JOHN W. MORGAN

The rational homotopy theory of smooth, complex projective varieties

Séminaire N. Bourbaki, 1977, exp. n° 475, p. 69-80

<http://www.numdam.org/item?id=SB_1975-1976__18__69_0>

The Rational Homotopy theory of smooth, complex projective Varieties

(following Deligne, Griffiths, Morgan, and Sullivan [2])

by John W. Morgan

The results on the rational homotopy types of smooth, projective varieties (or more generally Kähler manifolds) were motivated by Sullivan's theory relating the differential forms on a manifold to its homotopy type. Kähler manifolds present themselves as an example where interesting and highly nontrivial properties hold for the differential forms. Here we examine the homotopy theoretic consequences of these properties. We begin by recounting Sullivan's theory-first the general theory of homotopy types of differential algebras, then a little rational homotopy theory for spaces, and finally the connection between the two. After this we develop some of the properties of forms on a Kähler manifold and deduce the results.

We begin our discussion of Sullivan's theory [6] with the following question: How much of the algebraic topology (or homotopy theory) of a smooth manifold M is determined by its differential algebra of forms, $\mathcal{E}^*(M)$? (The differential algebra of forms is taken to mean the abstract graded commutative algebra with its differential.) Of course, the first result that comes to mind is de Rham's theorem that the singular cohomology ring, $H^*(M;\mathbb{R})$, is so determined (being the cohomology of the differential algebra). There is, however, more information available. For example suppose we have cohomology classes, a, b, and c such that $a\cdot b = 0$ and $b\cdot c = 0$. Pick closed representatives α, β, and γ for them. Both $\alpha \wedge \beta$ and $\beta \wedge \gamma$ are exact. Choose forms ρ and η so that $d\rho = \alpha \wedge \beta$ and $d\eta = \beta \wedge \gamma$. Then $\rho \wedge \gamma - (-1)^{\deg(a)}\ \alpha \wedge \eta$ is a closed form. Its cohomology class is well defined modulo the ideal generated by a and c . Its class in $H^*(M; \mathbb{R})/I(a,c)$ is called the Massey product of a, b, and c and is denoted $< a, b, c >$. This higher order product is not determined by the cohomology ring. But clearly, if $f : M \to N$ induces an isomorphism on cohomology with real coefficients then $< f^* a, f^* b, f^* c > = f^*(< a, b, c >)$.

<u>EXAMPLE</u>. $M = S^2 \times S^1 \# S^2 \times S^1$ and N = total space of the S^1 bundle over T^2 $(= S^1 \times S^1)$ with Chern class c_1 with $\int_{T^2} c_1 = 1$. The cohomology rings of M and N are identical:

$H^1 = \mathbb{R} \oplus \mathbb{R}$, $H^2 = \mathbb{R} \oplus \mathbb{R}$, $H^3 = \mathbb{R}$, and cup product from $H^1 \otimes H^1$ to H^2 is zero. In M we can find closed one forms α and β representing a basic for $H^1(M)$ so that α and β have disjoint support. As a result $\alpha \wedge \beta = 0$ <u>as a form</u> and $< [\alpha], [\alpha], [\beta] > = 0 = < [\alpha], [\beta], [\beta] >$. To calculate these products in N we begin with a model for T^2 . Let $\Lambda(x, y)$ be the exterior algebra on 2 one dimensional generators. This maps to $\mathcal{E}^*(T^2)$ inducing an isomorphism on cohomology. When we lift to N^3 it still gives an isomorphism on H^1 but $x \wedge y$ goes to an exact 2 form in N^3 . In fact there is a 1-form η in N^3 so that $d\eta = x \wedge y$ and $\int_{\text{fiber } S^1} \eta = 1$. From this one sees easily that $x\eta$ and ηy generate $H^2(N; \mathbb{R})$. But $[x\eta] = < [x], [x], [y] >$ and $[\eta y] = < [x], [y], [y] >$. Thus using the algebra of differential forms we distinguish these two manifolds even though they have the same cohomology ring.

There is, in a similar vein, a whole realm of higher order products which, with real coefficients, can be read off from the algebra of forms. All this information can be amalgamated in a "model" for the differential algebra. The crucial notion is that of a Hirsch extension of differential algebras. A <u>Hirsch extension of $\mathfrak{A}$</u> is a differential algebra $\mathfrak{B} = \mathfrak{A} \otimes \Lambda(V)^i$, ($V$ is a finite dimensional vector space; $\Lambda(V)^i$ means the free graded commutative algebra generated by V in degree $i > 0$.) with the property that $d : V \to (\mathfrak{A})^{i+1}$. Notice that if $\mathfrak{A}$ is free as an algebra then so is $\mathfrak{B}$. Also d induces a map $d : V \to H^{i+1}(\mathfrak{A})$. This map determines the extension up to isomorphism.

