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REPRESENTATION OF UNRAMIFIED FUNCTORS. APPLICATIONS

by J. P. MURRE

(according to unpublished results of A. GROTHENDIECK)

1. Introduction.

Let S be a prescheme which is locally noetherian, i. e. every point has a neighbourhood which is the spectrum of a noetherian ring. In the following all preschemes (and rings) are assumed to be over S (we say that a ring A is over S if $\mathrm{Spec}(A)$ is over S) and the morphisms are morphisms over S .

Let X be a prescheme ; X defines a contravariant functor

$$h_X : S' \to h_X(S') = \mathrm{Hom}_S(S' , X)$$

from the category (Sch/S) of preschemes over S to the category (Sets) . Many existence problems in algebraic geometry are of the following type : given a contravariant functor $F : (\mathrm{Sch}/S)^o \to (\mathrm{Sets})$, determine if the functor F is of the form h_X , i. e. if there exists a prescheme X together with a $\rho \in F(X)$ such that the map $\mathrm{Hom}_S(S' , X) \to F(S')$, given by $u \to F(u)(\rho)$, is bijective for all S' . If this is the case then we say that F is a representable functor, we write $F = X$ and we do not distinguish between F and X . Also we write in the following $\xi : S' \to F$ instead of $\xi \in F(S')$ when F is a contravariant functor, representable or not. Most of the solutions of problems of the above type are based on projective methods (in particular on the Hilbert schemes) ; here we want to discuss a non-projective construction.

We are going to list a number of necessary conditions for a functor in order to be representable. Let X be a prescheme which is locally of finite type over S . Consider $F(S') = h_X(S') = \mathrm{Hom}_S(S' , X)$, then F has the following properties :
(F_{1a}) Let $T \in (\mathrm{Sch}/S)$ and $\{U_\alpha\}$ an open covering of T in the Zariski topology, then

$$F(T) \to \prod_\alpha F\big(U_\alpha\big) \rightrightarrows \prod_{\alpha,\beta} F\big(U_\alpha \times_T U_\beta\big)$$

is exact (verification trivial ; the maps are the natural maps).
(F_{1b}) If $T' \to T$ is a faithfully flat, quasi-compact morphism, then

$$F(T) \longrightarrow F(T') \underset{p_2}{\overset{p_1}{\rightrightarrows}} F\big(T' \times_T T'\big)$$

is exact (this follows from the descent theorems, see [2], SGA, VIII, theorem 5.2). (F_{1a}) and (F_{1b}) are expressed together as :

(F_1) F is a sheaf in the faithfully flat, quasi-compact topology.

(F_2) F commutes with inductive limits of rings ; i. e. if A_α is an inductive <u>filtered</u> system of rings such that the $\mathrm{Spec}(A_\alpha)$ are all above an open affine piece of S and if $A = \varinjlim A_\alpha$, then the natural map

$$\varinjlim F\left(\mathrm{Spec}\ A_\alpha\right) \to F(\mathrm{Spec}\ A) \qquad \text{is bijective .}$$

The verification of this is easy but laborious, the essential point being that $F = X$ is locally of finite presentation over S ; see [1], EGA, IV, 8.8.2 (i).

(F_3) F commutes with projective adic limits of local artinian rings, i. e. if A is a complete, local noetherian ring with maximal ideal m , then the natural map

$$F(\mathrm{Spec}\ A) \to \varprojlim F\left(\mathrm{Spec}\ A/m^n\right) \qquad \text{is bijective .}$$

The verification of this is immediate.

Next we note that $F = X$ is (by definition) a left exact functor, i. e. if $T \leftarrow T' \leftleftarrows T''$ is an exact sequence of preschemes then $F(T) \to F(T') \rightrightarrows F(T'')$ exact; however we are going to list only a very special case of this property because the left exactness of the functor (when the functor is given before we know if it is representable) can usually only be established in this special case. We formulate this special case in terms of rings :

(F_4) If $A \to A' \rightrightarrows A' \otimes_A A'$ is an exact sequence of rings (exactness in terms of the underlying sets) with A local artinian, $\mathrm{length}_A(A'/A) = 1$, and trivial residue field extensions, then

$$F(\mathrm{Spec}\ A) \to F(\mathrm{Spec}\ A') \rightrightarrows F\left(\mathrm{Spec}\ A' \times_{\mathrm{Spec}\ A} \mathrm{Spec}\ A'\right) \quad \text{exact}$$

(i. e. (F_4) + (F_{1b}) means F <u>is universally prorepresentable</u> ; see [4], p. 9, theorem 1, the details of which can be found in [6]).

