\beta = 0 .$$

In *two dimensions*, the vortices are point-like objects. The interaction between two vortices of strength q_1 and q_2 , respectively, separated by a distance ℓ is approximately given by

(2.12) $$- q_1 q_2 \frac{1}{2\pi} \ln \ell$$

which is the Coulomb potential between two point charges, q_1 and q_2 , in two dimensions. Suppose now that $q_1 = - q_2 = 1$. The entropy, S , of the class of configurations of a + vortex and a - vortex separated by a distance ℓ , within some distance $\propto \ell$ from the origin is given by

(2.13) $$e^S \approx \text{const.}\ \ell^3$$

Thus, for $\beta > 8\pi$,

(2.14) $$e^{-\beta E} e^S \approx \text{const.}(\ell + 1)^{3-(\beta/2\pi)}$$

is summable in ℓ . This means that configurations of one vortex of strength +1 and one vortex of strength -1 , separated by a finite distance, are thermodynamically *stable*. In fact, it can be shown by a somewhat difficult, inductive construction [15], extending over an infinite sequence of length scales, that for sufficiently large values of β all vortices can be arranged in finite, neutral clusters of finite diameter and finite density. The conditions characterizing those clusters are scale-invariant. Our construction thus involves ideas of scale-invariance and self-similarity. Furthermore, it requires successive integrations over "fluctuations" on ever larger length scales, (a device reminiscent of renormalization group methods).

For small β , vortices unbind and form a plasma. Such Coulomb plasmas are studied rigorously in [39]. Thus, one expects a phase transition, as β is varied. It is non-trivial to show that the transition in the two-dimensional vortex gas just described corresponds, in the two-dimensional, two-component rotor model, to one from a small β phase in which $<\vec{\varphi}(0).\vec{\varphi}(x)>_\beta$ has exponential fall-off in $|x|$ to a large β phase in which $<\vec{\varphi}(0).\vec{\varphi}(x)>_\beta$ falls off like an inverse power (≤ 1) of $|x|$, as $|x| \to \infty$. This is proven rigorously in [15]. For details and further results on this and related models see [15, 35, 36, 39].

We now proceed to discussing the third general method in the theory of phase transitions.

(c) *Infrared bounds (rigorous spin wave theory)* [40]. This method which originated in [40] is rather general and is the only known method which gives satisfactory results in models where the spin takes values in a non-linear manifold and the symmetry group is non-abelian. (A review for mathematicians may be found e.g. in [41].) We describe it in terms of an example : Let $\vec{\varphi}$ be a lattice spin with $N = 1,2$, or 3 components. Let

$$d\lambda(\vec{\varphi}(x)) = e^{h\varphi^1(x)}\delta(|\vec{\varphi}(x)|^2 - 1)d^N\varphi(x) \,, \tag{2.15}$$

which is a measure on the $(N-1)$-dimensional unit sphere approaching the uniform measure, as $h \to 0$. The Hamilton function is given by

$$H_\Lambda(\vec{\varphi}) = \sum_{\substack{j,j'\in\Lambda \\ |j-j'|=1}} \{1 - \vec{\varphi}(j).\vec{\varphi}(j')\} \,, \tag{2.16}$$

and let $d\mu_{\beta,h}(\vec{\varphi})$ be an equilibrium state satisfying the DLR equations (1.10). It is known that for $h \neq 0$, $d\mu_{\beta,h}$ is unique (within some class of boundary conditions). We suppose that the underlieing lattice is *three* - or *higher dimensional*. Let Δ be a large, finite (hyper) cube,

$$\vec{\varphi}(\Delta) = \frac{1}{\text{vol.}\,\Delta} \sum_{j\in\Delta} \vec{\varphi}(j) \,.$$

The basic idea of spin wave theory is that for large β

$$\vec{\varphi}(\Delta) \approx M\vec{e}_1 + \delta\vec{\varphi}(\Delta) \tag{2.17}$$

where $\vec{e}_1$ is the unit vector in the 1-direction, i.e. the direction of the magnetic field (see (2.15)), $M > 0$ if $h > 0$, and $\delta\vec{\varphi}(\Delta)$ is the fluctuation of $\vec{\varphi}(\Delta)$ around $M\vec{e}_1$ which one expects to be $\propto \beta^{-1/2}$, for equilibrium configurations at low temperatures (large β).

These ideas can be formalized as follows : Let

$$\langle\vec{\varphi}(0).\vec{\varphi}(x)\rangle^c_{\beta,h} \equiv \langle\vec{\varphi}(0).\vec{\varphi}(x)\rangle_{\beta,h} - |\langle\vec{\varphi}(0)\rangle_{\beta,h}|^2 \,,$$

$h \neq 0$, and let $G^c_{\beta,h}(k)$ be the Fourier transform of $\langle\vec{\varphi}(0).\vec{\varphi}(x)\rangle^c_{\beta,h}$ in x which is a function on the d-dimensional torus,

$$B = [-\pi,\pi]^d \,, \quad \text{(the first Brillouin zone)}.$$

By using the so called transfer matrix method, Simon and the authors [40] have shown that

$$0 \leq G^c_{\beta,h}(k) \leq N\beta^{-1}[2d - 2\sum_{\alpha=1}^{d} \cos k_\alpha]^{-1} \,. \tag{2.18}$$

The upper bound in (2.18) which is called infrared -, or spin wave bound and our proof of this bound were inspired by known results (the Källen-Lehmann spectral representation of a two-point function, e.g. [42]) in relativistic quantum field theory. Mathematically, the proof is related to a proof of the Hölder inequality for traces ; (in fact one proof of (2.18) is based on the Hölder inequality applied to the trace of a product of integral operators.) By Fourier transformation

$$\begin{cases} 0 \leq \langle|\vec{\varphi}(0)|^2\rangle^c_{\beta,h} \leq N\beta^{-1}I_d \,, \text{ where} \\ I_d \equiv (2\pi)^{-d}\int_B d^dk[2d - 2\sum_{\alpha=1}^{d} \cos k_\alpha]^{-1}. \end{cases} \tag{2.19}$$

We note that I_d is divergent for $d = 1,2$, but finite in $d \geq 3$ dimensions, (with $I_d \propto d^{-1}$, for large d).

By (2.15) it is obvious that

$$<|\vec{\varphi}(0)|^2>_{\beta,h} = 1 .$$

Thus, for $\beta > NI_d$,

(2.20) $$M(\beta,h)^2 \equiv |<\vec{\varphi}(0)>_{\beta,h}|^2 \geq 1 - N\beta^{-1}I_d > 0 ,$$

uniformly in $|h| > 0$, i.e.

(2.21) $$\lim_{h \searrow 0} M(\beta,h) = M(\beta) > 0 .$$

It is easy to prove that $M(\beta) = 0$, for sufficiently small β. We therefore conclude that there is a phase transition.

It follows from the infrared bound (2.18) that

(2.22) $$\delta\vec{\varphi}(\Delta) \approx \sqrt{N\beta^{-1}|\Delta|^{(2-d)/d}} ,$$

in accordance with heuristic ideas based on spin wave theory. Note that for $d = 1,2$, $\delta\vec{\varphi}(\Delta)$ does not become small, as the volume $|\Delta|$ of Δ tends to ∞. This suggests that there is no spontaneous magnetization when $d = 1$ or 2. Indeed, for $N \geq 2$, there is no spontaneous magnetization and no symmetry breaking in two dimensions ; the well-known Mermin-Wagner theorem [38] ; (see also [43] for a proof which formalizes the above fluctuation argument).

The results reported here extend to a large class of spin systems, but the hypotheses required for the known proofs of the infrared bound (2.18) impose serious limitations on the class of Hamilton functions for which (2.18) is known to be valid [11. 3)].

We conclude this subsection by mentioning some recent results on the structure of the space of translation - invariant equilibrium states in the Ising - $(N = 1)$ and the two-component rotor model $(N = 2)$:

For $h \neq 0$, or for $h = 0$ but β so small that $M(\beta) = 0$, the (translation-invariant) equilibrium states are *unique* [44, 45]. Next, suppose that $h = 0$, $M(\beta) \neq 0$ (i.e. there is a non-zero spontaneous magnetization) and that β is a point of continuity of the internal energy density, $-\frac{\partial(\beta f)}{\partial\beta}$. (Since $\beta f(\beta)$ is concave in β, this is true for all, except perhaps countably many, values of β.) Then :

(i) In the Ising model, there exist precisely two extremal, translation-invariant equilibrium states, $<(.)>_{\beta,\pm}$, with

$$0 < <\varphi(0)>_{\beta,+} = - <\varphi(0)>_{\beta,-} .$$

See [46]. A deeper result, due to Aizenman [47], is that in the *two-dimensional* Ising model (i) is true *for all* $\beta > \beta_c$, *without* assuming translation invariance.