A <u>minimal model</u> for an algebra $\mathcal{E}^*$ is

$$\mathfrak{M} \xrightarrow{\varphi} \mathcal{E}^* \quad \text{such that}$$

1) $\mathfrak{M}$ can be written as an increasing union of Hirsch extensions

$$\text{ground field} \subset \mathfrak{M}_1 \subset M_2 \subset \ldots\ldots , \bigcup \mathfrak{M}_i = \mathfrak{M}$$

2) $d(x)$ is decomposable for all $x \in \mathfrak{M}$, and

3) φ_* is an isomorphism on cohomology.

(An $\mathfrak{M}$ satisfying 1) and 3) is a "model" for the space. Property 2) makes it minimal.)

If $H^1(\mathcal{E}^*) = 0$, (All our algebras are assumed to have H^0 = ground field.) then one can easily build such an $\mathfrak{M}$ by successive approximations. In general the construction is more complicated, and one can not dispense with one degree before going to the next.

<u>Theorem.- For any connected differential algebra with finite dimensional cohomology in each degree</u>, $\mathfrak{A}$, <u>there is a minimal model</u> $\mathfrak{M} \xrightarrow{\varphi} \mathcal{E}^*$.
<u>Such an</u> $\mathfrak{M}$ <u>is uniquely determined up to isomorphism.</u>

If $f : M \to N$ induces an isomorphism on rational cohomology, then the minimal models of $\mathcal{E}^*(M)$ and $\mathcal{E}^*(N)$ will be isomorphic. Thus not all of the homotopy information of a space will be captured in the minimal model of its forms. All torsion and divisibility questions escape, as well as things like perfect fundamental groups, since they have no effect on the rational cohomology. When one formally inverts maps inducing isomorphisms on rational cohomology what remains is rational, nilpotent homotopy theory. A space is replaced by its rational nilpotent completion, [9]. To describe the nature of this let us look first at the case of groups. If π is a group, let $\Gamma_n(\pi)$ be the n^{th} term in the lower central series; i.e. $\Gamma_1(\pi) = \pi$, $\Gamma_{i+1}(\pi) = [\Gamma_i(\pi),\pi]$. Define $N_n(\pi)$ to be $\pi/\Gamma_n(\pi)$. This gives us a tower of nilpotent groups, each being a central extension of the previous:

$$\cdots \to N_4(\pi) \to N_3(\pi) \to N_2(\pi) \to \{e\} \quad .$$

This tower is the <u>nilpotent completion of</u> π . Such towers can be tensored with $\mathbb{Q}$ to form the <u>rational nilpotent completion of</u> π . We tensor the abelianization of π , $N_2(\pi)$, with $\mathbb{Q}$ in the normal manner. Suppose inductively that we have $N_n(\pi) \to N_n(\pi) \otimes \mathbb{Q}$ which induces an isomorphism on $H^*(\ ;\mathbb{Q})$. The central extension $0 \to \Gamma_n/\Gamma_{n+1} \to N_{n+1}(\pi) \to N_n(\pi) \to 1$ is classified by $\alpha \in H^2(N_n(\pi);\ \Gamma_n/\Gamma_{n+1})$. Take the class $\alpha \otimes 1 \in H^2(N_n(\pi) \otimes \mathbb{Q};\ (\Gamma_n/\Gamma_{n+1}) \otimes \mathbb{Q}) = H^2(N_n(\pi);\Gamma_n/\Gamma_{n+1}) \otimes \mathbb{Q}$ to form a central extension $0 \to (\Gamma_n/\Gamma_{n+1}) \otimes \mathbb{Q} \to N_{n+1}(\pi) \otimes \mathbb{Q} \to N_n(\pi) \otimes \mathbb{Q} \to 1$. The former sequence maps to the latter, and a comparison theorem shows that on all terms the map is an isomorphism of cohomology with $\mathbb{Q}$ coefficients.