The geometric significance of (F_4) (+ (F_{1b})) is the following : it is possible to find topological rings which are, if the functor is representable, the completions $\hat{\mathcal{O}}_{X,a}$ of the local rings in the points a of X which are closed in their fibre.

The above conditions are necessary for a functor in order to be representable, but they are, as is easily seen, by no means sufficient. In fact, we don't have manageable sets of sufficient conditions for the general representability problem. Therefore we shall assume further special properties of X .

Let X (locally of finite type over S) be unramified over S ; this means that if $\tau : X \to S$, $x \in X$ and $s = \tau(x)$, then $m_{S,s}\, \mathcal{O}_{X,x} = m_{X,x}$ and $k(x)/k(s)$ separable algebraic for all $x \in X$. It turns out ([2], SGA, I, prop. 3.1) that $\tau : X \to S$ is unramified is equivalent with the fact that the diagonal $X \to X \times_S X$ is an open immersion and from this follows the following property of the functor (which as is easily seen, in turn implies unramification of X over S).

(F_5) $F\big(\mathrm{Spec}(A)\big) \to F\big(\mathrm{Spec}(A/I)\big)$ is injective, where A is a ring and I is a nilpotent ideal of A , and in fact we have to require this only for rings A of the type $k[\varepsilon]/(\varepsilon^2)$, where k is a field (we say : the functor is formally unramified over S).

Assume moreover that X is separated over S , by the valuative criterium ([1], EGA, II, prop. 7.2.3), this is equivalent with :

(F_6) If $\xi_1 , \xi_2 : \mathrm{Spec}(V) \rightrightarrows F$ with V a discrete valuation ring (over S), are equal in the generic point, then $\xi_1 = \xi_2$ (we say the functor is separated).

These conditions are still not sufficient for representability, we need two more conditions which follow if $F = X$ from the fact that X is unramified over S . Namely every point $x \in X$ has an open neighbourhood U such that $\tau/U : U \to S$ can be factored in $U \xrightarrow{j} V \xrightarrow{\sigma} S$ with a $\sigma : V \to S$ which is étale and j a closed immersion ([2], SGA, I, corollary 7.8). A morphism $\sigma : V \to S$ is said to be étale if it is unramified and flat over S . Now suppose we have a commutative diagram of the following type :

(☆)
$$\begin{tikzcd} F = X \arrow[d, "\tau"'] & T' = \mathrm{Spec}(A/I) \arrow[l, "\xi'"'] \arrow[d, "i"] \\ S & T = \mathrm{Spec}(A) \arrow[l] \end{tikzcd}$$

with I a nilpotent ideal in A . In case τ is étale, then ξ' can uniquely be lifted to a $\xi : \mathrm{Spec}(A) \to F$ (i. e. $\xi' = \xi . i$) ([2], SGA, I, corollary 5.6). In our case, τ is only unramified ; it follows from the above mentioned local factorization that there exists a largest subscheme between $\mathrm{Spec}(A/I)$ and $\mathrm{Spec}(A)$ to which ξ' can be lifted. Furthermore : the formation of this subscheme is compatible with localization (use (F_2)). From this follows in particular the following conditions on the functor :

(F_7) Let the situation be as in diagram (☆). Assume moreover that A is a complete, local noetherian ring with $\dim A = 1$, and that A has only one associated prime (in particular : T irreducible). Let N = nilradical A , and assume

$N.I = (0)$. Let t be the generic point of T and assume that

$$\xi'_t : \mathrm{Spec}(A/I)_t \longrightarrow \mathrm{Spec}(A/I) \xrightarrow{\xi'} F$$

can be lifted to a

$$\xi^* : \mathrm{Spec}(A_t) \to F .$$

Then ξ' itself can be lifted to a

$$\xi : \mathrm{Spec}(A) \to F .$$

(F_8) Let the situation be as in diagram (☆) with A noetherian. Assume T irreducible and $I.N = (0)$ (N nilradical of A). Let t be the generic point of T and assume that

$$\xi'_t : \mathrm{Spec}(A/I)_t \longrightarrow \mathrm{Spec}(A/I) \xrightarrow{\xi'} F$$

can <u>not</u> be lifted to any subprescheme of $\mathrm{Spec}(A_t)$ which is strictly larger than $\mathrm{Spec}(A/I)_t$. Then there exists a (non empty) open set W on T such that for all open subschemes W_1 of T contained in W the restriction $\xi'/W'_1 : W'_1 \to F$ of ξ' to W'_1 (with $W'_1 = W_1 \times_T T'$) can <u>not</u> be lifted to any subscheme of W_1 which is strictly larger than W'_1 .
(<u>Note</u> : using (F_2), it follows that for every $t_1 \in W$ the $\xi'_{t_1} : \mathrm{Spec}(A/I)_{t_1} \to F$ can <u>not</u> be lifted to a strictly larger subscheme of $\mathrm{Spec}(A_{t_1})$.)

