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ABSTRACT HOMOMORPHISMS OF SIMPLE ALGEBRAIC GROUPS

[after A. BOREL and J. TITS]

by Robert STEINBERG

§ 1. Introduction

Let G be a simple algebraic group (i.e. affine, connected, and having only finite, hence central, nontrivial normal subgroups) defined over an algebraically closed field k, and let $G(k)$ denote the group of rational points of G. According to Chevalley [6, exp. 7] :

1.1. A subgroup of G (identified with $G(k)$) is a Cartan subgroup if and only if it is maximal nilpotent and has every subgroup of finite index of finite index in its normalizer.

Similarly the structure of $G(k)$ as an abstract group determines the Borel subgroups, the maximal unipotent subgroups, ..., even the field k, up to isomorphism, and eventually almost completely determines the structure of G as an algebraic group. The end result (proved in § 2 below) can be stated thus :

1.2. THEOREM.- Let G, k be as above with G simply connected and let G' be an algebraic group over an algebraically closed field k'. Let $\alpha : G(k) \to G'(k')$ be an isomorphism of groups. Then there exists an isomorphism of fields $\varphi : k \to k'$ and a k'-isogeny $\beta : {}^{\varphi}G \to G'$ such that $\alpha = \beta \circ \varphi^{o}$ on $G(k)$.

Here ${}^{\varphi}G$ is the group over k' obtained by transfer of base field and $\varphi^{o} : G \to {}^{\varphi}G$ the corresponding map. (If G is given as a matrix group defined over k, then ${}^{\varphi}G$ is got by applying φ to the equations over k defining G and φ^{o} by applying φ to the matrix entries.)

Further a statement concerning the uniqueness of φ and β, especially simple when G is simply connected, can be given (see 2.3).

This theorem classifies, not only the possible structures of algebraic group for $G(k)$, but also the various groups of this type up to abstract isomorphism, thus also their abstract automorphisms. Further it does so in a precise way since the isogenies (i.e. rational homomorphisms, surjective with finite kernel) among the various simple groups are quite well known [6, Exp. 18].

The purpose of the article [4], of which we are giving an account here, is to prove a vast generalization of this result in which the class of groups is considerably extended and in which homomorphisms, not just isomorphisms, are considered, and then to apply this result to diverse situations concerning isomorphisms, automorphisms, continuity of homomorphisms, representations (homomorphisms into some $GL(V)$), etc..

First let us state this result. Let k be an infinite field and G a simple algebraic group defined over k and of positive k-rank, i.e. "isotropic". (Thus G contains nontrivial k-split tori and many rational unipotent elements.) Let H be a subgroup of $G(k)$ containing G^+, the group generated by the rational points of the unipotent radicals of the rational parabolic subgroups of G. (If k is perfect G^+ is the group generated by the rational unipotent elements.) Finally let G' be a simple algebraic group over an infinite field k' and $\alpha : H \to G'(k')$ a homomorphism. Then the generalization mentioned above is :

1.3. THEOREM ((A) of [4]).- Let everything be as just stated. Assume that G is simply connected or G' adjoint and that $\alpha(G^+)$ is dense in G' (in the Zariski topology). Then there exists a homomorphism $\varphi : k \to k'$, a k'-isogeny $\beta : {}^{\varphi}G \to G'$ with $d\beta \neq 0$ (called <u>special</u> for this reason), and a homomorphism $\gamma : H \to$ center of $G'(k')$, all three unique, such that $\alpha(h) = \gamma(h)\beta(\varphi^{o}(h))$ for all $h \in H$.

1.4. <u>Remarks</u>.- (a) G^+ is always dense in G. If P, P^- are opposed proper parabolic subgroups of G, then their unipotent radicals U, U^- generate G, as is easily seen. Taking P, P^- to be defined over k as we may since G is isotropic [3], we see that $\langle U(k), U^-(k)\rangle$, hence also G^+, is dense in G.

(b) It is conjectured that γ above is always trivial. This is certainly so in case G' is adjoint since then the center of $G'(k')$ is trivial. Assume that G is simply connected. Then there is a long standing conjecture, proved in most cases (for G split, quasi-split, a classical group,...), that $G(k) = G^+$ (and hence that the only choice for H in 1.3 is G^+). Now G^+ always equals its own derived group [4 : 6.4], hence has only trivial homomorphisms into Abelian groups. Thus this conjecture would imply the present one.

(c) Even if G is not simply connected, then the groups G^+ and $G(k)$ can be determined quite explicitly in most cases, and hence also the possibilities for H .

(d) An easy consequence of 1.3 is that if G and k are as there and k' is an algebraically closed field in which k has no imbedding, e.g. a field of a different characteristic, then a homomorphism from G^+ to any k'-group G' is necessarily trivial.

The proof of 1.3 and some related results will be discussed in § 3 and § 4.

In view of 1.4 (a) the theorem applies in case H' is a group constructed in the same way as H and $\alpha(H) = H'$. It therefore yields a classification of the groups of this type, and of their automorphisms. For the groups PSL_n over infinite fields, for example, the result can be formulated geometrically as follows, in view of the fundamental theorem of projective geometry : every isomorphism between two groups of this class is effected by a collineation of the underbying projective spaces and every automorphism by a collineation or a correlation. This is the original result of this type and was first proved in a classical memoir [11] of Schreier and van der Waerden who proved also that over finite fields the only exceptions are $PSL_2(\mathbb{F}_7) \cong PSL_3(\mathbb{F}_2)$ and $PSL_2(\mathbb{F}_4) \cong PSL_2(\mathbb{F}_5)$. Later it was extended by Dieudonné, Hua, Rickert, O'Meara and others (see [8 , 10 , 17] for further references) to include many of the other classical groups ((projective) symplectic, unitary, orthogonal, spin,...), on a case by case basis. Theorem 1.3 unifies these results as they apply to isotropic groups (for unitary and orthogonal groups, the defining form must have positive Witt index) and at the same time extends them to the exceptional groups. The earlier proofs, however, were decidedly more elementary.