In homotopy theory there is a formally analogous situation. The role of an abelian group is played by an Eilenberg-Mac Lane space, $K(\pi, n)$ with

π abelian. A central extension is replaced by a principal fibration, $K(\pi, n) \to E \to B$. Such a fibration is classified by its k-invariant, $k \in H^{n+1}(B; \pi)$. A tower of principal fibrations is a rational tower if each $K(\pi, n)$ which occurs has for group π a finite dimensional rational vector space. Given any tower of fibrations beginning from a point we can replace it by a rational tower. Each $K(\pi, n)$ is replaced by $K(\pi \otimes \mathbb{Q}, n)$. If inductively B has been replaced by $B \otimes \mathbb{Q}$ so that $B \to B \otimes \mathbb{Q}$ inducing an isomorphism on rational cohomology then

$$K(\pi,n) \to E \to B \quad \text{with } k \text{ invariant } k \in H^{n+1}(B; \pi)$$

is replaced by $K(\pi \otimes \mathbb{Q},n) \to E \otimes \mathbb{Q} \to B \otimes \mathbb{Q}$ with k invariant $k \otimes 1 \in H^{n+1}(B \otimes \mathbb{Q}; \pi \otimes \mathbb{Q}) = H^{n+1}(B; \pi) \otimes \mathbb{Q}$. Once again the first fibration maps to the second inducing an isomorphism of rational cohomology.

The rational nilpotent completion of a space is equivalent to all the maps from the space to towers of rational principal fibrations. This includes, for instance, the rational nilpotent completion of the fundamental group (by mapping to towers of $K(\pi \otimes \mathbb{Q}, 1)$'s). If the space is simply connected, then in the directed category of maps from it to rational towers there is a final object - namely the Postnikov tower of the space tensored with $\mathbb{Q}$. Thus this "rational Postnikov tower" is equivalent to the rational nilpotent completion of a simply connected space.

To connect algebras with homotopy theory we associate to every simplicial complex K, a rational differential algebra $E^*(K)$. This differential algebra consists of collections of forms ω_σ on σ for each simplex σ in K such that

1) in the barycentric coordinates for $\sigma^k, (x_o,\ldots,x_{k+1})$

$$\omega_\sigma = \Sigma\, p_I(x_o,\ldots,x_k)\, dx^{i} \wedge \ldots \wedge dx^{i_\ell} \quad \text{with } p_I \text{ a polynomial}$$

with $\mathbb{Q}$ coefficients, and

2) $\omega_\sigma|\tau = \omega_\tau$ whenever τ is a face of σ .

These forms are not continuous but can be integrated over piecewise linear chains in K . The result of integration over a simplicial chain in K is always a rational number. Thus we have a chain map $E^*(K) \longrightarrow C^*(K;\mathbb{Q})$. It induces an isomorphism on cohomology. Now we can connect rational differential algebras and towers of rational principal fibrations.

Lemma.- *Suppose* B *is a simplicial complex and* $A^* \to E^*(B)$ *inducing an isomorphism on cohomology. There is a natural one to one correspondence between Hirsch extensions of* A^* *and rational principal fibrations with base* B . The extension $\{A^* \otimes \Lambda(V)^{\ell+1}\ ,\ d:V \to H^{\ell+2}(A)\}$ corresponds to the fibration $K(V^*, \ell+1) \to N \to B$ whose k invariant is $d \in \mathrm{Hom}(V, H^{\ell+2}(B)) = H^{\ell+2}(B; V^*)$.

Under this correspondence $A^* \otimes \Lambda(V)^{\ell+1}$ *maps into* $E^*(N)$ *inducing an isomorphism on cohomology.*

Applying this lemma repeatedly shows that a rational tower of principal fibrations is equivalent to a sequence of rational Hirsch extensions.

Theorem.- *The minimal model of* $E^*(K)$ *is equivalent to the rational nilpotent completion of* K *in the following sense.*

1) *If* $\pi_1(K) = 0$, *then the minimal model for* $E^*(K)$ *has a canonical decomposition* $\mathfrak{m}_2 \subset \mathfrak{m}_3 \subset \dots \cup \mathfrak{m}_i = \mathfrak{m}$ ($\mathfrak{m}_k$ *is the subalgebra generated in degrees* $\leq k$). *This sequence of Hirsch extensions corresponds to the rational Postnikov tower of* K .

2) *In general, the subalgebra of the minimal model for* $E^*(K)$ *generated in degree* 1 *has a canonical decomposition as a sequence of Hirsch extensions. This decomposition corresponds to the rational nilpotent completion of* $\pi_1(K)$.

3) *Each possible decomposition of the minimal model of* $E^*(K)$ *into a sequence*

of Hirsch extensions gives rise to a map from K *into a tower of principal fibrations. These form a cofinal subset of all maps from* K *into such towers.*

We can compare the C^∞ differential forms on a manifold M with the rational forms on some C^1-triangulation K of M via the piecewise C^∞-forms on this triangulation. This gives the following result.