(ii) In the $N = 2$ rotor model (under the same hypotheses) there exist infinitely many extremal, translation-invariant equilibrium states

$$\{<(.)>_{\beta,\vartheta} : \vartheta \in [0,2\pi)\}$$

which can *all* be labelled by an angle ϑ and such that

$$\langle\vec{\varphi}(0)\rangle_{\beta,\vartheta} = |\langle\vec{\varphi}(0)\rangle_{\beta,0}| \begin{pmatrix}\cos\vartheta\\ \sin\vartheta\end{pmatrix} .$$

For the proof see [48].

The proofs of the results mentioned here are rather unintuitive and of very limited interest to the mathematician, although they involve some clever ideas.

2.2. Existence of critical points and inequalities for critical exponents

Almost all *rigorous* results concerning the existence of critical points and critical exponents known to us are results on the Ising - and the two - component rotor model, or the more general family of models defined in (1.11), (1.12), for $N = 1,2,(3,4)$ components. We therefore restrict our review to these models, but see [49] for a discussion of Dyson's hierarchical model.

The first rigorous results on the existence of critical points and estimates on critical exponents were proven by Glimm and Jaffe ; see [50] and refs. given there. As a consequence of the Lee-Yang theorem, the inverse correlation length $m(\beta,h)$ introduced in Sect. 1.4, e) is strictly positive, when $h \neq 0$. Let β_c be defined by

(2.23) $$\beta_c = \sup\{\beta : m(\beta) \equiv m(\beta,h=0) > 0\} .$$

Rosen and Glimm and Jaffe (see [50] for references) have shown that $m(\beta)$ tends to 0 continuously, as $\beta \nearrow \beta_c$. It has also been shown [20. 3)] that the magnetic susceptibility $\chi(\beta)$ diverges, as $\beta \nearrow \beta_c$.

Among rigorously established inequalities for critical exponents are (see (1.15), Sect. 1.4, for definitions) :

$$\nu \geq 1/2$$
$$\gamma \geq 1$$
$$0 \leq \eta \leq 1$$
$$d\nu \leq 2 - \alpha ,$$

etc. We refer the reader to [50] for a summary and references and to [21] for interesting generalizations.

Although the proofs of these results are quite clever, they are based on very special features of the Ising - and rotor model. They hardly involve mathematical arguments which are interesting in their own right and are therefore not paraphrased here.

There are now emerging two somewhat general, rigorous approaches towards a theory of the critical point and critical exponents [22, 23], [51, 52] which appear to give fairly complete results in five or more dimensions, for reasons we shall try to explain in the following.

§ 3. Scale transformations and scaling limit

In order to simplify our discussion, we consider a one-component spin field, φ, on the lattice $\mathbb{Z}^d$. Let $d\mu_\beta(\varphi)$ be an equilibrium state. (For simplicity, we imagine that β is the only thermodynamic parameter that is varied, but there could be dependence on a magnetic field, h, or other parameters, as well.) Let

(3.1) $$\varphi_x(j) = \varphi(j+x) \ , \quad x \in \mathbb{Z}^d \ ,$$

and assume that $d\mu_\beta(\varphi)$ is translation invariant, i.e.

(3.2) $$d\mu_\beta(\varphi_x) = d\mu_\beta(\varphi) \ .$$

As in Sect. 1.4, we define the correlation functions as the moments of $d\mu_\beta$, i.e.

(3.3) $$<\varphi(x_1)\ldots\varphi(x_n)>_\beta = \int \prod_{k=1}^{n} \varphi(x_k)\, d\mu_\beta(\varphi) \ .$$

By a trivial re-definition of φ it is always possible to assume that

$$<\varphi(x)>_\beta = 0 \ .$$

In the following we are interested in analyzing the long distance limit of the correlation functions defined in (3.3) and in relating existence and properties of this limit to the behaviour of the equilibrium state and the correlations, as β approaches a critical point β_c, defined as in (2.23). We assume that, for $\beta < \beta_c$, the state $<(.)>_\beta$ is extremal invariant (i.e. $d\mu_\beta$ is ergodic under the action of lattice translations, defined in (3.1)) and that $m(\beta)$ is positive, i.e. $<\varphi(x)\varphi(y)>_\beta$ tends to 0 exponentially fast, as $|x-y| \longrightarrow \infty$ with decay rate denoted $m(\beta)$; see Sect. 1.4, e). Furthermore, we assume that $m(\beta)$ tends to 0 continuously, as $\beta \nearrow \beta_c$. As mentioned in Sect. 2.2, these assumptions are known to hold in the Ising - and the $N = 2$ rotor model and in the family of models introduced in (1.11), (1.12), for $N = 1,2$.

We now define the *scaled correlations*

(3.4) $$G_\vartheta(x_1,\ldots,x_n) \equiv \alpha(\vartheta)^n <\varphi(\vartheta x_1)\ldots\varphi(\vartheta x_n)>_{\beta(\vartheta)} \ ,$$

where

(3.5) $$\begin{cases} 1 \le \vartheta < \infty \ , \\ \vartheta x_j \in \mathbb{Z}^d \iff x_j \in \mathbb{Z}^d_{\vartheta^{-1}} \equiv \{y : \vartheta y \in \mathbb{Z}^d\} \ , \end{cases}$$

and $\beta(\vartheta) < \beta_c$ and $\alpha(\vartheta)$ are functions of the *scale parameter* ϑ which one tries to choose in such a way that a non-trivial limit, as $\vartheta \to \infty$, exists. In the models mentioned above it suffices to impose the following *renormalization condition* : For $0 < |x-y| < \infty$,

(3.6) $$0 < \lim_{\vartheta\to\infty} G_\vartheta(x,y) \equiv G^*(x-y) < \infty \ .$$

It turns out that in our class of models (3.6) suffices to show that some limit

(3.7) $$G^*(x_1,\ldots,x_n) = \lim_{\vartheta_i\to\infty} G_{\vartheta_i}(x_1,\ldots,x_n)$$

exists, and $G^*(x_1,\dots,x_n)$ is a translation-invariant distribution, for all $n = 3,4,\dots$. It follows from (3.4) and (3.6) that

(3.8) $$\beta(\vartheta) \nearrow \beta_c \ , \quad \text{as} \quad \vartheta \to \infty \ .$$

If the limiting correlation $G^*(x-y)$ is required to have exponential fall-off in $|x-y|$ one would try to impose, in addition to (3.6),

(3.9) $$\vartheta m(\beta(\vartheta)) \longrightarrow m^* > 0 \ , \quad \text{as} \quad \vartheta \to \infty \ .$$

If $m(t)$, $t \equiv \frac{\beta_c - \beta}{\beta_c}$, is known to satisfy a scaling law

(3.10) $$m(t) \sim t^{\nu} \ ,$$

see (1.15), then (3.9) and this scaling law imply that

$$\vartheta t(\vartheta)^{\nu} = \vartheta\left(\frac{\beta_c - \beta(\vartheta)}{\beta_c}\right)^{\nu} \sim \text{const.} \ ,$$

i.e.

(3.11) $$\beta(\vartheta) \sim \beta_c - \text{const.}\ \vartheta^{-1/\nu} \ , \text{ as } \ \vartheta \to \infty \ .$$

Up to some technical finesse, it follows from (3.6) and (3.9) that

(3.12) $$\chi_\vartheta \equiv \sum_{x \in \mathbb{Z}^d_{\vartheta^{-1}}} \vartheta^{-d} G_\vartheta(0,x)$$

remains bounded, as $\vartheta \to \infty$. By (3.4)

(3.13) $$\chi_\vartheta = \alpha(\vartheta)^2\, \vartheta^{-d} \chi(\beta(\vartheta)) \ .$$

If $\chi(\beta)$ satisfies a scaling law

(3.14) $$\chi(t) \sim t^{-\gamma} \ ,$$

see (1.15), and

(3.15) $$\alpha(\vartheta)^2 \sim \vartheta^{d-2+\eta} \ ,$$

(this really *defines* the critical exponent η) then it follows, using (3.11)-(3.15), that

(3.16) $$(2 - \eta)\nu - \gamma = 0 \ .$$

This is one example of a relation between critical exponents. By (3.6), (3.9),

$$m(\beta(\vartheta)) \longrightarrow 0 \ , \text{ as } \ \vartheta \to \infty \ .$$

Recalling, in addition, the definition (3.4) of $G_\vartheta(x,y)$, we see that η is a measure of the fall-off of $\langle\varphi(x)\varphi(y)\rangle_\beta$ at an intermediate distance scale, $\propto \vartheta$ when $\beta_c - \beta \sim \vartheta^{-1/\nu}$.

