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HYPERBOLIC DIFFERENTIAL EQUATIONS AND ALGEBRAIC GEOMETRY (AFTER PETROWSKY)

by Michael F. ATIYAH

Introduction

Some twenty years ago Petrowsky [4] wrote a long and very interesting paper concerned with the support of the fundamental solution of hyperbolic equations. A substantial part of this paper was in fact concerned with some theorems on the topology of algebraic varieties, and to prove these Petrowsky based himself on the work of Lefschetz. The arguments were highly geometrical and involved a lot of intuitive topology which is not always easy to follow.

In the intervening twenty years algebraic geometry has progressed considerably and it has developed algebraic machinery to replace the older geometrical reasoning. It seems therefore reasonable that one should look again at Petrowsky's paper and try to prove his results by more modern methods. Leray and Gårding took some steps in this direction and last year Gårding drew the attention of Bott and myself to Petrowsky's paper. As a result of our joint efforts a modern version of Petrowsky's work is emerging, and it is this that I propose to report on here. I should say that a number of alternative approaches seem to be possible but the one I will present seems to be as simple as any.

§ 1. The Theorems of Petrowsky

Let $a(\xi_1, \dots, \xi_n)$ be a homogeneous polynomial of degree m with real coefficients. We say that a is hyperbolic with respect to ξ_1 if, for every non-zero real vector $(\xi_2, \dots, \xi_n)$, the equation in ξ_1

$$a(\xi_1, \xi_2, \dots, \xi_n) = 0 \tag{1.1}$$

has m real distinct roots. Note that this is an "open condition" so that the set of hyperbolic polynomials of degree m is open in the vector space of all homogeneous polynomials of degree m. We shall be interested in the fundamental solution $E(x_1, \dots, x_n)$ of the corresponding differential operator $a(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$, i.e. E is a distribution satisfying

$$a\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)E = \delta \tag{1.2}$$

where δ is the Dirac delta function. One can show (uniqueness of Cauchy problem) that there is a unique E with support in the half-space $x_1 \geqq 0$, and we shall be concerned only with this E. When we want to exhibit its dependence on a we write E_a.

The classical example is of course given by

$$a = -\xi_1^2 + \xi_2^2 + \dots + \xi_n^2$$

and (1.2) now defines the fundamental solution of the wave equation. In this case the support of E is contained in the cone

$$x_1^2 - x_2^2 - \dots - x_n^2 \geqq 0 \tag{1.3}$$

but there is a difference between n odd and n even. If n is odd supp.E is the whole solid cone (1.3) while if n is even (and > 2) supp.E is just the boundary of (1.3), namely

$$x_1^2 - x_2^2 - \dots - x_n^2 = 0 \, . \tag{1.4}$$

In the first case (n odd) one says that diffusion occurs. In the second case there is no diffusion and the interior of (1.3) is called a lacuna of E. These phenomena have well-known physical interpretations.

Petrowsky's programme was to investigate the existence of lacunas for general hyperbolic equations. Now the condition of hyperbolicity is easily seen to imply that (1.1) defines a non-singular hypersurface $A_{\mathbb{R}}$ in $P_{n-1}(\mathbb{R})$. The hyperplanes $\sum_{i=1}^{n} x_i \xi_i = 0$ which are tangential to $A_{\mathbb{R}}$ define a dual hypersurface with equation say

$$b(x_1, \dots, x_n) = 0 \tag{1.5}$$

It so happens that for the wave equation the polynomials a and b are of the same form but this is fortuitous and in general one must distinguish carefully between (1.5) in the x-space and (1.1) in the dual ξ-space. The complement (in the half-space $x_1 > 0$ of $\mathbb{R}^n$) of the cone (1.5) consists of a finite number of connected open sets, which generalize the interior and exterior of the quadratic cone (1.4). Now explicit formulae for E due to Herglotz show that E is in fact analytic on each of these components, so that the problem of the lacunas is to determine on which of these components E vanishes identically. Actually Petrowsky only investigates lacunas which are stable in a sense which we now describe.