Let us remark also that substantially the same results hold over finite fields, as was shown in a number of cases by various of the authors mentioned above and in the general case by the present author [13]. The proofs, indicated in § 2, are, at least in the split case, identical with those in the algebraically closed case from a certain point on.

Now let k , k' above be nondiscrete locally compact topological fields with

k' not isomorphic to $\mathbf{C}$. Then one can show (easily if k, k' are real) that every homomorphism $\varphi : k \to k'$ is necessarily a topological isomorphism of k onto a closed subfield of k', hence is continuous [4, § 2.3]. It follows, in this case, that if $G(k)$ and $G'(k')$ in 1.3 are viewed as topological groups in the natural way, then α must be continuous. In [4] this result is extended to semisimple groups and it is shown that the assumption of isotropicity is not needed. It then includes the result of E. Cartan [5] and van der Waerden [19] that every homomorphism of a compact connected semisimple Lie group into a compact Lie group is continuous, and that of Freudenthal [9] that every isomorphism of a connected Lie group with absolutely simple Lie algebra onto a Lie group is continuous. These results are discussed further in § 5.

Finally let us consider (abstract) representations.

1.5. THEOREM ((B) of [4]).- Let k, G, G^+, H be as in 1.3, k' an algebraically closed field, and $\rho : H \to PGL_n(k')$ $(n \geq 2)$ a projective representation which is irreducible on G^+. Then there exist irreducible rational projective representations π_i of G and distinct homomorphisms $\varphi_i : k \to k'$ $(1 \leq i \leq m < \infty)$ such that ρ is the restriction to H of the tensor product of the representation ${}^{\varphi_i}\pi_i \circ \varphi_i^o$.

This result, conjectured by the present author in case k is algebraically closed in [14], together with a statement of uniqueness, will be proved in § 6 below. Since the only continuous homomorphisms of the complex field into itself are the identity and ordinary complex conjugation the above result overlaps the classical result that every irreducible differentiable complex representation of a connected complex Lie group is the tensor product of a holomorphic representation and an antiholomorphic one (see, e.g., [12, p. 22-12]).

§ 2. Algebraically closed fields and finite fields

We start with 1.2 since its proof, which is quite simple, will serve as a model for that of 1.3, which is not. Let everything be as in 1.2 and identify G with $G(k)$. For the standard facts about affine algebraic groups, many to be used without explicit reference, we cite [2 , 3 , 6].

2.1. In G one has the following abstract characterizations.

(a) The maximal tori : as in 1.1.

(b) The Borel subgroups : those that are maximal solvable and without proper subgroups of finite index.

(c) The maximal connected unipotent subgroups : the derived subgroups of the Borel subgroups.

(d) car k : if p is a prime then car $k = p$ if and only if there is no p-torsion in any maximal torus.

Here G need not be simple, only connected reductive (and nontrivial in the case of (d)). The proof of (a) is given in [6, Exp. 7]. If B satisfies the properties listed in (b), then it is closed by the maximality, connected since it has no subgroups of finite index, hence Borel by the maximality. Conversely, let B be Borel. Then B is maximal solvable [6, p. 9-05]. Write $B = TU$ (semidirect) with T a torus and U the subgroup of unipotent elements of B . Let B' be a subgroup of finite index. Since T is divisible it has no proper subgroup of finite index. Hence $B' \supseteq T$. Since G is reductive, U is the product of one-parameter subgroups each normalized by T according to a nontrivial character (root). If t is a regular element of T (at which no root vanishes) it follows that the map $u \to t^{-1}u^{-1}tu$ on U is surjective (in fact, bijective). Thus if u' is arbitrary, then tu' is conjugate to t , hence semisimple, hence contained in a maximal torus of B , hence contained in B' by the earlier argument. Thus $B' \supseteq U$, $B' \supseteq B$, $B' = B$, which yields (b). Since U above is maximal unipotent connected, (c) follows, for example, from the surjectivity above. Finally since a torus is diagonalizable (d) is clear.

2.2. Proof of 1.2. We observe first that G' is simple. For since B above does not have a proper subgroup of finite index neither does G , which is generated by its Borel subgroups, hence neither does G', which is thus connected. And since G does not have nontrivial infinite normal subgroups, neither does G'. In G let B, B^- be opposite Borel subgroups and U, U^- their unipotent radicals, so that $B \cap B^- = T$ is a maximal torus and one has $B = TU$, $B^- = TU^-$. It follows from 2.1 that the groups $B' = \alpha(B)$, $B^{-\prime} = \alpha(B^-)$, etc., have the same properties in G', and that car k = car k'. Further α matches up the normalizers of T and T', hence also the corresponding Weyl groups. Let a be a simple root for G relative to T, U_a the corresponding one-parameter subgroup of U, and n_a an element of the normalizer of T representing the reflection corresponding to a. Since the union of B and another B, B double coset can be a group only if the coset has the form Bn_aB (a simple) and since $U_a = U \cap {}^{n_a}U^-$, it follows that $\alpha(U_a) = U'_{a'}$ and $\alpha(n_a) = n_{a'}$ for some simple root a' for G'. Since ${}^{n_a}U_a = U_{-a}$ and similarly for a' one obtains from α an isomorphism from $\langle U_a, U_{-a}\rangle$ to $\langle U'_{a'}, U'_{-a'}\rangle$ which may be viewed as an isomorphism from $SL_2(k)$ to $SL_2(k')$ or to $PSL_2(k')$ (see, e.g., [6] and recall that G is simply connected) preserving superdiagonal, subdiagonal, and diagonal elements. Let φ and χ on k and k^* be defined by