We now claim that in our class of models, see (1.11), (1.12),

(3.17) $$\eta \geq 0 \ .$$

For these models, the infrared bound (2.18) holds. From that bound one can deduce that, for $d \geq 3$,

(3.18) $$0 \leq \langle\varphi(0)\varphi(x)\rangle_\beta \leq c_d \beta^{-1} |x-y|^{2-d} \ ,$$

(at least for one - or two-component fields ; see [53]). Here c_d is a geometric constant. Since $\beta_c \leq \infty$ is strictly positive, we conclude from (3.4), (3.8) and (3.18) that

$$\alpha(\vartheta)^2 \geq \text{const. } \vartheta^{d-2} , \tag{3.19}$$

whence (3.17).

Quite generally, control of the two-point function in the form of an inequality (2.18) or (3.18) is required in order to determine the choice of $\alpha(\vartheta)$.

We now must focus our attention on the question of why we are interested in the large scale behaviour of a lattice spin system, i.e. in studiing the limit where $\vartheta \to \infty$. Here are some answers.

1) Suppose we are able to *construct* the limiting correlation functions, $G^*(x_1,\ldots,x_n)$, of the rescaled correlations, $G_\vartheta(x_1,\ldots,x_n)$, as $\vartheta \to \infty$, such that the renormalization conditions (3.6) and (3.9) hold. Then we must have, in particular, a way of determining the functions $\beta(\vartheta)$ and $\alpha(\vartheta)$. But, by (3.10) and (3.11), the choice of $\beta(\vartheta)$ determines the critical exponent ν , and, by (3.15), the choice of $\alpha(\vartheta)$ determines η . Thus an explicit *construction* of the $\vartheta \to \infty$ limit determines, in principle, the critical exponents ν , γ and η .

2) As our derivation of relation (3.16) shows, proving merely *existence* of a $\vartheta \to \infty$ limit yields non-trivial relations between critical exponents.

3) But perhaps the main interest in constructing the limits, $G^*(x_1,\ldots,x_n)$, of the rescaled correlation functions comes from the fact that

these limits may be the Euclidean Green's functions of a relativistic quantum field theory, i.e.

$$G^*(x_1,\ldots,x_n) \equiv S_n(x_1,\ldots,x_n) , \tag{3.20}$$

for some quantum field theory satisfying the Wightman axioms (W0)-(W5).

Indeed, in the models considered above, this is true if we can prove that the distributions $G^*(x_1,\ldots,x_n)$ are invariant under simultaneous rotations of their arguments - but even if this property failed, the G^* 's *are* the Euclidean Green's functions of a quantum field theory with a vacuum state that would then not be Lorentz invariant.

For some scaling (= continuum) limits of the models introduced in (1.11), (1.12) in two and three dimensions and of the two-dimensional Ising model it has been shown (see e.g. [12, 50], [24] respectively) that the distributions, G^* , are the Euclidean Green's functions of relativistic quantum field theories satisfying all Wightman axioms (W0)-(W5).

§ 4. Renormalization group (block spin) transformations

In this section we briefly sketch a specific idea how to accomplish the construction of the scaling (≡ continuum) limits, $G^*(x_1,\ldots,x_n)$ of the rescaled correlations $G_\vartheta(x_1,\ldots,x_n)$, as $\vartheta \to \infty$, the *Kadanoff block spin transformations.*

They may serve as a typical example of "*renormalization (group) transformations*". Clearly there are other examples of this general idea, including ones in the context of dynamics (in particular, the Feigenbaum theory [2]). We also try to indicate how mathematical control of renormalization group transformations leads to the calculation of critical exponents.

4.1. Block spin transformations

We define a function $\varkappa \equiv \varkappa^{\varepsilon}$ on $\mathbb{R}^d$ as follows :

$$\varkappa(y) = \begin{cases} \varepsilon^{-d} , & -\frac{\varepsilon}{2} \le y^{\mu} \le \frac{\varepsilon}{2} , \quad \mu = 1,\dots,d \\ 0 & , \text{ otherwise,} \end{cases}$$

where $y = (y^1,\dots,y^d) \in \mathbb{R}^d$ and ε is an arbitrary positive number. Let

$$\varkappa_x(y) = \varkappa(y - \varepsilon x) , \quad x \in \mathbb{Z}^d .$$

Let $G_\vartheta(x_1,\dots,x_n)$ be the rescaled correlation function defined in (3.4). Then

$$(4.1)\quad G_\vartheta(\varkappa_{x_1},\dots,\varkappa_{x_n}) = \sum_{y_1,\dots,y_n \text{ in } \mathbb{Z}^d_{\vartheta^{-1}}} G_\vartheta(y_1,\dots,y_n) \prod_{k=1}^{n} \vartheta^{-d}\varkappa_{x_k}(y_k)$$
$$= (\alpha(\vartheta)\vartheta^{-d})^n \sum_{z_1,\dots,z_n \text{ in } \mathbb{Z}^d} \langle\varphi(z_1)\dots\varphi(z_n)\rangle_{\beta(\vartheta)} \cdot \prod_{k=1}^{n} \varkappa_{x_k}(\vartheta^{-1}z_k) .$$

We now set

$$\vartheta = \vartheta_m = \varepsilon^{-1}L^m ,$$

where L is some positive integer and $m = 1,2,3,\dots$, and define

$$(4.2)\qquad r_m(\varphi(x)) = \alpha(\varepsilon^{-1}L^m).L^{-dm} \sum_{\substack{z\in\mathbb{Z}^d \\ -\frac{1}{2}\le L^{-m}z^{\mu}-x^{\mu}\le\frac{1}{2}}} \varphi(z) ,$$

$x \in \mathbb{Z}^d$, $\mu = 1,\dots,d$. Then

$$(4.3)\qquad G_{\vartheta_m}(\varkappa_{x_1},\dots,\varkappa_{x_n}) = \langle r_m(\varphi(x_1))\dots r_m(\varphi(x_n))\rangle_{\beta(\vartheta_m)} .$$

Let $d\mu(\varphi)$ be an arbitrary, translation-invariant, finite, positive measure on the space of all configurations $\{\varphi(x) : x \in \mathbb{Z}^d\}$. We define a transformation R_m of μ by the equation

$$\int \prod_{k=1}^{n} r_m(\varphi(x_k))d\mu(\varphi) = \int \prod_{k=1}^{n} \varphi(x_k)d(R_m\mu)(\varphi) ,$$

for all $x_1,\dots,x_n$ in $\mathbb{Z}^d$, $n = 1,2,3,\dots$.

Note that r_m (resp. R_m) consists of a transformation increasing the scale size (taking the average over all spins in a block) followed by a (in the present example : linear) coordinate transformation in spin space. Further more, we note that if μ is extremal invariant then so is $R_m\mu$.

In order to arrive at an interesting concept we now suppose that $\alpha(\vartheta)$ is proportional to some power of ϑ , i.e.

$$(4.4)\qquad \alpha(\vartheta)^2 \sim \vartheta^{d-2+\eta} ,$$

for some η . We then define

(4.5) $$r(\varphi(x)) = L^{(\eta-d-2)/2} \sum_{\substack{z\in\mathbb{Z}^d \\ -\frac{1}{2}\le L^{-1}z^\mu - x^\mu \le \frac{1}{2}}} \varphi(z)$$

Then

$$r_m(\varphi(x)) = \alpha(\varepsilon^{-1})\underbrace{r\circ\dots\circ r}_{m\text{ times}}(\varphi(x)) .$$

Let $R\mu$ be the unique measure such that

(4.6) $$\int \prod_{k=1}^{n} r(\varphi(x_k))d\mu(\varphi) = \int \prod_{k=1}^{n} \varphi(x_k)d(R\mu)(\varphi) ,$$

for all $x_1,\dots,x_n$, $n = 1,2,3,\dots$. [Note, R maps extremal invariant measures to extremal invariant ones.] Then

(4.7) $$d(R_m\mu)(\varphi) = d(\underbrace{R\circ\dots\circ R}_{m\text{ times}}\mu)(\alpha(\varepsilon)\varphi) \equiv d(R^m\mu)(\alpha(\varepsilon)\varphi) .$$

If we now choose $d\mu = d\mu_\beta$, where $\{\mu_\beta\}$ is a family of Gibbs states of our spin system indexed by β we obtain, setting $\beta = \beta(\vartheta_m)$,

(4.8) $$G^*(\varkappa_{x_1},\dots,\varkappa_{x_n}) = \lim_{m\to\infty} G_{\vartheta_m}(\varkappa_{x_1},\dots,\varkappa_{x_n}) = \lim_{m\to\infty} \int \prod_{k=1}^{n} \varphi(x_k)d(R^m\mu_{\beta(\vartheta_m)})(\alpha(\varepsilon)\varphi) ,$$

provided the limit exists.

