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THE HEAT EQUATION IN RIEMANNIAN GEOMETRY

[after PATODI, GILKEY, etc.]

by M. F. ATIYAH

1.   Heat Equation approach to the index theorem.

As explained in the preceding lecture the idea of
using the heat equation to solve the index problem is
as follows.   Let $D : C^\infty(X,E) \to C^\infty(X,F)$ be an
elliptic operator on the compact manifold X.   Using
metrics on E,F and X we may form its adjoint D^*
and the self-adjoint operators

$$\Delta_E = D^* D \quad , \quad \Delta_F = DD^* \ .$$

For $t > 0$ the heat operators $e^{-t\Delta_E}, e^{-t\Delta_F}$ have C^∞
kernels so that their traces are well-defined.   Now
except for their zero eigenspaces the operators Δ_E and
Δ_F are conjugate $(\Delta_E = D^{-1}\Delta_F D)$, while their zero-
eigenspaces coincide with the null spaces $\mathcal{N}(D), \mathcal{N}(D^*)$
of D and D^* respectively.   Hence, for all $t > 0$,

(1.1) $\operatorname{Tr} e^{-t\Delta_E} - \operatorname{Tr} e^{-t\Delta_F} = \dim \mathcal{N}(D) - \dim \mathcal{N}(D^*) = \operatorname{index} D.$

But we have asymptotic expansions as $t \to 0$ for the terms
on the left, namely

(1.2) $\operatorname{Tr} e^{-t\Delta_E} \sim \sum_{k=-n}^{\infty} a_k t^{k/m}$

where $n = \dim X$, $m = \operatorname{order} \Delta_E$.   If b_k are the
coefficients of the corresponding expansion for Δ_F we
deduce from (1.1)

$$a_k = b_k \quad \text{for} \quad k \neq 0$$

$$(1.3) \qquad a_0 - b_0 = \text{index } D \ .$$

Now the point about the expansion (1.2) is that the coefficients a_k are locally computable. More precisely we have

$$a_k = \int_X \alpha_k(x)$$

where $\alpha_k(x)$ is a C^∞ measure on X given (in principle) as an explicit function of the coefficients of Δ_E. In terms of a complete orthonormal set $\{\phi_n\}$ of eigenfunctions of Δ_E with eigenvalues λ_n, the α_k are defined by an asymptotic expansion

$$\Sigma \, e^{-\lambda_n t} \, |\phi_n(x)|^2 \, |dx| \ \sim \ \sum_{k=-n}^{\infty} \alpha_k(x)^{k/m}$$

(as $t \to 0$). Thus we derive, in principle, a general formula

$$(1.4) \qquad \text{index } D = \int_X \{\alpha_0(x) - \beta_0(x)\}$$

where $\beta_k(x)$ is the measure associated to b_k.

This approach to the index problem was put forward in [1] but it was not until the recent work of Patodi [6] that any progress was made. The difficulty lies in the fact that the α_k are very complicated functions of the coefficients of Δ_F. In particular they involve many derivatives. Patodi's contribution was to show that, in the particular case $D = d + d^* :$ even forms $\to$ odd forms (so that index D = Euler characteristic of X), remarkable

cancellation took place between α_0 and β_0 so that
finally (1.4) reduced to the Gauss-Bonnet formula.

Subsequently Gilkey [5] gave an alternative proof
of Patodi's result based on an intrinsic identification
of the integrand in (1.4). This was later extended to
cover the important case of the Hirzebruch signature
theorem. [2] presents a simplified version of Gilkey's
argument and generalizes it to cover more general
operators arising in Riemannian geometry. This provides
enough examples of the index theorem so that the general
case can be deduced from them by topological arguments
(K-theory). Here we follow the presentation in [2].

§2. The Hirzebruch signature theorem.

Suppose now X is a compact oriented Riemannian
manifold of dimension 2ℓ, and let $d : \Omega^i \to \Omega^{i+1}$
$d^* : \Omega^i \to \Omega^{i-1}$ be the exterior derivative on forms and
its adjoint. Then $d + d^*$ acts on the space Ω of
all forms and anti-commutes with the involution τ
defined by

$$\tau(\alpha) = i^{p(p-1)+\ell} *\alpha \qquad \alpha \ \varepsilon \ \Omega^p$$

where $* : \Omega^p \to \Omega^{2\ell-p}$ is the duality operator. Thus
$d + d^*$ switches the ± 1-eigenspaces $\Omega_{\pm}$ of τ and hence
is of the form $\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$ where $A : \Omega_+ \to \Omega_-$ is called
the signature operator. One verifies easily (using Hodge
theory) that, if $\ell = 2k$, index $A = $ sign (X) is the
signature of the quadratic form on $H^{2k}(X)$. The heat

equation method of §1 then gives us an integral formula

(2.1)   sign (X) = $\int_X \omega$.