$\alpha(1 + xE_{12}) = 1 + \varphi(x)E'_{12}$ and $\alpha(\mathrm{diag}(y,y^{-1})) = \mathrm{diag}(\chi(y), \chi(y)^{-1})$, with E_{12}, E'_{12} the appropriate matrix units. Here χ is well-defined even if the second group is $PSL_2(k')$ since then $SL_2(k)$ has no center, hence car $k = 2$ and car $k' = 2$. Here one can normalize the identification of $\langle U_a, U_{-a}\rangle$ with $SL_2(k)$ so that $\varphi(1) = 1$ (or one can do this for a'). We claim that then φ is an isomorphism of fields and that $\chi = \varphi/k^*$. One verifies that for given $y \neq 0$ the product $(1 + yE_{12})(1 + zE_{21})(1 + yE_{12})$ normalizes the diagonal subgroup only if $z = -y^{-1}$. Let $w(y)$ denote this product when $z = -y^{-1}$. It follows that $\alpha(w(y)) = w'(\varphi(y))$. Since also $\mathrm{diag}(y,y^{-1}) = w(y)w(1)^{-1}$ and $\varphi(1) = 1$ it follows that $\chi = \varphi/k^*$. Finally since φ is additive and χ is multiplicative, φ is an isomorphism as asserted. It depends on a. Now any root r is simple relative to some ordering of the roots. We write φ_r for the corresponding isomorphism. Also we let $u_r : k \to U_r$ denote a parametrization. Let a and b be simple roots with $a \neq b$ and $(a,b) \neq 0$. Then one has a commutator relation

$(u_a(x), u_b(y)) = \prod_{i,j>0} u_{ia+jb}(C_{abij}\, x^i y^j)$ with C_{abij} fixed in k and $C_{ab11} \neq 0$. On applying α and comparing with the corresponding relation in G' we get $\varphi_{a+b} = \varphi_a^m$ and $\varphi_{a+b} = \varphi_b^n$ with $(a+b)' = ma' + nb'$. Since φ_{a+b}, φ_a and φ_b are isomorphisms it follows that m and n are powers of the characteristic exponent p of k. Since G is simple and its root system irreducible it follows that there exists an isomorphism $\varphi : k \to k'$ and integers m_r, all nonnegative, some equal to 0, such that $\varphi_r = \mathrm{Fr}^{m_r} \circ \varphi$ (Fr = Frobenius), first for all simple roots r and then for all roots as we see by making the Weyl group act. We then set $\beta = (\varphi^o)^{-1} \circ \alpha : {}^{\varphi}G \to G'$. Identifying ${}^{\varphi}G$ with G according to φ^o we have a normalization in which $\varphi = \mathrm{id}$. Then β is a morphism on each U_r and on each $T_r = \langle U_r, U_{-r}\rangle \cap T$, by the above. Hence β is a morphism on U^-TU since this set is naturally isomorphic to the Cartesian product of all of the groups U_r (r a root) and all of the groups T_a (a simple), arranged in some order. Finally since this set is open in G and β is a homomorphism of groups β is a morphism on G, which proves 1.2.

2.3. Uniqueness. One can choose φ and β in 1.2 so that β is special. Then they are unique. More precisely, if φ and β are so chosen and if $\bar{\varphi}$ and $\bar{\beta}$ satisfy the conclusions of 1.2 then there exists $m \geq 0$ such that $\varphi = \mathrm{Fr}^m \circ \bar{\varphi}$ and $\bar{\beta} = \beta \circ (\mathrm{Fr}^m)^o$.

Proof. Choose r so that $m_r = 0$ above. Then $\beta : {}^{\varphi}U_r \to U'_r$ is an algebraic isomorphism. Thus $d\beta \neq 0$ and β as constructed above is special. For the other assertions we replace G by ${}^{\bar{\varphi}}G$, thus normalize to the case $\bar{\varphi} = \mathrm{id}$. Then on U_r, imbedded as k in SL_2 as above, we have $\bar{\beta} = \beta \circ \varphi$. Since $\bar{\beta}$ is a morphism and β an algebraic isomorphism, it follows that φ is a morphism, hence is of the form Fr^m $(m \geq 0)$. Then $\bar{\beta} = \alpha = \beta \circ (\mathrm{Fr}^m)^o$. Hence if $\bar{\beta}$ is also special, $d\bar{\beta} \neq 0$, then $m = 0$, $\beta = \bar{\beta}$, and $\varphi = \mathrm{id} = \bar{\varphi}$.

2.4. A slight extension. If we assume that α is surjective instead of bijective in 1.2 and 2.3 then the conclusions still hold.

Proof. $\ker \alpha$ is central in G since G is simple so that $G/\ker \alpha$ is also a simple algebraic group. Applying 2.1 to this group we see that the properties of

B , T ,... are preserved by α , which is all that is needed for the rest of the proof.