Let a_o be a fixed hyperbolic polynomial with dual b_o, let C be a component of $b_o(x) \neq 0$ $(x_1 > 0)$ and let $x \epsilon C$. We say C is a stable lacuna for a_o if, for all polynomials a sufficiently close to a_o, the fundamental solution E_a vanishes identically near x. It is easy to see that this notion is independent of the chosen point $x \in C$, and depends only on C.

If one attempts to pass from constant coefficient differential operators to variable coefficients only the stable lacunas are of any significance. This is really Petrowsky's motivation.

Petrowsky's theorems give necessary and sufficient condition for stable lacunas in terms of the homology of the algebraic hypersurface (1.1). There are a number of different cases depending on the values of n and m. In fact one must distinguish between

(i) n odd or n even

(ii) m odd or m even

(iii) $m \geqq n$ or $m < n$.

Partly for brevity and partly because I have not worked through the details I will omit the case of odd n. The parity of m makes only a very slight difference and for simplicity I will take m even. The two alternatives in (iii) are however more substantial and I will describe the results in both cases.

We denote by A the hypersurface in <u>complex</u> projective space $P_{n-1}(\mathbb{C})$ given by the equation

$$a(\xi_1,\ldots,\xi_n) = 0$$

and by A_R the set of real points of A, i.e.

$$A_R = A \cap P_{n-1}(\mathbb{R}) .$$

We already know that A_R is non-singular, but A might have singularities which are not real. However since we are only interested in <u>stable</u> lacuna we can perturb the coefficients of A and so assume that A itself is non-singular.

From the condition of hyperbolicity one easily deduces that A_R consists of $k = \frac{m}{2}$ components $A_R^1, \dots, A_R^k$ each homeomorphic to S^{n-2}. Moreover if B^i denotes the component of $P_{n-1}(\mathbb{R}) - A_R^i$ containing the point $(1,0,\dots,0)$ then B^i is an $(n-1)$-cell, $A_R^i = \partial B^i$ and (for a suitable ordering)

$$B^1 \subset B^2 \subset \dots \subset B^k .$$

Thus if we choose an orientation of the tangent space to $P_{n-1}(\mathbb{R})$ at $(1,0,\dots,0)$ it induces an orientation on each A_R^i and so on A_R.

Suppose now that C is a component of $b(x) \neq 0$ (in $x_1 > 0$), let $x \in C$, H_x the hyperplane $\Sigma\, x_i\, \xi_i = 0$ in $P_{n-1}(\mathbb{C})$ and $(H_x)_R = H_x \cap P_{n-1}(\mathbb{R})$. We denote by A_x the intersection $A \cap H_x$. Since $b(x) \neq 0$, A and H_x cut transversally and so A_x is non-singular. In particular

$$(A_x)_R = A_x \cap P_{n-1}(\mathbb{R}) = A_R \cap (H_x)_R$$

is non-singular.

Since A_R is contained in the contractible set B^k the restriction of the Hopf bundle L of $P_{n-1}(\mathbb{R})$ to A_R is trivial. Hence, if we choose an isomorphism

$$L_{(1,0,\dots,0)} \cong \mathbb{R}$$

$(A_x)_R$ may be viewed as the zero-set of a function $A_R \to \mathbb{R}$ and so it inherits an orientation from the orientation of A_R. Hence $(A_x)_R$ defines a cycle in A_x. Its homology class will be denoted by β_x : thus

$$\beta_x \in H_{n-3}(A_x\ ;\ \mathbb{C})$$

(to avoid special cases we shall assume $n \neq 2$).

The linear form in ξ

$$x.\xi = \sum_{i=1}^{n} x_i \xi_i$$

is a section of the Hopf bundle L. Its restriction to A_R may as above be regarded as a function $A_R \to \mathbb{R}$. Then $\xi \mapsto \text{sign}(x.\xi)$ defines a map

$$A_R - (A_x)_R \to \{\pm 1\}$$

and we may consider $\text{sign}(x.\xi)\, A_R$ as a relative cycle of A mod (A_x). Its homology class will be denoted by α_x : thus

$$\alpha_x \in H_{n-2}(A, A_x; \mathbb{C}).$$

If we denote by ∂ the boundary homomorphism

$$H_{n-2}(A, A_x \,; \mathbb{C}) \to H_{n-3}(A_x \,; \mathbb{C})$$

then clearly we have

$$\partial \alpha_x = 2\beta_x \,.$$

In particular this shows that the image of β_x in $H_{n-3}(A \,; \mathbb{C})$ is zero.