In order to prove existence of the limit in (4.8), one must analyze the transformation R on (the boundary of) a suitably chosen cone of finite measures. In particular, one has to construct fixed points of R , study the spectrum of the linearization of R at the fixed points (the linearization of R acts on a linear space of measurable (or continuous, or analytic) functions of spin configurations, φ), and construct the stable and unstable manifold of R near a fixed point. We shall discuss some examples below.

Remarks.— 1) By (4.4)-(4.6), the transformation $R \equiv R_\eta$ depends on the exponent η . The condition that the limit in (4.8) exist and be non-trivial fixes η .

2) We shall see that the critical exponents ν and γ are determined by positive eigenvalues > 1 of the linearization of R_η at the appropriate fixed point of R_η .

3) It is usually expected that if a measure μ is a Gibbs measure (i.e. μ satisfies the DLR equations for some Hamilton function H - more precisely some interaction [13, 14] - see (1.10)) then $R_\eta\mu$ is again a Gibbs measure. This, however, is not true in general. But if it is true on a suitably chosen space of Gibbs states then R_η uniquely determines a transformation $\mathcal{R}_\eta$ acting on a space of (equivalence classes of) Hamilton functions, or interactions. The simplifying feature of this set-up is that the derivative of $\mathcal{R}_\eta$ acts on the linear hull of the same space.

4) Below, we shall briefly indicate how these ideas are applied to dynamics.

4.2. Fixed points of block spin transformations, stable and unstable manifolds, critical exponents

Let M be some cone of finite measures, μ, on some measure space of spin configurations $\varphi = \{\varphi(j)\}_{j\in\mathbb{Z}^d}$. Let R_η be a renormalization (block spin) transformation acting on M, as discussed in Sect. 4.1. (One ought to assume probably that M can be given a topology such that the action of R_η on M is smooth.) Of particular interest are the fixed points, μ^*, of R_η. It is usually not so hard to convince oneself that there exists at least one fixed point. Supposing, for example, that $\varphi(j) \in \mathbb{R}$, $j \in \mathbb{Z}^d$, and that R_η is given by (4.4)-(4.6), it is easy to show that R_η has at least a one-dimensional manifold of fixed points, μ_t^*, $t \in \mathbb{R}$, which are *Gaussian* measures. Gaussian measures are uniquely characterized by their mean and their covariance. The mean of μ_t^* is 0, the covariance is of the form $e^t C^*$, where

$$\int d\mu_{t=0}^*(\varphi)\varphi(x)\varphi(y) \equiv C^*(x,y) = c^*(x-y) \sim |x-y|^{2-d-\eta} . \tag{4.9}$$

See [54] and refs. given there. (Non-Gaussian fixed points have been constructed, too, but no non-Gaussian fixed points interesting for statistical physics or relativistic quantum field theory appear to be known, in the sense of rigorous mathematics, except in the two-dimensional Ising model.)

There is an intimate mathematical connection between fixed points, μ^*, of R_η and "stable distributions" in probability theory. It is worthwhile to note that fixed points, μ^*, cannot be strongly mixing. See e.g. [54, 55] and refs. given there for a discussion of these probabilistic aspects. We stress, however, that the main concepts of the renormalization group are more general than their probabilistic formulation !

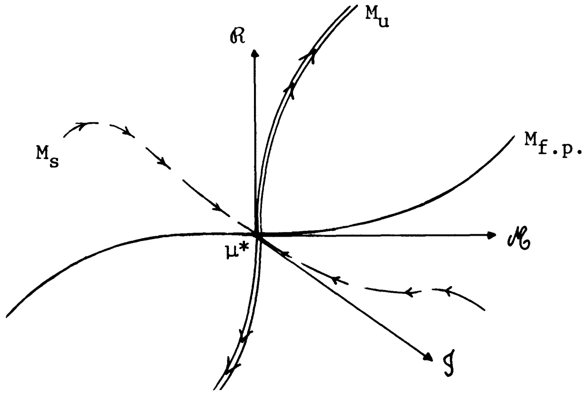
We now choose some fixed point, μ^*, of R_η. We define $M_{f.p.} = M_{f.p.}(R_\eta,\mu^*)$ to be the manifold of all fixed points of R_η passing through μ^*. Since a certain class of coordinate transformations, like

$$\varphi(j) \longrightarrow \alpha\varphi(j) , \quad \text{for all } j \in \mathbb{Z}^d ,$$

for some positive α independent of j, commute with R_η, the fixed points of R_η are not isolated, and the linearization of R_η at some fixed point μ^* will generally have an eigenvalue 1 (and possibly further eigenvalues) corresponding to coordinate transformations.

Under suitable hypotheses on R_η and M, one can decompose M in the vicinity of $\mu^* \in M_{f.p.}(R_\eta,\mu^*)$ into a stable manifold, $M_s(\mu^*)$, and an unstable manifold $M_u(\mu^*)$:

Fig. 1



States on $M_s(\mu^*)$ are driven towards μ^* , states on $M_u(\mu^*)$ are driven away from μ^* , under the action of R_η . The tangent space, $\mathcal{R}$, to $M_u(\mu^*)$ at μ^* is the linear space spanned by eigenvectors of $DR_\eta(\mu^*)$ (the derivative of R_η at μ^*) corresponding to eigenvalues of modulus > 1 . It is called the space of "*relevant perturbations*". The space $\mathcal{I}$ of "*irrelevant perturbations*" is the tangent space to $M_s(\mu^*)$ and is spanned by eigenvectors of $DR_\eta(\mu^*)$ corresponding to eigenvalues of modulus < 1 . The space $\mathcal{M}$ of "*marginal perturbations*" is spanned by eigenvectors of $DR_\eta(\mu^*)$ corresponding to eigenvalues of modulus 1 . Generically $\mathcal{M}$ will be the tangent space, $\mathcal{C}$, to $M_{f.p.}(R_\eta,\mu^*)$, and, in a neighborhood of μ^* , each point in $M_{f.p.}(R_\eta,\mu^*)$ can be reached by applying a coordinate transformation to μ^* . However, it may happen that the dimension of $\mathcal{M}$ is larger than the one of $\mathcal{C}$. In that case, linear analysis is insufficient. It may happen that one can enlarge M_s , (or M_u , or both,) by submanifolds of points which are driven towards (away from) μ^* with "asymptotically vanishing speed". This is precisely what appears to happen in the Ising - and rotor models (more generally, in the models introduced in (1.11), (1.12)) in four dimensions : $\dim \mathcal{M} = 2 = \dim \mathcal{C} + 1$; (moreover, $\dim \mathcal{R} = 1$). However, all fixed points are scale-invariant Gaussian measures, and M_s can be enlarged by a one dimensional submanifold tangent to a direction in $\mathcal{M}$ at μ^* .

In the situation described here one expects logarithmic corrections to scaling laws.

[Another possibility compatible with $\dim \mathcal{M} > \dim \mathcal{C}$ is the appearence of a stable, periodic cycle. For the transformation R_η defined in (4.4)-(4.6) one should be able to rule out this possibility.]