Here ω is a measure depending skew-symmetrically on
the orientation of X (because reversing orientation
takes τ into -τ , hence A into -A) and hence may
be regarded as a 4k-form canonically associated to the
metric g on X. The explicit results for second-order
operators assert that ω(g) is given by a universal
polynomial in the g_{ij}, their derivatives of various
orders and $(\det g)^{-\frac{1}{2}}$. Moreover under a change of scale
in the metric $g \mapsto k^2 g$ (k a positive constant)
$A \mapsto k^{-1} g$ and so the constant terms in the expansion of
$e^{-tA^* A}$, e^{-tAA^*} are unaltered. Hence $\omega(k^2 g) = \omega(g)$.
Now the main discovery of Gilkey was that these
qualitative properties of ω(g) are enough to identify
it as a Pontrjagin form. More precisely we have

THEOREM (Gilkey)   Let ω(g) be any differential form
functionally associated to a Riemannian metric g and
such that

    (i) ω(g) is given in local coordinates by a
    universal polynomial in g_{ij}, their derivatives
    of all orders and $(\det g_{ij})^{-\frac{1}{2}}$

    (ii) $\omega(k^2 g) = \omega(g)$.

Then ω(g) = f(p_1, ..., p_ℓ) where $p_i(g)$ are the
coefficients of the characteristic polynomial of the
curvature matrix $\mathcal{R}$:

      $\det(1 + t\mathcal{R}) = \Sigma \, p_i t^{2i}$

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Applying this theorem to the form $\omega(g)$ in (2.1) we deduce that

(2.2) $\text{sign}(X) = \displaystyle\int_X f(p_1, \ldots, p_\ell)$

for some suitable polynomial f. Computing sufficiently many examples (products of complex projective spaces) then enables us to identify f with the Hirzebruch L-polynomial (after normalizing the p_i by appropriate powers of 2π) exactly as in Hirzebruch's original proof. Thus Gilkey's theorem plays in local Riemannian geometry the part played by Thom's cobordism theory: it characterizes the Pontrjagin numbers.

We now indicate how Gilkey's theorem is proved. There are three steps of which the first two are classical and well-known:

(1) By using geodesic coordinates we argue that $\omega(g)$ must be a polynomial in the components of the curvature tensor and its covariant derivatives together with $(\det g)^{-\frac{1}{2}}$

(2) By the theorem on tensorial invariants of the orthogonal group we deduce that $\omega(g)$ is a linear continuation of basic 'monomials' of the form

$$m(R) = \Sigma_q^* \; R_{\alpha_1} \; R_{\alpha_2} \; \cdots \; R_{\alpha_r} \; .$$

Here each R_{α_i} denotes a covariant derivative (of order ε_i) of the curvature tensor, Σ_q means that we contract all but q of the suffixes and $*$ indicates that we skew-symmetrize the q suffixes left (so as to get a skew-symmetric q-tensor where $q = \deg \omega$).

(3) The homogeneity condition (ii) on $\omega(g)$ implies

that we only need monomials $m(R)$ with $q = 2r + \varepsilon$,

where $\varepsilon = \Sigma\varepsilon_i$ is the total number of covariant derivative

indices. If $\varepsilon > 0$ this implies that in some factor R_{α_i}

we must skew-symmetrize at least three out of the first

five indices. The Bianchi identities then show that

$m(R) = 0$. If $\varepsilon = 0$, so that $m(R)$ involves only

the curvature tensor itself (and no higher derivatives),

a similar calculation shows that we must use precisely

two suffixes in each R_{α_i} for contraction. The symmetries

of R show that we may take these as the first two

suffixes. It is now clear that $m(R)$, as a form, is

an exterior product

$$(\mathrm{Tr}\ R)^{k_1} \wedge (\mathrm{Tr}\ R)^{k_2} \wedge \ \cdots$$

and so is a Pontrjagin form.

Note The square root factor involving det g allowed in

the hypothesis of the Gilkey theorem turns out to be

illusory in the end. However it is necessary to allow

for it because it appears, a priori, in the heat equation

formula. This point was in fact overlooked in [2] as

pointed out by Colin de Verdière. In fact in the Gauss-

Bonnet theorem $(\det g)^{\frac{1}{2}}$ enters into the final formula

but that is because we have there a measure independent

of orientation (so not a form) and so the theorem on

'tensorial' invariants has to be modified. However the

characterization of the "Euler form" is more complex than

that of the Pontrjagin forms and we shall avoid it.