Theorem.- The minimal model of $\mathcal{E}^*(M)$ *is isomorphic to the minimal model of* $E^*(K)$ *tensored with* $\mathbb{R}$.

Corollary.- The minimal model for the C^∞*-forms on* M *is the real form of a rational differential algebra which is equivalent to the rational homotopy type of* M .

This completes our description of Sullivan's theory. We now turn our attention to the special case of interest - projective varieties or more generally Kähler manifolds.

On any complex manifold M there is the operator $J : TM \to TM$, $J^2 = -1$ which induces the almost complex structure. We define $d_c = J^{-1}dJ$. Then $d_c^2 = 0$, and $dd_c = -d_c d$. The operator J is actually part of an S^1 action on TM . This gives a decomposition of the complex valued differential forms $\mathcal{E}^*(M; \mathbb{C}) = \bigoplus_{p,q\geq 0} \mathcal{E}^{p,q}(M)$. It is the usual decomposition according to type, e.g. $\mathcal{E}^{p,q}(M)$ are sums of forms which in any local complex coordinate system are expressable as

$$f(z, \bar{z})\, dz^{i_1}\wedge\ldots\wedge dz^{i_p} \wedge d\bar{z}^{j_1}\wedge\ldots\wedge d\bar{z}^{i_q} \quad .$$

This also decomposes d into $\partial + \bar{\partial}$ with ∂ of type (1, 0) and $\bar{\partial}$ of type $(0, 1)$, with $\partial^2 = 0 = \bar{\partial}^2$, and $\partial\bar{\partial} = -\bar{\partial}\partial$.

The extra information we use about the projective variety is that the underlying complex manifold carries a Kähler metric induced from the natural one on $\mathbb{C}P^N$. In this metric $\Delta_d = \Delta_{d_c} = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$, [4], [7], [8]. Thus in this metric the complex Harmonic forms for d, d, ∂, and $\bar{\partial}$ are all the same. This leads to results about the differential algebra involving the complex structure but not the particular Kähler metric.

Lemma.- 1) For a Kähler manifold every complex cohomology class has a representative which is closed under ∂ and $\bar{\partial}$. (The harmonic one for some Kähler metric.)

2) If a class $\alpha \in \bigoplus_{r \geq p} \mathcal{E}^{r,s}$ is exact, then $\alpha = d\beta$ for some $\beta \in \bigoplus_{r \geq p} \mathcal{E}^{r,s}$.

These two conditions are equivalent to either of the following.

A) Let $F^p(\mathcal{E}^*(M; \mathbb{C})) = \bigoplus_{r \geq p} \mathcal{E}^{r,s}$. This filtration leads to the Fröhlich spectral sequence [3] abutting to $H^*(M; \mathbb{C})$. The spectral sequence collapses at E_1, i.e. $E_1 = E_\infty$. Furthermore the induced filtration on cohomology together with its complex conjugate induces a Hodge structure of weight n on H^n, i.e. $H^n(M; \mathbb{C}) = \bigoplus_{p+q=n} F^p(H^n) \cap \bar{F}^q(H^n)$.

B) If a class $\alpha \in \mathcal{E}^*(M)$ is closed under both d and d_c and is exact under either then $\alpha = dd_c\beta$ for some β.

(Of course in B we could replace d and d_c by ∂ and $\bar{\partial}$.)

The results are proved by decomposing into types, using the equality of the harmonic functions and the Hodge theorem that each cohomology class has a unique harmonic representative ([4]). Condition B is called the dd_c-lemma. All our homotopy theoretic results emanate from these properties.

Before studying the full minimal model let us return to the case of Massey products. Suppose we have classes a and b of types $(1, 0)$ and $(0, 1)$ with $a \cdot b = 0$. Pick closed representatives α of type $(1, 0)$ and β of type $(0, 1)$. Then $\langle a, a, b\rangle$ is represented by $\alpha \wedge \eta$ where $d\eta = \alpha \wedge \beta$. By the above conditions we can pick such an η of type $(1,0)$ and such an η' of type $(0,1)$. The form $\eta - \eta'$ is closed. By adding an appropriate closed $(1,0)$ form to η and a closed $(0,1)$ form to η' we can assume that $\eta - \eta'$ is exact. Then $\alpha \wedge \eta$ is cohomologous to $\alpha \wedge \eta'$. But $\alpha \wedge \eta$ is of type $(2, 0)$ whereas $\alpha \wedge \eta'$ is of type $(1,1)$. This implies that both must be exact.