It is clear that α_x and β_x depend only on the component C and not on the point $x \in C$ chosen. We may therefore write α_C and β_C instead of α_x, β_x. The theorems of Petrowsky (for n and m both even) are :

THEOREM 1. Let $m < n$. Then C is a stable lacuna if and only if $\beta_C = 0$.

THEOREM 2. Let $m \geqq n$. Then C is a stable lacuna if and only if $\alpha_C = 0$.

Remark. It is well-known that the "external component" - corresponding to hyperplanes $x.\xi = 0$ meeting all the components of A_R - is always a stable lacuna. Thus theorems 1 and 2 imply that $\beta_C = 0$ in this case : a topological result which is by no means obvious.

§ 2. Proof of Theorem 2

The proofs of Theorems 1 and 2 are on similar lines but with a number of important differences. I will restrict myself here to giving the proof of Theorem 2.

By use of the Radon-transform one obtains explicit integral formulae for the fundamental solution E_a, known as the Herglotz-Petrowsky formulae (cf. [2, p.140] for a modern derivation using distributions). To describe these in a suitably invariant form we recall first that the sheaf Ω of holomorphic $(n-1)$-forms on $P_{n-1}(\mathbb{C})$ is isomorphic to(*) $\underline{O}(-n)$ and so

$$\Omega(n) \cong \underline{O}$$

has a one-dimensional space of sections. Let

$$\omega \in \Gamma(P_{n-1}(\mathbb{C}), \Omega(n))$$

be a generator, then

$$\frac{\omega}{a} \text{ is a meromorphic section of } \Omega(n-m)$$

and taking the residue on A we get

$$\omega_a = \mathrm{Res}_A(\frac{\omega}{a}) \in \Gamma(A, \Omega_A(n-m))$$

where Ω_A is the sheaf of $(n-2)$ forms on A. Similarly we have an element

$$\omega_{a,x} = \mathrm{Res}_{A_x}(\frac{\omega_a}{(x.\xi)}) \in \Gamma(A_x, \Omega_{A_x}(n-m-1)).$$

The Herglotz-Petrowsky formula for the case m, n both even and $m \geqq n$ is then

$$E_a(x) = \text{Const.} \int_{A_R} \mathrm{sign}\,(x.\xi)(x.\xi)^{m-n}\,\omega_a \qquad (2.1).$$

Let $s = (s_1, \ldots, s_n)$ be a multi-index with $|s| = \Sigma\, s_i = m-n+1$ and put

$$D^s = \partial_1^{s_1}\,\partial_2^{s_2}\ldots\partial_n^{s_n} \qquad (\partial_i = \frac{\partial}{\partial x_i}) .$$

(*) We use the standard notations of algebraic geometry.

Applying this operator to (2.1) we find

$$D^s E(x) = \text{Const.} \int_{(A_x)_R} \xi^s \omega_{a,x} \tag{2.2}$$

Note that $\xi^s \omega_{a,x}$ is a holomorphic $(n-3)$-form on A_x and so the integral in (2.2) depends only on the homology class of $(A_x)_R$ in A_x-denoted in § 1 by β_x. To exhibit this fact we will rewrite (2.2) as

$$D^s E_a(x) = \text{Const.} < \xi^s \omega_{a,x} , \beta_x > \tag{2.3}$$

where $< \ , \ >$ denotes the pairing between homology and cohomology (or harmonic forms). Suppose now that $\beta_x = 0$ so that

$$(A_x)_R = \partial C$$

where C is a chain in A_x. Then

$$\Gamma_x = \text{sign } (x.\xi) A_R + 2C$$

is a cycle in A_R defining a homology class, say γ_x. Since $(x.\xi)^{m-n}\omega_a$ is a holomorphic $(n-2)$-form on A its restriction to A_x vanishes identically and so

$$\int_{\Gamma_x} (x.\xi)^{m-n}\omega_a = \int_{A_R} \text{sign } (x.\xi)(x.\xi)^{m-n}\omega_a .$$