The main result is the following :

THEOREM 1. - S <u>locally noetherian. A contravariant functor</u> $F : (\mathrm{Sch}/S)^{\circ} \to (\mathrm{Sets})$ <u>is representable by a prescheme</u> X <u>which is locally of finite type, unramified and separated over</u> S <u>if and only if</u> F <u>satisfies</u> (F_1) , (F_2) , (F_3) , (F_4) , (F_5) , (F_6) , (F_7) , <u>and</u> (F_8).
(<u>Note</u> : neither (F_7) nor (F_8) can be omitted.)

COROLLARY 1. - <u>The above theorem holds true if condition</u> (F_4) <u>is replaced by the condition</u> (F'_4) <u>below (which is easier to verify) provided</u> (F_5) <u>is taken in the strong form (at least for local artinian rings</u> A) :

(F'_4) <u>Given a local artinian ring</u> A <u>with residue field</u> k <u>and</u> $\xi_0 \in F\big(\mathrm{Spec}(k)\big)$. <u>Let</u> $A \to A'$ <u>be injective and</u> $\mathrm{length}_A(A'/A) = 1$, <u>assume that the image</u> ξ'_0 <u>of</u> ξ_0 <u>in</u> $F\big(\mathrm{Spec}(A' \otimes_A k)\big)$ <u>can be lifted to a</u> $\xi' \in F\big(\mathrm{Spec}(A')\big)$. <u>Then</u> ξ_0 <u>can be lifted to</u> $\xi \in F\big(\mathrm{Spec}(A)\big)$.

In the above situation, $A \to A' \rightrightarrows A' \otimes_A A'$ is exact (see [6], page 39, lemme 7). The equivalence of (F_4) and (F'_4) is straightforward by using (F_{1b}) and (F_5).

We have immediately :

COROLLARY 2. - A functor $F : (Sch/S)^o \to (Sets)$ is representable by a prescheme X which is locally of finite type, étale and separated over S, if and only if F satisfies (F_1), (F_2), (F_3), (F_6), and

(F'_5) $F\big(\mathrm{Spec}(A)\big) \to F\big(\mathrm{Spec}(A/I)\big)$ bijective (I nilpotent ideal of A).

Note : it is possible to prove a statement similar to theorem 1 with "unramified" replaced by "locally quasi-finite" over S'.

In section 2, we give an outline of the proof of the theorem, and in section 3 some applications. In the proof of the theorem, we need a technical result from descent theory ; for the sake of completeness, we have sketched a proof of this result in the appendix.

2. Outline of the proof.

First of all, we note that the above conditions on F are stable under base change.

CASE I. - dim S finite.

We proceed by induction on dim S.

Step 1. - $\dim S = 0$. We can assume $S = \mathrm{Spec}(\Lambda)$, Λ local artinian ring. Using the prorepresentability (F_4), we find a system of complete local rings A_i $(i \in I)$ and (by (F_3)) morphisms $\rho_i : \mathrm{Spec}(A_i) = X_i \to F$. Put $X = \coprod_i X_i$ (disjoint union), by (F_{1a}), we get $\rho : X \to F$ such that $\rho/X_i = \rho_i$. From (F_5) follows that the A_i are unramified and finite over Λ. The following lemma concludes the proof in step 1 :

LEMMA 1 (Λ noetherian ring). - Let X and F be two contravariant functors from (Sch/Λ) to $(Sets)$ satisfying both (F_1), (F_2), and (F_3). Let $\rho : X \to F$ be a functor morphism. Assume that the map $X(T) \to F(T)$, defined by ρ, is injective (resp. bijective) for preschemes T of type $T = \mathrm{Spec}(B)$ with B a local Λ-algebra which is artinian and the residue field of which is a finite algebraic extension of $k(s)$ (s is the point of $S = \mathrm{Spec}(\Lambda)$ below the closed point of $\mathrm{Spec}(B)$). Then ρ is a monomorphism (resp. isomorphism).