2.5. Special isogenies and central isogenies. We recall some facts about isogenies of connected semisimple algebraic groups. An isogeny $\pi : G \to G'$ is central if $\ker d\pi$ is central, or, equivalently, if π is an algebraic isomorphism on unipotent subgroups, or, again, if, when restricted to corresponding maximal tori, π^* maps one root system onto the other. Every central isogeny is special and conversely for simple groups a special isogeny can be noncentral only in the exceptional case that G is of type B_n , C_n , F_4 , G_2 and $\operatorname{car} k = 2$, 2 , 2 , 3 , resp. . The central isogenies are those that figure in the definition of universal covering, hence of simpleconnectedness. If G is simply connected and F the quotient of the weight lattice by the root lattice then the central isogenies $G \to \cdot$ are in correspondence with the subgroups of F , and also with the subgroups of $Z(G)$ in case $(\operatorname{car} k, |F|) = 1$. For all this see [4, § 3 ; 6, Exp. 18].

2.6. The simple connectedness of G . This assumption can not be entirely dropped in 1.2 since then α need not be a morphism even if it is a morphism on each $\langle U_r , U_{-r}\rangle$ and each such group is isomorphic to SL_2 , as is shown by the example π (nat) : $SL_4 \to PSL_4$, $\operatorname{car} k = 2$, $\alpha = \pi^{-1}$. However such examples are the only ones possible :

2.7. Corollary.- In 1.2 as extended in 2.4 drop the assumption that G is simply connected. Then $\alpha = \beta \circ \varphi^{o} \circ \gamma$ with φ and β as in 1.2 and γ the inverse of purely inseparable central isogeny. If G is simply connected or G' adjoint, then γ may be omitted.

Proof. Let $\pi : \widetilde{G} \to G$ be the universal covering of G . By 2.4 one has $\alpha \circ \pi = \widetilde{\beta} \circ \varphi^{o}$ with $\varphi^{o} : \widetilde{G} \to {}^{\varphi}\widetilde{G}$. By replacing $\widetilde{G}$ by ${}^{\varphi}\widetilde{G}$ and G by ${}^{\varphi}G$ we may assume that $\varphi = \mathrm{id}$. Factor π thus : $\widetilde{G} \xrightarrow{\pi_s} G_1 \xrightarrow{\pi_i} G$ with π_s separable and π_i purely inseparable, both central. Then π_s is a quotient map and $\widetilde{\beta}$ is constant on its fibres. Hence there is a (unique) morphism $\beta : G_1 \to G'$ such that $\widetilde{\beta} = \beta \circ \pi_s$. Then $\alpha = \beta \circ \gamma$ with $\gamma = \pi_i^{-1}$, as required. Further if $d\widetilde{\beta} \neq 0$ then $d\beta \neq 0$ so that β can be chosen to be special. Finally, if G' is adjoint then $\widetilde{\beta}$ factors through π itself so that α is a morphism and γ may be omitted. We have used here (and elsewhere) a theorem of Chevalley [6, p. 18-07] which gives conditions under which one isogeny emanating from a (connected) semisimple group can be factored through another. These conditions are seen to be verified here since G' is adjoint and π is central.

The discussion of uniqueness here, which is very easy, will be omitted.

2.8. Automorphisms. Let G be simple (and k still algebraically closed) and α an (abstract) automorphism of G. Then there exist φ, β, γ as in 2.7 (with $k' = k$ and $G' = G$) such that $\alpha = \gamma^{-1} \circ \beta \circ \varphi^{o} \circ \gamma$. Here γ is necessary only if $\mathrm{car}\, k = 2$ and G is of type D_{2n} corresponding to a semispinorial representation (thus not simply connected and not adjoint).

Proof. By 2.7 we may suppose that G is not simply connected and not adjoint. Thus we are not in the exceptional cases of 2.5 and every special isogeny is central. Let $\pi : \widetilde{G} \to G$ be the universal covering. By 2.4 applied to $\alpha \circ \pi : \widetilde{G} \to G$ we have $\alpha \circ \pi = \widetilde{\beta} \circ \varphi^{o}$ with φ an automorphism of k and $\widetilde{\beta} : {}^{\varphi}\widetilde{G} \to G$ a central isogeny. Since ${}^{\varphi}\widetilde{G}$ is simply connected, $\widetilde{\beta}$ is thus a universal covering, thus equivalent to π. Thus $\mathrm{dg}\, \widetilde{\beta} = \mathrm{dg}\, \pi = \mathrm{dg}\, {}^{\varphi}\pi$. Let G, hence also $\widetilde{G}$, etc., not be of type D_{2n}. Then the group F of 2.5 is cyclic [6]. Thus there is at most one subgroup of a given index, thus at most one central isogeny ${}^{\varphi}G \to \cdot$ of a given degree. Thus ${}^{\varphi}\pi$ and $\widetilde{\beta}$ are equivalent and there exists an algebraic isomorphism $\beta : {}^{\varphi}G \to G$ such that $\widetilde{\beta} = \beta \circ {}^{\varphi}\pi$. Then $\alpha \circ \pi = \beta \circ {}^{\varphi}\pi \circ \varphi^{o} = \beta \circ \varphi^{o} \circ \pi$. Since π is surjective, $\alpha = \beta \circ \varphi^{o}$. Now let G be of type D_{2n}. In any case $\widetilde{\beta} = \alpha \circ \varphi^{o^{-1}} \circ {}^{\varphi}\pi$. Thus $\ker \widetilde{\beta} = \ker {}^{\varphi}\pi$. If $\mathrm{car}\, k \neq 2$ then again $\widetilde{\beta}$ is equivalent to ${}^{\varphi}\pi$ for now the central isogenies coming from ${}^{\varphi}G$ correspond to the central subgroups of ${}^{\varphi}G$ since F is now a $(2,2)$ group, of order prime to $\mathrm{car}\, k$. Finally if $\mathrm{car}\, k = 2$ then π is purely inseparable, hence an abstract isomorphism. If we apply what has been proved to ${}^{\pi^{-1}}\alpha$ on $\widetilde{G}$ we get 2.8 with $\gamma = \pi^{-1}$, which ends the proof.