Suppose now that R_η depends on a continuous parameter, δ , and that δ_o is some "critical" value of δ such that

$$\dim \mathcal{M} = \dim \mathcal{C} , \quad \text{for } \delta > \delta_o ,$$
$$\dim \mathcal{M} > \dim \mathcal{C} , \quad \text{for } \delta = \delta_o .$$

Then δ_0 is a bifurcation point, and one expects the emergence of new fixed points (or periodic cycles) for $\delta < \delta_0$. In the study of the models mentioned above, it was proposed by Wilson [1] to identify δ with the *dimension* d and to interpolate analytically in d [1]. The critical dimension, corresponding to δ_0, is 4, and above four dimensions the fixed points governing the critical behaviour of those models are Gaussian, and $\eta = 0$. There are partial results towards showing that the "relevant" fixed points in dimension 4 are Gaussian, as well ; [22, 23].

Next, we discuss how critical exponents are related to the spectrum of $DR_\eta(\mu^*)$, where R_η is the transformation defined in (4.5)-(4.7). We consider a simple case : In a neighborhood of μ^*, $M_{f.p.}(R_\eta,\mu^*)$ is obtained by applying suitable coordinate transformations in spin space to μ^*. By adopting some normalization condition which fixes the choice of coordinates we can project out the marginal directions associated with $M_{f.p.}$. We assume that, after this reduction, the tangent space at μ^* splits into a *one-dimensional* space of relevant perturbations and a *co-dimension-one* space of irrelevant perturbations, (in particular, there are no further marginal perturbations). Taking smoothness properties of R_η in some neighborhood of μ^* for granted, we conclude that in some neighborhood of μ^* there exist a one-dimensional unstable and a co-dimension-one stable manifold passing through μ^*.

Next, let $\{\mu_\beta\}_{\beta>0}$ be a family of Gibbs measures of some spin system crossing the stable manifold, $M_s(\mu^*)$, transversally at some value β_c of the parameter β. We assume that, for all $\beta < \beta_c$, μ_β is extremal invariant, and that the inverse correlation length, (or mass - see Sect. 1.4, e)), $m(\beta)$, is positive and continuous in β, with

(4.10) $$m(\beta) \searrow 0 , \quad \text{as} \quad \beta \nearrow \beta_c ,$$

as discussed at the beginning of Sect. 3. (The class of all spin systems whose Gibbs states have these properties, for given R_η and μ^*, is called a *universality class.*)

Let $M(j,m^*)$ be the manifold of extremal, translation-invariant probability measures, μ, on the measure space of spin configurations, φ, which have the property that

$$\int d\mu(\varphi)\varphi(0)\varphi(x)$$

has exponential decay rate $m(j,m^*)$, as $|x| \to \infty$, where

(4.11) $$L^j m(j,m^*) \equiv m^* > 0 ,$$

for all j. If the space M of measures on which R_η acts is chosen appropriately, $M(j,m^*)$ will typically be of co-dimension 1, and $M(\infty,m^*) = M_s(\mu^*)$, in some neighborhood of μ^*. Hence, for j large enough, $M_u(\mu^*)$ will typi-

[1] Another possibility is to identify δ with the *range of the interaction.*

cally cross $M(j,m^*)$ transversally at some point μ_j . We assume that $\{\mu_\beta\}_{\beta<\beta_c}$ crosses $M(j,m^*)$ transversally at a point μ_{β_j} , for large enough j - which is consistent with (4.10). Clearly the sequence $\{\beta_j\}$ converges to β_c , as $j \to \infty$. Furthermore, by the definition of R_η , see (4.4)-(4.6), Sect. 4.1, and the definition of $M(j,m^*)$, see (4.11),

(4.12)
$$R_\eta M(j,m^*) = M(j-1,m^*) ,$$

for all j .

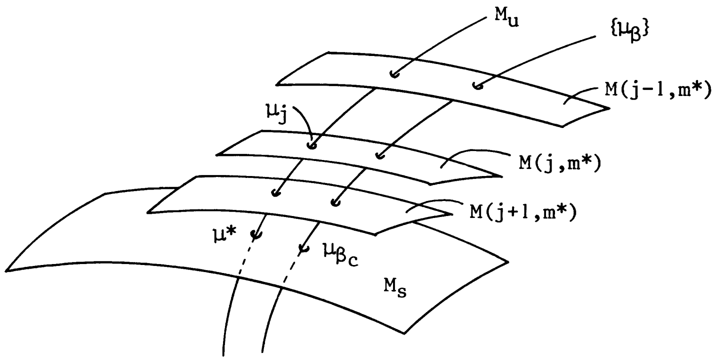
Let λ be the unique, simple eigenvalue of $DR_\eta(\mu^*)$ which is larger than 1 . In a neighborhood of μ^* , $M_u(\mu^*)$ can be given a metric such that

(4.13)
$$\text{dist}(\mu_j,\mu^*)/\text{dist}(\mu_{j+1},\mu^*) \longrightarrow \lambda , \quad \text{as} \quad j \to \infty ,$$

as follows from (4.12). Thus if μ_{β_c} is sufficiently "close" to μ^* it follows from our assumptions on $\{\mu_\beta\}_{\beta>0}$ (see Fig. 2) that

(4.14)
$$\beta_j - \beta_c \sim \lambda^{-j} \quad , \quad \text{as} \quad j \to \infty .$$

Fig. 2



By the definition of $M(j,m^*)$, see (4.11),

(4.15)
$$m(\beta_j) = L^{-j} m^* .$$

Thus, if we set $t = \frac{\beta_c - \beta}{\beta_c}$ and $m(t) \equiv m(\beta)$, $\beta < \beta_c$, we obtain from (4.14) and (4.15)

(4.16)
$$\begin{cases} m(t) \sim t^{\ln L/\ln\lambda} , & \text{as} \quad t \to 0 , \text{ i.e.} \\ \nu = \ln L/\ln\lambda . \end{cases}$$

Thanks to relation (3.16), the exponent γ of the susceptibility is determined by η and ν .

This concludes our general discussion of the basic renormalization group strategy.

Remarks.— 1) The ideas and concepts discussed here have other interesting applications to relativistic quantum field theory and statistical mechanics : As we have argued in Sect. 4.1, (4.4) through (4.8), one can use renormalization transformations, R_η , and their fixed points in order to construct the scaling limits, $G^*(x_1,\dots,x_n)$, of the correlation functions of a spin system which, under general and explicit conditions [8], can be shown to be the Euclidean Green's functions of a relativistic quantum field theory. So far, constructive quantum field theory has - in this language - been mostly concerned with the analysis of *Gaussian fixed points* of the transformations R_η , with $\eta = 0$, and the action of $R_{\eta=0}$ in a small neighborhood of those fixed points.

2) Another application of those ideas concerns the phenomenon of *asymptotic symmetry enhancement*. One example of this phenomenon is found in the fact that in many models the scaling limits, $G^*(x_1,\dots,x_n)$, of the correlation functions of some spin system are invariant under *all* simultaneous *Euclidean motions* of their arguments, although the functions $G_\vartheta(x_1,\dots,x_n)$ are only invariant under translations by an arbitrary vector $a \in \mathbb{Z}^d_{\vartheta^{-1}}$. Other examples concern the generation of internal symmetries in the scaling limit. See e.g. [15, 35] for such examples. (Symmetry enhancement arises whenever a fixed point, μ^* , and the *marginal* and *relevant* perturbations of μ^* have a large, "accidental" symmetry group.)

3) Renormalization group methods can also be applied to *dynamics* : Let ϕ_t denote a smooth flow on a finite dimensional manifold, M . Consider the following mapping on the space of all such flows on M :

$$R_{\vartheta,\Lambda} : \phi_t \longrightarrow (R_{\vartheta,\Lambda}\phi)_t \equiv \Lambda^{-1} \circ \phi_{\vartheta t} \circ \Lambda ,$$

where Λ is a smooth mapping from M into M , (a coordinate transformation). The mapping $R_{\vartheta,\Lambda}$ is the analogue of the transformation R_η defined in (4.5)-(4.7). When time is discrete, i.e. $t = n = 1,2,3,\dots$, and

$$\phi_t = \phi^n ,$$

for some mapping ϕ from M into M , one would study, for example,

$$R_\Lambda : \phi \longrightarrow R_\Lambda\phi = \Lambda^{-1} \circ \phi \circ \phi \circ \Lambda .$$

This is the Feigenbaum map. It poses very interesting, mathematical problems and serves to understand phenomena like the period doubling bifurcations and the onset of turbulence ; see [2]. (This is one among few examples where non-trivial fixed points have been constructed.)

