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by John G. THOMPSON

I will primarily limit this lecture to a discussion of results obtained by two students, Goldschmidt and MacWilliams.

For each group X , let $m(X)$ be the minimal number of generators of X and let $d(X) = \max\{m(A)\}$, where A ranges over all the normal abelian subgroups of X .

Suppose G is simple and T is a Sylow 2-subgroup of G . In studying the minimal simple groups, it became clear that the case $d(T) \leq 2$ was anomalous. I handled the problem by first determining all the possibilities for a Sylow 2-subgroup and then using techniques available in any minimal simple group.

For further work in simple groups, it is desirable to classify all simple G such that $d(T) \leq 2$. The case $d(T) = 1$ is non trivial, but seems well on the way to a solution, so we assume $d(T) = 2$.

The most naive way to tackle this problem is first to classify all 2-groups T with $d(T) = 2$. This is difficult, but one result about 2-groups is helpful.

LEMMA 1.- *If* T *is a* 2-*group with* $d(T) \leq 2$, *then every subgroup of* T *is generated by* 4 *elements*.

This result then leads fairly rapidly to

THEOREM 1 (MacWilliams).- Suppose T is a Sylow 2-subgroup of the simple group G, $d(T) = 2$ and $T.C(T) \subset N(T)$. Then $|T| = 4$, 64 or 128 and T is determined by $|T|$.

If $|T| = 64$, T is isomorphic to a Sylow 2-subgroup of $U_3(4)$ and if $|T| = 128$, T is isomorphic to a Sylow 2-subgroup of the new simple groups of Janko of orders 604 800 and 50 232 960.

The structure of T in case $T.C(T) = N(T)$ is not yet determined. Several of the families of known simple groups satisfy these hypotheses.

Goldschmidt's work had a different origin. Initially, he studied simple groups with a Sylow 2-subgroup whose class of nilpotency is 2. One of the results obtained is that a Sylow 2-subgroup has exponent 4. However, this emerges from a more general set up, the starting point being

LEMMA 2.- Suppose p is a prime and P is a Sylow p-subgroup of a group G. Let n be the smallest integer such that $n(p - 1) \geq c - 1$, where c is the class of nilpotency of P. Let $W = \langle x^{p^n} \mid x \in Z(P) \rangle$, where $Z(P)$ is the center of P. Then W is weakly closed in P (that is, $g \in G$ and $W^g \subseteq P$ imply $W = W^g$).

This is elementary, but clever. The crucial result is

THEOREM 2 (Goldschmidt).- Suppose T is a Sylow 2-subgroup of G, W is a weakly closed subgroup of T, $1 \subset W \subseteq Z(T)$ and t is an involution of $T - W$. If $W \subseteq O_{2',2}(C(x))$ for every involution x of Wt, then G is not simple.

Here $O_{2',2}(X)$ is the largest normal subgroup of X with a normal 2-complement. The proof is character-theoretic.

If one couples this result with work of Gorenstein, we get

THEOREM 3 (Goldschmidt).- *If* W *is weakly closed in* T *and* $W \subseteq Z(T)^2$ *where* T *is a Sylow* 2-*subgroup of* G, *then* $W \subseteq O_{2',2}(G)$.

All these results are fragmentary, but given the state of finite group theory, this is not surprising.