Conversely, if G is of this exceptional type then γ may be needed : let $\widetilde{\alpha}$ be an algebraic automorphism of $\widetilde{G}$ which maps the semispinorial representation defining G onto another one, and then $\alpha = {}^{\pi}\widetilde{\alpha}$.

2.9. Split groups and quasisplit groups. For split groups the proofs of 1.2 and the later results work equally well over arbitrary fields, with $G(k)$, $G'(k')$ replaced by G^+ , G'^+ , once it is known that α must preserve the properties of B , T , U ,... . For quasisplit groups (those having a rational Borel subgroup) only a little more work is needed (see [13] where automorphisms are considered). For infinite fields, this preservation will be shown in § 3 below in a very general setting.

2.10. Finite groups. For finite fields the preservation can be proved in most cases as follows. First G is quasisplit by a theorem of Lang, so that rational B and U exist. One then shows that except for a few cases of small rank car k is that prime which makes the largest contribution to the order of G^+ . This involves an exhaustive analysis of the group orders $|G^+|$ for the various types of simple groups and finite fields (see [1]). Thus G^+ determines $p = \text{car } k$, hence also, up to conjugacy, the Sylow p-subgroup $U(k)$, and finally $B(k)$, the normalizer of $U(k)$. The excluded cases then involve further considerations, and eventually one gets the result as stated for algebraically closed fields with a few exceptions, e.g. $PSL_2(\mathbf{F}_7) \cong SL_3(\mathbf{F}_2)$,... . If we are interested only in automorphisms then this development can be dispensed with since we have the preservation a priori (see [13]).

§ 3. Proof of 1.3

We give only several indications, supposing first G' adjoint. If X is a subgroup of G , let us write $\bar{\alpha}(X)$ for $\overline{\alpha(X \cap H)}$ (closure in G').

3.1. (cf. 7.1 of [4]) Let S be a k-subtorus of G and U a connected unipotent subgroup normalized by S such that $Z_G(S) \cap U = \{1\}$. Suppose that $H \cap S$ is dense in S and that $H \supseteq U(k)$. Then U' is a connected unipotent k'-subgroup of G' .

Proof. The set A of $s \in S$ such that $Z(s) \cap U = \{1\}$ is dense open in U , and for each $s \in A \cap H$ the map $u \to (s,u)$ on U is a k-isomorphism of varieties, thus (*) it maps $U(k)$ onto itself [2 : 9.3]. SU is solvable, thus $L = (SU)'$ is also. Let us choose E of finite index in $S \cap H$ such that E' is connected, hence contained in L^o. Since $S \cap H$ is dense in S, there exists $s \in A \cap E$. By $(*)$, $\alpha(U(k)) = (\alpha(s), \alpha(U(k))) \subset (L^o, L) \subset L^o$ and then $\alpha(U(k)) \subset DL^o$, the derived group of L^o. But $DL \subset U'$ since $D(SU) \subset U$. Thus $V = DL^o$, a connected unipotent group by the theorem of Lie-Kolchin.

3.2. One has $\operatorname{car} k = \operatorname{car} k'$.

Proof. Since G is isotropic, one can realize the situation in 3.1 with S and U nontrivial [3]. Then U' is nontrivial, for G^+ is generated by $U(k)$ as a normal subgroup of itself [16] and $\alpha(G^+)$ is dense in G'. If $\operatorname{car} k = 0$ then $G(k)$ is divisible, hence $\alpha(G(k))$ is also, hence $\operatorname{car} k' = 0$. If $\operatorname{car} k = p \neq 0$, then $U(k)$ has only p-elements, thus $\alpha(U(k))$ has also, and $\operatorname{car} k' = p$.

3.3. (7.2 of [4]) Let P, P^- be opposite parabolic k-subgroups of G, U, U^- their unipotent radicals, $Z = P \cap P^-$.

(a) U' and $U^{-\prime}$ are connected unipotent k'-groups and $U^{-\prime}Z'U'$ is dense open in G'.

(b) P', $P^{-\prime}$ are opposed parabolic k'-subgroups of G', U', $U^{-\prime}$ their unipotent radicals and $Z' = P' \cap P^{-\prime}$.

Proof. (a) There exists a split torus S normalizing U and such that $Z_G(S) \cap U = \{1\}$, and it can be imbedded in a split semisimple k-subgroup of G [3]. It follows that $(S \cap G^+)$ is dense in S, and U' and $U^{-\prime}$ are connected unipotent by 3.1. Let $\Omega' = U^{-\prime}Z'U'$. Now G is the union of a finite number of translates of U^-ZU by elements of G^+ [4 : 6.11] ; thus H and $U^-(k)(Z \cap H)U(k)$ have the same property. Thus $G' = H'$ is the union of a finite number of translates of Ω'. Thus Ω' contains a nonempty open subset of G, thus is itself open since it is a double coset : every double coset AcB is an orbit for the action $g \to agb^{-1}$ of $A \times B$ on G, thus is locally closed.