4.3. Rigorous uses of block spin transformations

The first mathematically rigorous analysis of a specific example to which the renormalization group strategy outlined in the previous sections can be applied is the one by Bleher and Sinai [49] who analyzed Dyson's hierarchical model. The Hamilton function of this model is chosen in such a way that the renormalization group transformations can be reduced to non-linear transformations acting on some space of densities, f, of the single spin distribution,

$$d\lambda(\varphi) = f(\varphi)d\varphi .$$

Their work was reconsidered and extended in [56] and in [49. 3)] and refs. given there. It had a stimulating influence on the development of the probabilistic approach to the renormalization group, initiated by Jona-Lasinio and his colleagues in Rome [57, 55] and continued by Sinai and Dobrushin, [54, 55] and refs. given there. It was Gallavotti and collaborators [58] who first applied the renormalization group method to (the ultraviolet problem in) *constructive quantum field theory* in a systematic and transparent way, although ideas and techniques related to it - and developed independently - can already be found in work of Glimm and Jaffe [59]. These applications concern the construction of the $\lambda\varphi^4$ model - see (1.11), (1.12) - in the continuum limit in three dimensions. [This problem is equivalent to the study of a renormalization group transformation analogous to R_η in the vicinity of a *Gaussian* fixed point.] The work in [58] motivated further applications to constructive quantum field theory, notably by Balaban [60], and to statistical mechanics [61]. These developments are evolving towards a rigorous mathematical theory of renormalization group transformations in the vicinity of Gaussian fixed points, (usually with a one-dimensional, unstable manifold consisting of Gaussian measures). Such a theory is relevant for the analysis of dipole gases in dimension $d \geq 2$ and of the models considered in these notes - see (1.11), (1.12) - in dimension $d \geq 5$. This work is carried out by Gawedzki and Kupiainen [62] and by Magnen and Sénéor [63]. A looser interpretation of the renormalization group strategy partially motivated the work in [15, 36].

First applications of renormalization group methods to *dynamics* were made in [2], although the idea to use them in the study of dynamics is certainly older; see e.g. [3. 5)].

All the work quoted here involves very intricate analytical and combinatorial methods and can therefore not be sketched here.

In the remaining section we outline another much more special but quite successful approach to critical phenomena which gives rather good results for the models discussed in these notes, near Gaussian fixed points, [23]. It was inspired by a formalism first developed in [64] and made rigorous in [23. 1)] relating the theory of classical spin systems to the theory of *random walks*. A related, slightly prior approach, due to Aizenman, may be found in [22].

But mathematically rigorous results on critical phenomena in equilibrium statistical mechanics still do not nearly measure up to the practical successes of the renormalization group. This ought to be a challenge !

§ 5. Random walks and critical phenomena in the Ising - and the $\lambda\varphi^4$ models in $d \geq 5$ dimensions

While the emphasis in Sections 3 and 4 was on general ideas and principles it is on specific results and special methods, contained in [23, 65], in the present section. These methods are motivated by an approach developped in [64]. The main results are related to some prior results of Aizenman [22]. We limit our review to examples illustrating the flavour of those methods, emphasizing the relevance of the theory of random walks in the analysis of Ising - and $\lambda\varphi^4$ - models in dimension $d \geq 4$. The basic fact about random walks which motivates our analysis can be summarized in the following theorem : *In four or more dimensions, two random walks in the continuum limit* $(\vartheta \to \infty)$ *, i.e. two Brownian paths, starting at different points,* $x_1 \neq x_2$ *, of* $\mathbb{E}^d$ *will never intersect each other, with probability* 1 .

In four dimensions the proof of this result is somewhat subtle, but in five dimensions it is easy : Consider two random walks, ω_1 and ω_2, on the lattice $\mathbb{Z}^d_{\vartheta^{-1}}$, starting at x_1, x_2, respectively. The probability, $P_{z,i}$, that ω_i, $i = 1,2$, will visit some lattice site z is a harmonic function of x_i $(x_i \neq z)$ bounded by

$$P_{z,i} \leq \text{const}.\vartheta^{2-d}|z - x_i + \vartheta^{-1}|^{2-d} ,$$

where $|x-y|$ is the Euclidean distance (distance in lattice units $\times\, \vartheta^{-1}$) between x and y. For ω_1 and ω_2 to intersect each other at least once, it is necessary that ω_1 and ω_2 visit a common site $z \in \mathbb{Z}^d_{\vartheta^{-1}}$. The probability of this last event is bounded by

$$P_{z,12} = P_{z,1}\cdot P_{z,2} \leq \text{const.}\ \frac{\vartheta^{4-2d}}{|x_1 - z + \vartheta^{-1}|^{d-2}\cdot|x_2 - z + \vartheta^{-1}|^{d-2}}$$

Thus, the probability, $P_{\text{int.}}$, that ω_1 and ω_2 intersect each other somewhere is bounded above by

$$P_{\text{int.}} \leq \sum_z P_{z,12} \leq \text{const.}\vartheta^{4-d} \sum_{z\in\mathbb{Z}^d_{\vartheta^{-1}}} \vartheta^{-d} \frac{1}{|x_1 - z + \vartheta^{-1}|^{d-2}\cdot|x_2 - z + \vartheta^{-1}|^{d-2}}$$

which, for $|x_1 - x_2| > 0$, clearly tends to 0, as $\vartheta \to \infty$, provided

$$d \geq 5 .$$

In four dimensions, the last estimate is poor and has to be refined. We shall apply a refined argument to spin systems, (Sect. 5.2).

5.1. Rigorous results on the existence of the scaling limit of the $d \geq 5$ dimensional Ising - and $\lambda\varphi^4_d$ - models

The Hamilton function of the models considered in this section is defined by

$$H_\Lambda(\varphi) = - \sum_{\substack{j,j'\in\Lambda \\ |j-j'|=1}} \varphi(j)\varphi(j') , \tag{5.1}$$

where Λ is some finite region in $\mathbb{Z}^d$, and $d > 4$. We consider the following family of single spin distributions :

(5.2) $$d\lambda(\varphi) = \exp[-\frac{\lambda}{4}\varphi^4 + \frac{\mu^2}{2}\varphi^2 + \text{const.}]d\varphi ,$$

see (1.11), (1.12) ; (there is no magnetic field, i.e. $h = 0$). Formally, a Gibbs state μ_β which solves the DLR equations (1.10) for this model is given by

(5.3) $$d\mu_\beta(\varphi) = Z_\beta^{-1} e^{-\beta H(\varphi)} \prod_j d\lambda(\varphi(j)) ,$$

where Z_β is the so-called partition function chosen so that $\int d\mu_\beta(\varphi) = 1$. The r.s. of equation (5.3) has to be understood as the thermodynamic limit of measures associated with finite sublattices, Λ. The limit, $\Lambda \nearrow \mathbb{Z}^d$, exists by correlation inequalities [67]. As remarked in Sect. 1, we obtain the standard Ising model if we set $\mu = \lambda$ and let $\lambda \to \infty$. All results in this section remain true in this limit. By the infrared bound [inequality (2.18) of Sect. 2.1, (c)] and correlation inequalities, see [53], we have

(5.4) $$0 \le \langle\varphi(x)\varphi(y)\rangle_\beta \le c_d \beta^{-1} |x-y|^{2-d} ,$$

for $\beta \le \beta_c$ and $d \ge 3$, where c_d is a geometrical constant, and

$$\langle(.)\rangle_\beta \equiv \int (.) d\mu_\beta(\varphi) .$$

See also (3.18). Furthermore, as remarked in Sect. 2.2,

(5.5) $$\begin{cases} m(\beta) \searrow 0 , & \text{as } \beta \nearrow \beta_c , \text{ and } \nu \ge 1/2 , \\ \chi(\beta) \nearrow \infty , & \text{as } \beta \nearrow \beta_c , \text{ and } \gamma \le 1 . \end{cases}$$

For proofs, see [20]. Let

$$G_\vartheta(x_1,\dots,x_n) \equiv \alpha(\vartheta)^n \langle\varphi(\vartheta x_1)\dots\varphi(\vartheta x_n)\rangle_{\beta(\vartheta)} ,$$

with $\lambda = \lambda(\vartheta)$, $\mu = \mu(\vartheta)$. We choose $\alpha(\vartheta)$, $\beta(\vartheta)$, $\lambda(\vartheta)$ and $\mu(\vartheta)$ such that

$$G^*(x-y) = \lim_{\vartheta\nearrow\infty} G_\vartheta(x,y)$$

exists and satisfies

(5.6) $$0 < G^*(x-y) < \infty , \quad \text{for } 0 < |x-y| < \infty .$$

Whether (5.6) can be fulfilled or not is a rather difficult question and is not analyzed here. [Note that by renormalizing $\lambda(\vartheta)$ and $\mu(\vartheta)$ we can always require that $\beta_c = 1$; see [40].]