Thus if $\beta_x = 0$ the fundamental solution $E_a(x)$, given by (2.1), is a period of a holomorphic form :

$$E_a(x) = \text{Const.} < (x.\xi)^{m-n}\omega_a , \gamma_x > \tag{2.4}$$

Note that the image of γ_x in the homomorphism

$$H_{n-2}(A ; \mathbb{C}) \to H_{n-2}(A, A_x ; \mathbb{C})$$

is just α_x and that (2.4) holds for any class γ_x with this property - because the period of $(x.\xi)^{m-n}\omega_a$ on a cycle in A_x is zero.

One half of Theorem 2 is now easy. Suppose in fact that $\alpha_x = 0$. Then

$$\beta_x = \tfrac{1}{2}\,\partial\alpha_x = 0,$$

and so (2.4) holds. Moreover since $\alpha_x = 0$ we can take $\gamma_x = 0$ and so, from (2.4), $E_a(x) = 0$. Thus the component of x is a stable lacuna for a as required.

The converse is more difficult and needs a non-trivial result from algebraic geometry. To explain this let us consider

$$F(a,p) = \langle p\omega_a , \alpha \rangle \tag{2.5}$$

where p is a homogeneous polynomial of degree $m-n$ and α is any homology class on A. We make no restrictions beyond assuming that A is non-singular In a sufficiently small neighbourhood of a given polynomial a_0 we may identify the homology group $H(A)$ with $H(A_0)$. Thus for fixed $\alpha \epsilon H(A_0)$, $F(a,p)$ is a function of a and p. Thus for fixed $\alpha \in H_{n-2}(A_0)$:

PROPOSITION 1. Let $F(a,p)$ be defined by (2.5) and assume that it vanishes identically for all p and for all a near a_0. Then

(i) if n is odd $\alpha = 0$.

(ii) if n is even α is a multiple of the homology class obtained by intersecting A with a general linear space of dimension $\frac{n}{2}$.

We defer the proof of this to the next section and proceed now to show how Proposition 1 leads to the second half of Theorem 2. We assume therefore that the component of x is a stable lacuna. Thus by (2.3)

$$\langle \xi^s \omega_{a,x} , \beta_x \rangle = 0 \qquad |s| = m-n+1.$$

Applying Proposition 1 to the hyperplane $x.\xi = 0$ we deduce at once that $\beta_x = 0$. Hence we can use formula (2.4) for $E_a(x)$. Applying the operator D^r when $|r| = m-n$ to (2.4) we get

$$0 = D^r E_a(x) = \langle \xi^r \omega_a , \gamma_x \rangle .$$

Applying Proposition 1 again we deduce that γ_x is the class of the intersection $A \cap P_{n/2}$. In particular therefore γ_x is homologous to a class in A_x and so α_x, the image of γ_x in $H_{n-2}(A, A_x ; \mathbb{C})$, is zero. This completes the proof of Theorem 2.

§ 3. Proof of Proposition 1

We shall deduce Proposition 1 from the following result

PROPOSITION 2. Let X be a projective non-singular variety, Y a non-singular hyperplane section. Then the groups $H^*(X-Y ; \mathbb{C})$ are isomorphic to the de Rham groups of the complex of rational differential forms on X with poles on Y.

Remarks. Proposition 2 is proved in [1 ; Theorem 4]. It is in fact part of the theory of "integrals of the second kind", and its proof is not difficult. Actually Grothendieck [3] has recently extended Proposition 2 to any affine variety $X-Y$ - but this needs the resolution of singularities and is more difficult.