(b) Let T' be a maximal torus of Z'^o and V^-, V opposed maximal connected unipotent subgroups of Z'^o normalized by T' and such that $V^-.T'.V$ contains a non empty open subset of Z'^o. By (a) and the density of $\alpha(H)$ in G'. $U^{-\prime}.V^-.T'.V.U'$ contains (thus is) a nonempty open subset of G', and $U^{-\prime}.V^-$, $V.U'$ are connected unipotent groups normalized by T'. As G' is simple $U^{-\prime}.V^-.T'$ and $T'.V.U'$ are opposed Borel subgroups of G'. From this the assertions of (b) follow without trouble.

In (b) G' can be reductive, and in (a) arbitrary.

Now let S_m be a maximal k-split torus of G, a_m the maximal root on S_m relative to some ordering, $\check{a}_m$ the coroot of a_m, and $S = \check{a}_m$ (Mult) the corresponding one-dimensional torus. Let $Z = Z_G(S)$ and U (resp. U^-) = the connected unipotent subgroup of G corresponding to the positive (resp. negative) weights on S relative to some ordering. Then Z is connected reductive and $P = ZU$, $P^- = ZU^-$ are opposite parabolic subgroups. Further all of these groups are k-groups. For all of this see [3]. This is the set-up in which 3.2 and 3.3 will be used.

3.4. Definition of φ. One uses the action of S on U in much the same way as in 2.2 where the group SL_2 was considered (there S = diagonal subgroup, U = superdiagonal unipotent subgroup), the multiplicative structure of k being embodied in S and the additive structure in U. But now the situation is more complicated since U is not just the group Add. However, U *is* the extension of one vector space by another such that S acts according to a character a on one and according to $2a$ on the other [4, § 8]. From this and a suitable "preservation theorem", refining 3.2 and 3.3, and a good deal of further work, one can construct a homomorphism $\varphi : k \to k'$ such that if G is replaced by ${}^{\varphi}G$ then α becomes on $U(k)$ the restriction of a special morphism $\beta_U : U \to U'$, and similarly for U^-.

3.5. Completion of proof. The group Z acts on U (*via* a rational representation, in fact, if one of the above vector spaces is trivial), as does Z' on U', and the last action is faithful since G' is adjoint. From this one gets a morphism β_Z on Z whose restriction to $Z \cap H$ agrees with α. Since the map $U^- \times Z \times U \to U^-ZU = \Omega$ is an isomorphisms of varieties we deduce a morphism β_Ω on Ω which agrees with α on $\Omega \cap H$. Finally we define β on G thus. Write $x \in G$ as gy with $g \in G^+$, $y \in \Omega$, and then set $\beta(x) = \alpha(g)\beta_\Omega(y)$. This defines $\beta(x)$ uniquely since for fixed $g \in G^+$, $\beta_\Omega(gx) = \alpha(g)\beta_\Omega(x)$ for $x \in \Omega \cap g^{-1}\Omega$ for this holds on the dense set $\Omega \cap g^{-1}\Omega \cap H$. Since β is a morphism of varieties on Ω, it is so on each $g\Omega$, hence also on G. But β is also a homomorphism on the dense subgroup H since $\alpha = \beta|_H$, clearly. Thus β

is a morphism of groups. Further β is special since β_U is.

3.6. G <u>simply connected</u>. Consider this case now. Let $\pi : G' \to \mathrm{Ad}\, G'$ be the natural map. Applying the case just proved to $\pi \circ \alpha$ we get φ and β as before such that $\pi \circ \alpha = \beta_1 \circ \varphi^o$ (on H). Since G , hence ${}^{\varphi}G$, is simply connected and π is central there exists an isogeny $\beta : {}^{\varphi}G \to G'$ such that $\beta_1 = \pi \circ \beta$, thus $\pi \circ \alpha = \pi \circ \beta \circ \varphi^o$. Since π is central, α and $\beta \circ \varphi^o$ agree on H up to a map μ into the center of G' , and clearly μ is a homomorphism.

3.7. <u>Uniqueness</u>. This can be proved as in 2.3.

3.8. <u>An example</u>. Consider π (nat) : $SL_3(R) \to PSL_3(R)$, $\gamma = \pi^{-1}$. This shows that, even if car k = 0 , if G is not simply connected and G' is not adjoint then 1.3 may fail (cf. 2.7).

3.9. <u>A complement</u>. Suppose that G' , H' are like G , H in 1.3 and that $\alpha(H) = H'$. Then $\varphi : k \to k'$ in 1.3 is an isomorphism, not just a homomorphism.

This is proved in [4 : 8.11].

3.10. <u>Automorphisms</u>. The result is like that in 2.8 except that now a homomorphism $\mu : H \to Z(G) \cap H$ must be included and in the exceptional case of type D_{2n} of 2.8 one does not know (but one supposes) that car k = 2 since H may not contain Z(G) (as in the example of 3.8 ; it does so if G is split, quasisplit,...). The proof is similar.

§ 4. Extension and reformulation

We wish to extend 1.3 to the case where G' is reductive. This can not always be done : Let $\alpha : SL_n(C) \to SL_{2n}(R)$ be the map obtained by replacing each complex coordinate by two real coordinates. Clearly there is no homomorphism $\varphi : C \to R$. The image group is semisimple, not simple (as an algebraic group). This process which produces from a group defined over C (SL_n in this case) a corresponding group defined over R (the image) is called restriction of scalars and works whenever we have a finite dimensional field (or even algebra) k' separable over a given field k and a group G defined over k' . We write $R_{k'/k}\, G$ for the resulting group over k . There exists a natural isomorphism

$R^{o}_{k'/k} : G(k') \rightarrow (R_{k'/k}G)(k)$ (for this see [16 : I, § 1, 6.6]).