By (5.4) and (5.6), and because $\beta(\vartheta) \nearrow \beta_c < \infty$, as $\vartheta \nearrow \infty$,

(5.7) $$\alpha(\vartheta) \ge \text{const.}\vartheta^{(d/2)-1} .$$

We now define the four-point Ursell function, $u_{4,\beta}$:

(5.8) $$u_{4,\beta}(x_1,x_2,x_3,x_4) = \langle\varphi(x_1)\dots\varphi(x_4)\rangle_\beta - \sum_p \langle\varphi(x_{p(1)})\varphi(x_{p(2)})\rangle_\beta \langle\varphi(x_{p(3)})\varphi(x_{p(4)})\rangle_\beta ,$$

where $\sum_p$ ranges over all three pairings of $\{1,2,3,4\}$. The four-point Ursell function of the model defined in (5.1)-(5.3) satisfies the following remarkable inequalities

(5.9) $$0 \geq u_{4,\beta}(x_1,\ldots,x_4) \geq -3\beta^2 \sum_{z_1\ldots z_4}{}' \prod_{k=1}^{4} \langle\varphi(x_k)\varphi(z_k)\rangle_\beta ,$$

where z_1 ranges over $\mathbb{Z}^d$, $|z_\ell - z_1| \leq 1$, $\ell = 2,3,4$. [For a more precise statement see [23].] The upper bound on $u_{4,\beta}$ is the Lebowitz inequality [68], the lower bound is the new inequality of [23] closely related to Aizenman's inequality [22].

We define the re-scaled four-point Ursell function

(5.10) $$u_{4,\vartheta}(x_1,\ldots,x_4) = \alpha(\vartheta)^4 u_{4,\beta(\vartheta)}(\vartheta x_1,\vartheta x_2,\vartheta x_3,\vartheta x_4) .$$

From the definitions of $G_\vartheta(x,y)$ and $u_{4,\vartheta}$ and from (5.9) it follows that

(5.11) $$0 \geq u_{4,\vartheta}(x_1,\ldots,x_4) \geq \alpha(\vartheta)^{-4}\,\vartheta^d 3\beta(\vartheta)^2 . \{ \sum_{z_1\ldots z_4}{}' \vartheta^{-d} \prod_{k=1}^{4} G_\vartheta(x_k,\vartheta^{-1}z_k)\} .$$

Note that the upper and lower bound on $u_{4,\vartheta}$ do *not explicitly depend* on $\lambda(\vartheta)$ and $\mu(\vartheta)$! Now by (5.7), and since $\beta(\vartheta) \nearrow \beta_c < \infty$,

(5.12) $$\alpha(\vartheta)^{-4}\,\vartheta^d \beta(\vartheta)^2 \leq \text{const.}\vartheta^{4-d}$$

which tends to 0 , as $\vartheta \to \infty$, in dimension

$$d > 4 .$$

One can use inequality (5.4) to prove that

(5.13) $$\sum_{z_1\ldots z_4}{}' \vartheta^{-d} \prod_{k=1}^{4} G_\vartheta(x_k,\vartheta^{-1}z_k) \leq K_\delta ,$$

provided $|x_i - x_j| \geq \delta > 0$, for $i \neq j$, and some arbitrarily small $\delta > 0$, and K_δ is a constant which is finite for each $\delta > 0$.

By (5.11)-(5.13),

(5.14) $$\lim_{\vartheta\to\infty} u_{4,\vartheta}(x_1,\ldots,x_4) = 0 ,$$

provided $x_i \neq x_j$, for $i \neq j$ and $d > 4$.

Hence

$$G^*(x_1,\ldots,x_4) = \sum_p G^*(x_{p(1)} - x_{p(2)})G^*(x_{p(3)} - x_{p(4)}) ,$$

if $x_i \neq x_j$. Inequalities analogous to (5.9) can be proven for arbitrary $2n$ point functions. As in (5.11)-(5.13), they can be used to show that, in dimension $d > 4$ and for $x_i \neq x_j$, $i \neq j$,

(5.15) $$G^*(x_1,\ldots,x_{2n}) = \sum_p \prod_{\ell=1}^{n} G^*(x_{p(2\ell-1)} - x_{p(2\ell)}) .$$

Thus the scaling (= continuum) limits of the correlation functions of the models defined in (5.1)-(5.3), in particular of the Ising model, (at non-coinciding arguments) in five or more dimensions are *Gaussian*. (This result is expected to hold in four dimensions, too, but there are only partial results [22, 23]. See also Sect. 5.2.)

We now show how to use inequalities like (5.9) to prove that the critical exponent, γ , of the susceptibility, $\chi(\beta)$, takes the value 1 , in five or more dimensions :

It is not hard to derive the equation

(5.16) $\frac{d\chi(\beta)}{d\beta} = \sum_j \sum_{|j'-j|=1} \sum_x \{<\varphi(0)\varphi(j)>_\beta <\varphi(x)\varphi(j')>_\beta + \frac{1}{2} u_{4,\beta}(0,x,j,j')\}$;

(use (5.1), (5.3).) By using (5.9) and the fact that $<\varphi(0)\varphi(x)>_\beta$ is square-summable in x, for $d \geq 5$ and $\beta \leq \beta_c$, which follows from (5.4), we obtain

(5.17) $$c_-\chi(\beta)^2 \leq \frac{d\chi(\beta)}{d\beta} \leq c_+\chi(\beta)^2$$

for some finite, positive constants c_-, c_+ and all $\beta < \beta_c$. Integrating over $\beta < \beta_c$ we find

$$\gamma = 1 .$$

(One expects that $\nu = 1/2$, $\eta = 0$, in $d \geq 5$, but the proof is incomplete.) For results in four or less dimensions see [22, 23, 65, 66].

5.2. The random walk representation of classical spin systems

In the following we sketch some ideas that go into the proof [23] of an identity representing the classical spin systems as *gases of random walks interacting via soft core repulsion*. This representation was first proposed by Symanzik in [64]. It has many nice features which are useful for a qualitative understanding of critical phenomena. A different, but related representation has been used in [22].

The following calculations are formal. For a rigorous justification see [65]. We assume that

(5.18) $$d\lambda(\varphi) = g(\varphi^2)d\varphi ,$$

where g is continuous on $\mathbb{R}^+$ and has stronger than exponential decay at infinity. (A general class of even single spin distributions, in particular the one of the Ising model, will be obtained from the one satisfying (5.18) by taking weak limits.) Let

(5.19) $$g(\varphi^2) = \int \hat{g}(a)e^{-ia\varphi^2}da$$

be a Fourier decomposition of g. Let $F(\varphi)$ be a function depending smoothly on only finitely many $\varphi(j)$'s. We consider the correlation function

$$<\varphi(x)F(\varphi)>_\beta .$$

If we insert (5.19) into (5.3), with H given by (5.1) we obtain

(5.20) $$<\varphi(x)F(\varphi)>_\beta = Z_\beta^{-1}\int\prod_j \hat{g}(a(j))da(j) \cdot \int\varphi(x)F(\varphi)e^{-\frac{1}{2}<\varphi,(\beta P+2ia)\varphi>}\prod_j d\varphi(j) ,$$

where $<.,.>$ is the scalar product on $\ell_2(\mathbb{Z}^d)$, and

$$(Pf)(j) = -\sum_{|j'-j|=1} f(j') .$$

The φ-integral on the r.s. of (5.20) is Gaussian, and we obtain

(5.21) $$<\varphi(x)F(\varphi)>_\beta = Z_\beta^{-1}\sum_y\int\prod_j \hat{g}(a(j))da(j) \cdot (\beta P + 2ia)^{-1}_{xy}\int \frac{\partial F(\varphi)}{\partial\varphi(y)} e^{-\frac{1}{2}<\varphi,(\beta P+2ia)\varphi>}\prod_j d\varphi(j) .$$