We shall apply Proposition 2 with $X = P_{n-1}(\mathbb{C})$ embedded in a high projective space by the polynomials of degree m, so that we can take Y to be the hypersurface $A \subset X$ given by $a(\xi_1, \ldots, \xi_n) = 0$. A rational $(n-1)$-form on $P_{n-1}(\mathbb{C})$ with poles on A is then of the form $\frac{p\omega}{a^q}$ where q is an integer $\geqq 1$ and p is a homogeneous polynomial in $(\xi_1, \ldots, \xi_n)$ of degree $qm-n$. As a corollary of Proposition 2 we therefore deduce :

COROLLARY. Let $\eta \in H_{n-1}(P_{n-1}(\mathbb{C}) - A ; \mathbb{C})$ and assume that

$$< \frac{p\omega}{a^q}, \eta > = 0$$

for all $q \geqq 1$ and all p homogeneous of degree $qm-n$. Then $\eta = 0$.

Before proceeding to the proof of Proposition 1 we make some remarks about the homology of A and of $P_{n-1} - A$. For any $\alpha \epsilon H_k(A)$ we can associate a class $T(\alpha) \epsilon H_{k+1}(P_{n-1} - A)$. Geometrically if α is represented by a cycle C then $T(\alpha)$ is represented by a "tube" with axis C - i.e. the normal circle bundle of A in P_{n-1} restricted to C. Thus if $\deg p = m-n$, and $k = n-2$, we have

$$< p\omega_a \ , \alpha > = < \mathrm{Res}_A \frac{p\omega}{a} \ , \alpha > = \frac{1}{2\pi i} < \frac{p\omega}{a} \ , T(\alpha) > \tag{3.1}$$

Note also that if we apply Poincaré duality on A and $P_{n-1} - A$ to make identifications

$$H_k(A) = H^{2n-4-k}(A)$$

$$H_{k+1}(P_{n-1} - A) = H^{2n-3-k}(P_{n-1} \ , A)$$

then $\alpha \longmapsto T(\alpha)$ becomes the coboundary homomorphism

$$H^{2n-4-k}(A) \xrightarrow{\delta} H^{2n-3-k}(P_{n-1} \ , A) \tag{3.2}$$

The exact sequence to which (3.2) belongs shows that

$$\mathrm{Ker}\ \delta = \mathrm{Im}\ H^{2n-4-k}(P_{n-1}).$$

Taking $k = n-2$ this is zero for odd n and generated by the dual of $A \cap P_{n/2}$ for n even. Thus the conclusion of Proposition 1 is just the assertion $T(\alpha) = 0$.

We pass now to the proof of Proposition 1. We consider the variable polynomial

$$a_\lambda = a_o + \lambda b$$

where b is fixed (of degree m) and λ is a small parameter. By hypothesis therefore

$$< p\omega_{a_\lambda} , \alpha > = 0 \quad \text{identically in} \quad p,\lambda$$

where degree $p = m-n$. Applying (3.1) this gives

$$< \frac{p\omega}{a_o+\lambda b} , T(\alpha) > = 0.$$

Applying $(\frac{\partial}{\partial\lambda})^\nu$ and then putting $\lambda = 0$ we get

$$< \frac{pb^\nu\omega}{a_o^{\nu+1}} , T(\alpha) > = 0 \qquad (3.3).$$

But this holds for all p and b of the given degrees and the pb^ν span the space of all homogeneous polynomials of degree $m-n+\nu m$. Thus (3.3) shows that $\eta = T(\alpha)$ satisfies the hypotheses of the Corollary to Proposition 2 and so we have $T(\alpha) = 0$. As pointed out above this is precisely the required conclusion of Proposition 1.

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COMPLEMENTS (*)

A slight variation of the proof given above leads to some improvements on the results of Petrowsky. Thus one can prove :

(1) If $m < n$ every lacuna is stable.

Moreover a simple topological lemma combined with Theorem 2 above yields :

(2) If $m \geq n$ there are no (non-trivial) stable lacunas.

The "outside" of the cone $b(x) = 0$ is always a stable lacuna : this is the trivial lacuna. The condition that n is even, which held throughout our exposition, is essential in the proof of (2) but probably unnecessary for (1).

(*) Added June 1967.