4.1. THEOREM (8.16 of [4]).- Assume as in 1.3 except that G' is reductive. Let G'_i $(1 \leq i \leq m)$ be the normal k'-subgroups of G' that are k'-simple (perhaps not absolutely simple). Then there exist finite separable extensions k_i $(1 \leq i \leq m)$ of k', field homomorphisms $\varphi_i : k \rightarrow k_i$, and a special k'-isogeny $\beta : \prod_{i=1}^{m} R_{k_i/k'}({}^{\varphi_i}G) \rightarrow G'$ and a homomorphism $\mu : H \rightarrow Z(G')(k')$ such that $\beta(R_{k_i/k'}({}^{\varphi_i}G)) = G'_i$ and $\alpha(h) = \mu(h).\beta(\prod_{i=1}^{m} R^{o}_{k_i/k'}(\varphi^{o}_i(h)))$ for all $h \in H$.

We give the proof in case G' is adjoint.

Then G' is the direct product of the G'_i 's, and $G'_i = R_{k_i/k'}G''_i$ with k_i/k' finite separable and G''_i absolutely simple. Let $\pi_i : G' \rightarrow G'_i$ be the natural projection. One then applies 1.3 to each of the maps $(R^{o}_{k_i/k'})^{-1} \circ \pi_i \circ \alpha : G \rightarrow G''_i$ and collects the results to get 4.1.

4.2. Uniqueness. We consider only the case : k' algebraically closed. Then 4.1 simplifies since the G'_i themselves are absolutely simple, each $k_i = k'$, and each R and each $R^{o} = \mathrm{id}$. Then the possibility of making β special and the resulting uniqueness easily follow from that of 1.3. We see further that $\varphi_i = Fr^{m}\varphi_j$ $(i \neq j)$ could never occur since then the image of ${}^{\varphi_i}G \times {}^{\varphi_j}G$ in $G'_i \times G'_j$ would be the graph of a morphism $G'_j \rightarrow G'_i$ and thus not dense.

4.3. A reformulation of 4.1. Under the hypotheses of 4.1 there exists a finite dimensional separable commutative k'-algebra L, a homomorphism $\varphi : k \rightarrow L$, a k'-isogeny $\beta : R_{L/k'}{}^{\varphi}G \rightarrow G'$ and a homomorphism $\mu : H \rightarrow$ center $G'(k')$ such that $\alpha(h) = \mu(h).\beta(R^{o}_{L/k'}(\varphi^{o}(h)))$ for all $h \in H$.

Proof. In 4.1 let $L = \dot{\Sigma}\, k'_i$, $\varphi = \dot{\Sigma}\, \varphi_i$, ...

4.4. A conjecture. The result 4.3 remains true if G' is arbitrary, with, perhaps, some mild changes (like dropping the separability).

The authors of [4] indicate that they have proved this in a number of cases and expect to return to it later. It holds, for example, if G is split, simply connected, semisimple and k is infinite and not nonperfect of car 2 .

§ 5. Continuity of homomorphisms

There are many results in [4] on this subject. We discuss here only one or two of them related to the development given so far. We now assume that k is given a nondiscrete locally compact topology which makes it into a topological field, hence $G(k)$ into a Lie group, and similarly for k' and G' , and further that G is semisimple and G' reductive.

5.1. DEFINITION.- Given a connected normal k-subgroup G_1 of G , we say that $G_1(k)$ is a complex factor of $G(k)$ if either (1) $k \cong \mathbf{C}$ or (2) $k \cong \mathbf{R}$ and G_1 is isogenous to a group of the form $R_{\bar{k}/k} G_2$.

5.2. THEOREM (9.8, 9.13 (ii) of [4]).- Let G , G' , k , k' be as above. Suppose that G possesses no nontrivial normal k-anisotropic factor and that G' possesses no nontrivial complex factor. Let H and α be as in 1.3 ($G(k) \supset H \supset G^+$, $\alpha(H)$ Zariski-dense in G') . Then α is continuous. In particular $k \not\cong \mathbf{C}$. Further each surjective homomorphism of $G(k)$ onto $G'(k')$ is continuous and each such isomorphism is a topological isomorphism.

If G is simple then the first statement follows from 4.1 and the fact, mentioned above, that if $k' \neq \mathbf{C}$ then every homomorphism $\varphi : k \to k'$ is continuous. The general case follows from this case by a series of simple reductions. The last statement then follows since φ is then necessarily surjective by 3.9.

As just seen, the assumption of isotropicity on G has been used to deduce 5.2 from 4.1, but as shown in [4] it is, in fact, not needed here and in many other results. For example :

5.3. THEOREM (9.13 (i) of [4]). In 5.2 replace the assumption of isotropicity on G by : the universal covering of G is separable (which holds if G is simply connected or if $\operatorname{car} k = 0$, and in many other cases). Then the last conclusion there holds.

The proof of 5.3 is based on a line of reasoning (due to van der Waerden [19]) not in the spirit of the above development and will be omitted.

In closing, let us mention that in particular this result implies the result of Freudenthal in the introduction, extended to other types of groups and fields.

5.4. Added remark. One of the authors of [4] has proved, over $\mathbf{R}$, a very general theorem [18, § 4] of the type we have been discussing. It implies that 4.4 holds if $k = k' = \mathbf{R}$ and G is simply connected as an algebraic group (but not necessarily semisimple) and equal to its own derived group. The two basic cases are $G = SL_2(\mathbf{R})$ and $G = Spin_3(\mathbf{R})$. From these the general case is deduced.