§3. The general index theorem.

If ξ is a complex vector bundle over X with a
hermitian inner product and a unitary connection we can
define an elliptic operator A_ξ, acting on forms with
coefficients in ξ, which generalizes the signature
operator A of §2. To compute index A_ξ by the
heat equation method we need to extend Gilkey's theorem
to include auxiliary vector bundles. Essentially the
same proof as before enables us now to characterize
the mixed Pontrjagin-Chern forms of (g,ξ) as the only
functorially defined forms satisfying the analogues
of (i) and (ii) in the Gilkey theorem: we now require
invariance also under change of scale in the inner
product of ξ.

Again the explicit polynomial in the Pontrjagin
classes of X and the Chern classes of ξ must be found
by computing some suitable examples.

We now have sufficiently many explicit examples to
deduce the general index theorem. Roughly speaking every
elliptic operator can be deformed into a generalized
signature operator A_ξ. More precisely we recall (see [4])
that every elliptic operator P defines via its symbol
an element $\sigma_P \varepsilon K(TX)$, where K denotes the K-theory
with compact supports and TX is the cotangent bundle
of X. The stability of the index under deformation,

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and the existence of (pseudo-differential) operators with given symbol, imply that $P \mapsto$ index P induces a homomorphism

(3.1)   index : $K(TX) \to \mathbb{Z}$.

Now $K(TX)$ is a module over $K(X)$ in such a way that

$$\xi \cdot \sigma_A = \sigma_{A_\xi} .$$

Moreover, modulo 2-primary groups, $K(TX)$ is a free $K(X)$-module with generator σ_A. Hence (3.1) is determined by its values on σ_{A_ξ} , for all ξ . Thus the generalized signature theorem implies the general index theorem for X even-dimensional and oriented. The other cases follow easily by passing to the oriented double covering and multiplying by a circle.

Included in the general index theorem is the Riemann-Roch theorem for compact complex manifolds. For Kähler manifolds it can be given a direct treatment on the lines of the signature theorem. One uses the Dirac operator of the associated Spin^c-structure. It should be emphasized that this is still essentially a theorem about Riemannian manifolds: it is not necessary to try to extend the Gilkey theorem to complex Hermitian manifolds. The first heat equation proof of the Riemann-Roch theorem is due to Patodi [7].

§4.   Further developments.

The heat equation proof of the signature theorem (and others of the same type) gives more information than previous proofs, in that the integrand is given an analytical

interpretation. In other words we have a local version
of the signature theorem, from which the global version
follows by integration. Not surprisingly this local
version gives rise to interesting results for manifolds
with boundary [3]. To explain these results let us
consider a compact oriented 4k-dimensional Riemannian
manifold X with boundary Y and assume that, near
the boundary, X is isometric to the product Y × I.
We define sign (X) now as the signature of the quadratic
form induced by the cup-product on the image of $H^{2k}(X,Y)$
in $H^{2k}(X)$. Then the difference

$$\text{sign } (X) \; - \; \int_X L$$

(where L denotes the L-polynomial in the Pontrjagin
forms) need not vanish but, as an easy consequence of the
signature theorem for closed manifolds, it depends only
on Y (as oriented Riemannian manifold). Heat equation
methods enable us to identify this invariant of Y as
a 'spectral invariant', that is as a function of the
eigenvalues of the Laplacian on forms. Precisely define
the self-adjoint operator B on even forms on Y by

$$B\phi = (-1)^{k+p+1} (d* - *d)\phi \qquad (\phi \in \Omega^{2p})$$

and put

(4.1) $\qquad \eta(s) = \sum_{\lambda \neq 0} \text{sign } \lambda \; |\lambda|^{-s}$

where λ runs over the eigenvalues of B (with
multiplicities counted). Note that B^2 is just the
Laplacian Δ so that λ^2 runs over the eigenvalues of Δ .

For Re(s) large (4.1) converges absolutely. One
proves that η(s) can then be continued meromorphically
to the entire s-plane, that s = 0 is not a pole and
finally that

$$(4.2) \qquad \text{sign } (X) - \int_X L = - \eta(0)$$

There are similar results for the other operators
arising in Riemannian geometry. Moreover as a
consequence of these Riemannian cases one deduces,
by topological methods, that for any elliptic self-
adjoint operator the η-function analogous to (4.1)
is finite at s = 0.

Formula (4.2) has many interesting extensions and
applications for which we refer to [3].

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