We now expand $(\beta P + 2ia)^{-1}_{xy}$ in a Neumann series in βP. (This expansion converges under our assumptions on g ; see [65].) Each term in the series is labelled by a random walk, ω , on $\mathbb{Z}^d$ starting at x and ending at y . Let $n_j(\omega)$ be the total number of visits of ω at site j . Then

$$(\beta P + ia)^{-1}_{xy} = \sum_{\omega: x\to y} \beta^{|\omega|} \prod_j (2ia(j))^{-n_j(\omega)} , \tag{5.22}$$

where $|\omega|$ is the total number of nearest neighbor steps made by ω . We define

$$d\nu_n(t) = \begin{cases} \delta(t)dt , & \text{if } n = 0 \\ \frac{1}{(n-1)!} \chi_{\mathbb{R}^+}(t) t^{n-1} dt , & n = 1,2,3,\dots . \end{cases} \tag{5.23}$$

By inserting (5.22) and the identity

$$(2ia)^{-n} = \int e^{2iat} d\nu_n(t)$$

into the r.s. of (5.21) and carrying out the $a(j)$-integrals we obtain

$$\langle\varphi(x)F(\varphi)\rangle_\beta = \sum_y \sum_{\omega: x\to y} Z_\beta^{-1} \int \prod_j d\nu_{n_j(\omega)}(t(j)) . \int e^{-\beta H(\varphi)} \frac{\partial F(\varphi)}{\partial\varphi(y)} \prod_j d\lambda(\varphi(j) + 2t(j)) . \tag{5.24}$$

The variables $t(j)$ have the interpretation of waiting times for the jump process ω . (Indeed when $d\lambda$ is Gaussian, one obtains a standard Poisson jump process.) Identity (5.24) is the basic formula relating spin systems to random walks. It can be iterated by writing

$$\frac{\partial F(\varphi)}{\partial\varphi(y)} = \begin{cases} \text{const.} , \text{ or} \\ \varphi(z)G(\varphi) , \quad \text{for some } z \in \mathbb{Z}^d , \end{cases}$$

where G is a function of φ with the same properties as F . We define

$$z_\beta(t) \equiv Z_\beta^{-1} \int e^{-\beta H(\varphi)} \prod_j d\lambda(\varphi(j) + 2t(j)) , \tag{5.25}$$

and

$$z_\beta(\omega_1,\dots,\omega_n) \equiv \int \prod_j \prod_{k=1}^n d\nu_{n_j(\omega_k)}(t_k(j)) z_\beta(t_1 + \dots + t_n) \tag{5.26}$$

where $\omega_1,\dots,\omega_n$ are some given random walks. The functions $z_\beta(\omega_1,\dots,\omega_n)$ can be interpreted as correlation functions of n random walks, $\omega_1,\dots,\omega_n$, immersed in a gas of closed random walks (random loops) with soft core repulsion. See [23, 64, 65, 66]. It follows easily from (5.24) through (5.26) that

$$\langle\varphi(x)\varphi(y)\rangle_\beta = \sum_{\omega: x\to y} \beta^{|\omega|} z_\beta(\omega) , \tag{5.27}$$

and

$$u_{4,\beta}(x_1,\dots,x_4) = \sum_p \sum_{\substack{\omega_1: x_{p(1)}\to x_{p(2)} \\ \omega_2: x_{p(3)}\to x_{p(4)}}} \beta^{|\omega_1|+|\omega_2|} \{z_\beta(\omega_1,\omega_2) - z_\beta(\omega_1)z_\beta(\omega_2)\} ; \tag{5.28}$$

analogous formulas can be derived for arbitrary 2n-point functions.

The point is now that one can prove the following inequalities on $z_\beta(\omega_1,\dots)$:

A) If $\omega_1 \cap \omega_2 = \emptyset$

$$z_\beta(\omega_1,\omega_2) \geq z_\beta(\omega_1)z_\beta(\omega_2) .$$

B)
$$\sum_\omega \beta^{|\omega|} z_\beta(\omega,\omega_1,\dots) \leq (\sum_\omega \beta^{|\omega|} z_\beta(\omega)) z_\beta(\omega_1,\dots) .$$

It is quite remarkable that these inequalities go in opposite directions. They follow from (5.25) and (5.26) by applying standard correlation inequalities, due to Griffiths and Ginibre [67]. (See [65] for more general results.)

If we insert B) into the r.s. of (5.28) we obtain

(5.29) $$u_{4,\beta}(x_1,\ldots,x_4) \le 0 \ .$$

Inserting A) into the r.s. of (5.28) and noticing that $z_\beta(\omega_1,\omega_2) \ge 0$ one concludes that

(5.30) $$u_{4,\beta}(x_1,\ldots,x_4) \ge -\sum_p G_\beta(x_{p(1)},x_{p(2)}|x_{p(3)},x_{p(4)}) \ ,$$

where

(5.31) $$G_\beta(x_1,x_2|x_3,x_4) \equiv \sum_{\substack{\omega_1:x_1\to x_2\\ \omega_2:x_3\to x_4\\ \omega_1\cap\omega_2\neq\emptyset}} \beta^{|\omega_1|} z_\beta(\omega_1)\beta^{|\omega_2|} z_\beta(\omega_2) \ .$$

If we require that some point z belongs to $\omega_1 \cap \omega_2$ and then sum over all choices of $z \in \mathbb{Z}^d$ we obtain

(5.32) $$G_\beta(x_1,x_2|x_3,x_4) \le \beta^2 \sum_{\substack{z\in\mathbb{Z}^d\\ z',z''}} \sum_{\substack{\omega_1':x_1\to z\\ \omega_2':x_3\to z}} \beta^{|\omega_1'|+|\omega_2'|} \sum_{\substack{\omega_1'':z'\to x_2\\ \omega_2'':z''\to x_4}} \beta^{|\omega_1''|+|\omega_2''|} z_\beta(\omega_1',\omega_1'') z_\beta(\omega_2',\omega_2'') \ ,$$

where $|z'-z| = |z''-z| = 1$. (As argued below, this estimate is very poor in dimension $d \le 4$.) Applying B) to the r.s. of (5.32) and inserting the final result into (5.30) we obtain our basic inequality (5.9). See [23, 65], and [22] for related results.

We now suggest a substantial improvement of (5.32). (The inequality in (5.30) is expected to be quite accurate.) Let $\beta = \beta(\vartheta) \nearrow \beta_c$, as $\vartheta \to \infty$, and let

$$x_i = \vartheta y_i \ , \quad y_i \in \mathbb{Z}^d_{\vartheta^{-1}} \ , \quad i = 1,\ldots,4$$
$$|y_i - y_j| \ge \delta > 0 \ , \quad \text{for } i \neq j \ ,$$

independently of ϑ . In order to construct the scaling (= continuum) limit of u_4 we must study the behaviour of the r.s. of (5.31) for large ϑ , i.e. for walks ω_1 and ω_2 which join points that are separated by a distance $\propto \vartheta$ and which make large excursions (i.e. have "large Hausdorff dimension"), because $\beta(\vartheta) \sim \beta_c$. Now on the r.s. of (5.31), the *only* walks ω_1 and ω_2 which contribute must *intersect each other*. We may then choose the point z on the r.s. of (5.32) to be the *first* intersection of ω_1 with ω_2 , (in the orientation of ω_1). In that case, the walks ω_1' and ω_2' which end at the same point, z , are *not* permitted to *intersect* each other, except once : at z . Now, for $|x_1 - z| \sim \vartheta \sim |x_3 - z|$, the probability $p_\vartheta(\omega_1',\omega_2')$ for two walks, ω_1' and ω_2' , *not* to intersect each other in expected to behave like

(5.33) $$p_\vartheta(\omega_1',\omega_2') \lesssim \begin{cases} \vartheta^{d-4} \ , & d < 4 \\ (\log\vartheta)^{-\varkappa} \ , & \text{for some } \varkappa > 0 \ , \ d = 4 \ , \end{cases}$$

with probability 1 , as $\vartheta \to \infty$. If on the r.s. of (5.32) the trivial upper bound is replaced by (5.33) one predicts that

$$u_{4,\vartheta}(x_1,\ldots,x_4) \longrightarrow 0 \ , \quad \text{as} \quad \vartheta \to \infty \ ,$$

like $(\log\vartheta)^{-\varkappa}$, in *four dimensions*. See [65]. (For $d \leq 3$, conjecture (5.33) is consistent with known upper bounds on u_4 [69].)

These arguments can be made rigorous for standard random walks with independent increments [70] and yield a new proof of the well-known theorem [71] that, in $d \geq 4$ dimensions, two Brownian paths starting at different points of $\mathbb{R}^d$ never intersect each other, with probability 1 .

Arguments similar to the ones described here (see [66, 70]) have also been considered by Aizenman [22].

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