This line of argument can be embellished to a systematic study of the minimal model. There is however a more direct argument. Consider the diagram of differential algebras

$$\left\{ \frac{\mathrm{Ker}\, d_c}{\mathrm{Im} d_c}, d \right\} \longleftarrow \{\mathrm{Ker}\, d_c, d\} \longrightarrow \{\mathcal{E}^*(M), d\} .$$

Applying the dd_c lemma repeatedly one shows in a straightforward fashion that both maps induce isomorphisms on cohomology and that $d : \frac{\mathrm{Ker}\, d_c}{\mathrm{Im}\, d_c} \to \frac{\mathrm{Ker}\, d_c}{\mathrm{Im}\, d_c}$ is the zero differential. This proves the following theorem.

<u>Theorem</u>.- <u>The minimal model of the differential forms on a compact Kähler manifold is isomorphic to the minimal model of its cohomology ring with</u> 0 <u>differential</u>.

<u>Corollary</u>.- <u>On a compact Kähler manifold all Massey products vanish</u>.

<u>Proof</u>. Massey products clearly vanish in a differential algebra with $d=0$, and they are invariant under maps of differential algebras inducing isomorphisms on cohomology.

Now applying Sullivan's theory of differential forms and homotopy theory we deduce consequences. Let M be a compact Kähler manifold.

Corollary.- <u>If M is simply connected, then its rational Postnikov tower can be read off from its rational cohomology ring. In particular one can deduce the rational homotopy groups and the rational k-invariants of the space from the cohomology ring.</u>

Corollary.- <u>In general the rational nilpotent completion of $\pi_1(M)$ is determined by $H^1(M; \mathbb{Q})$ and the cup product map $H^1 \wedge H^1 \to H^2$.</u>

Corollary.- <u>The tower of rational nilpotent Lie algebras associated with $\pi_1(M)$ is the same as the tower of nilpotent completions of a graded Lie algebra. In the graded Lie algebra the generators have weight -1 and the relations weight -2. Thus the relations are homogeneous quadratic relations. (In case of a Riemann surface this Lie algebra can be taken to be $\mathfrak{L}(X_1,\ldots,X_{2g})/(\Sigma[X_{2i-1}, X_{2i}])$.)</u>

Perhaps a word is in order on why all the results are stated over $\mathbb{Q}$ when we were working with the real C^∞-forms. The results on the C^∞-forms yield immediately from Sullivan's theory the "real versions" of all the above corollaries. But it is an elementary fact from linear algebraic groups that such results automatically descend to the rationals. This is similar to the result that a finite dimensional filtered algebra over $\mathbb{Q}$ which can be split into a graded algebra over $\mathbb{R}$ or $\mathbb{C}$ can be split into a graded algebra over $\mathbb{Q}$.

Lastly, these results have their generalization to smooth open varieties just as Hodge structures can be generalized to give mixed Hodge structures on the cohomology, [1]. However, the generalizations are not as strong as the results here. As a sample we offer the following.

<u>Theorem</u> ([5]).- <u>If</u> $\mathcal{U}$ <u>is an open smooth variety, then the tower of rational nilpotent quotients associated to</u> $\pi_1(\mathcal{U})$ <u>is the same as the Lie algebras of a graded Lie algebra. The generators have weights</u> -1 <u>and</u> -2 <u>and the relations have weights</u> -2, -3 <u>and</u> -4 .

References

[1] P. DELIGNE, Théorie de Hodge mixte, II, Publ. Math. IHES 40 (1971), 5-57.

[2] P. DELIGNE, P. GRIFFITHS, J. MORGAN, and D. SULLIVAN, Real homotopy theory of Kähler manifolds, Inventiones 29 (1975), 245-274;

[3] A. FRÖHLICHER, Relations between the cohomology groups of Dolbeault and topological invariants, Proc. Nat. Acad. Sci. USA 41 (1955), 641-644.

[4] W.V.D. HODGE, The Theory and Application of Harmonic Integrals", Cambridge University Press, Cambridge, G.B., 2nd edition 1959.

[5] J. MORGAN, The homotopy theory of open, smooth, varieties, (to appear)

[6] D. SULLIVAN, Infinitesimal calculations in topology, (to appear) Ann. of Math.

[7] A. WEIL, "L'Introduction à l'Etude des Variétés kähleriennе", Hermann, Paris, 1958.

[8] R. WELLS, Jr., "Differential Analysis on Complex Manifolds", Printice-Hall, Englewod Cliffs, N.J., 1973.

[9] A. BOUSFIELD and D. KAN, "Homotopy limits, completions, and localizations", Lecture Notes in Mathematics 304, Berlin-Heidelberg-New York, Springer, 1972.