§ 6. Irreducible representations

We recall that a projective (resp. linear) representation of a group is a homomorphism into some [finite-dimensional] $PGL(V)$ (resp. $GL(V)$). We shall identify isomorphic representations. The principal result in [4] in this area is the following result and its refinement in 6.4.

6.1. THEOREM (10.3 of [4]).- Let k , G , H be as in 1.2, and let k' be an algebraically closed field. Let $\rho : H \to PGL_n(k')$ $(n \geq 2)$ be a projective representation irreducible on G^+ . Then there exist homomorphisms $\varphi_i : k \to k'$, finite in number, and irreducible rational projective representations π_i of ${}^{\varphi_i}G$ such that, on H , ρ is equivalent to the tensor product of the $\pi_i \circ \varphi_i^o$.

Let $G' = \overline{\rho(G^+)}$. By 3.1 this group is connected, thus by the lemma of Schur it is also reductive, semisimple, adjoint. Let $\{G'_i\}$ be its simple factors. The identity representation of $G' = \prod G'_i$ is irreducible, thus can be written as a tensor product $\prod \lambda_i$ with λ_i an irreducible rational projective representation of G'_i . By 4.1 there exist homomorphisms $\varphi_i : k \to k'$ and special isogenies $\beta_i : {}^{\varphi_i}G \to G'_i$ such that $\rho = \prod(\lambda_i \circ \beta_i \circ \varphi_i^o)$ on G^+ . If ρ' denotes this product and $h \in H$, then $\rho'(h)\rho(h)^{-1}$ centralizes $\rho(G^+)$ (G^+ is normal in H) and is thus equal to 1 by Schur's lemma. This gives 6.1 with $\pi_i = \lambda_i \circ \beta_i$.

6.2. Refinement and uniqueness. In 6.1 uniqueness does not hold if $\operatorname{car} k = p \neq 0$, for, if β is an irreducible rational projective representation then ${}^{Fr}\beta \circ Fr^o$ is one also. The situation is, in fact, more complicated than this, for one has the following result [14, th. 6.1].

6.3. THEOREM.- Assume car $k = p \neq 0$. Let $M(G)$ denote the set of irreducible rational projective representations of G for which the dominant weight is a linear combination of the fundamental weights with all coefficients between 0 and $p - 1$. (Up to isomorphism, there are p^{ℓ} such, ℓ = rank G .) Then every irreducible rational projective representation of G is isomorphic to a finite tensor product of the form $\prod_i \pi_i \circ Fr^i$ with $\pi_i \in M(Fr^i_G)$, uniquely up to trivial factors.

For simplicity we shall write $M(G)$ in this situation. If car $k = 0$, then $M(G)$ is defined as above with no restriction on coefficients.

6.4. THEOREM.- In 6.1 it can be arranged that the φ_i are distinct and that each π_i is nontrivial and in $M({}^{\varphi_i}G)$. Then the decomposition is unique. Conversely, if the φ_i and π_i are such, then the resulting product is irreducible.

Proof. The first statement follows easily from 6.1, 6.3 and the last remark of 4.2. For the uniqueness, we put together in blocks the terms of the product for which the corresponding φ_i 's differ only by a power of Fr . We get a coarser factorization $\Pi = \Pi^1 \Pi^2 \ldots$ with $\Pi^j = \pi^j \circ \varphi^j$ and $\pi^j = \Pi_i \pi^j_i \circ Fr^i$ with $\pi^j_i \in M({}^{\varphi^j}G)$. Now $\Pi^j(G)$ is the image of ${}^{\varphi^j}G$ under π^j , hence is simple, equal to one of the G'_i . It follows that the $\Pi^j(G)$ form a permutation of the G'_i . The uniqueness of the φ^j and π^j follows from 4.2, and then the uniqueness of the π^j_i from 6.3. The final statement is proved in [14, th. 5.1].

6.5. Linear representations. The preceding results extend to linear representations if one assumes that G is simply connected and adds a homomorphism into the center of $GL_n(k')$. The proof is rather easy.

We thus see that the theory of abstract irreducible representations of H is very much like that of rational ones, e.g. in case k is algebraically closed so that $H = G$ $(= G(k))$: Let B be a Borel subgroup. Then B fixes a unique line of V and acts on it according to some character, the "highest weight", which conversely determines ρ uniquely.

We close with a reformulation of the conjecture 4.4 in terms of representations. We recall that a function $f : H \to k'$ is called a representative function if the space generated by its translates over k' (left or right) is finite dimensional. As one sees these functions are the matrix coefficients of the finite-dimensional representations of H over k'. They form a k'-algebra. Let L be a finite-dimensional commutative k'-algebra and $\varphi : k \to L$ a homomorphism. Now if f is a polynomial function on G defined over k, and g is a k'-linear function from L to k' one sees easily that $g \circ \varphi \circ f : H \to k'$ is a representative function. For example, if $d : k \to k$ is a derivation, then $d \circ f$ is of this form, with L the algebra of dual numbers over k and φ the map $x \to x + dx \cdot \epsilon$. This case is related to a number of examples given in [4 : 8.18 (b), 9.15 (a)] to show the pathology that can occur if various assumptions are omitted.

6.6. <u>Reformulation of 4.4</u>. Under the assumptions of 4.4 every representative function on H is a polynomial in functions of the above form.

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