The proof is straightforward. For the injectivity of the map $X(T) \to F(T)$ one first restricts to a local T, obtained from localizing a Λ-algebra of finite type and such that the residue field of the closed point of T is a finite algebraic extension of the residue field of the image point in S. Next one restricts to a T of finite type over Λ, and finally one takes T arbitrary. For the surjectivity one proceeds in the same order.

Step 2. - The induction step from $\dim S = n - 1$ to $\dim S = n$.

Step 2a. - Let $S = \mathrm{Spec}(\Lambda)$ with Λ noetherian local ring which is complete and $\dim \Lambda = n$. Let s be the closed point $S' = S - \{s\}$. By the induction assumption, $F_{S'} = F \times_S S'$ is representable by a S'-scheme $F' = X'$ which is locally of finite type and unramified over S'. Put $\Lambda_n = \Lambda/\mathfrak{m}^{n+1}$ ($\mathfrak{m}$ maximal ideal of Λ),

$$S_n = \mathrm{Spec}(\Lambda_n) , \qquad F_n = F \times_S S_n .$$

According to step 1, F_n is representable by a scheme Y_n, clearly

$$Y_n = Y_{n+1} \times_{S_{n+1}} S_n .$$

The scheme Y_n is of type $\coprod_{i\in I} \mathrm{Spec}(B_{ni})$ ($i \in I$ = set of points of Y_o) ; for each fixed i, we put $B_i = \varprojlim B_{ni}$ (Note : the B_i are the local rings prorepresenting F above S). From (F_5), we get that the B_i are unramified and finite over Λ. Using (F_3), we find τ_i : $\mathrm{Spec}(B_i) \to F$; put $Y = \coprod_i \mathrm{Spec}(B_i)$, there exists $\tau : Y \to F$ (extending the τ_i). Consider the morphism

$$\tau' = \tau_{S'} : Y' = Y_{S'} \longrightarrow F_{S'} = X' .$$

We want to show that τ' is an open immersion and that the images of $(\mathrm{Spec}\ B_i)_{S'}$ and $(\mathrm{Spec}\ B_j)_{S'}$ under τ' are disjoint for $i \neq j$. Write for shortness $Z_i = \mathrm{Spec}\ B_i$ and $Z_i' = (Z_i)_{S'}$. In order to see that the images are disjoint, we have to show that under the morphism $\tau_i' \times \tau_j' : Z_i' \times_{S'} Z_j' \to X' \times_{S'} X'$ the inverse image of the diagonal of $X' \times_{S'} X'$ is empty. If this is not the case, then we can find a morphism $\mathrm{Spec}(V) \to Z_i \times_S Z_j$ of a discrete valuation ring V having its closed point above a point $b \in Z_i \times_S Z_j$ which is itself above the closed points of the factors Z_i and Z_j and having its open point above a point a which is in the above mentioned inverse image of the diagonal (Note : $Z_i \times_S Z_j$ is proper over S) (see [1], EGA, II, prop. 7.1.9). The maps $\mathrm{Spec}(V) \to Z_i \times_S Z_j \rightrightarrows F$ contradict (F_6). There remains to show that $\tau_i' : Z_i' \to X'$ is an open immersion ; we drop the index i (and write therefore $Z = \mathrm{Spec}(B)$). We may assume that

$Z \to S$ is a closed immersion, because otherwise we make ([1], EGA, III, chap. 0, 10.3.2) a faithfully flat finite base extension killing the residue field extensions of B over Λ (note this does not increase $\dim S$, cf. [1], EGA, IV, corollary 6.1.3) ; it suffices to show that τ' is open immersion after such a base change and then $Z \to S$ is a closed immersion because B is unramified over Λ. If $Z \to S$ is a closed immersion, then also $Z' \to X'$ is a closed immersion. If $z \in Z$ is a <u>closed</u> point of Z', then it suffices to show that the ideal J which defines Z' at the point z in $\mathcal{O}_{F',z}$ is zero. If $J \neq (o)$, then there exists an integer n_o such that $J \subset \mathfrak{m}_{F',z}^{n_o}$, but $J \not\subset \mathfrak{m}_{F',z}^{n_o+1}$. Let $\mathfrak{a}_o$ (resp. $\mathfrak{a}_1$) be the inverse images of $\mathfrak{m}_{F',z}^{n_o+1}$ (resp. of $\mathfrak{m}_{F',z}^{n_o+1} + J$) in Λ under the map $\Lambda \to \mathcal{O}_{S,z} \to \mathcal{O}_{F',z}$. We note that $\mathfrak{a}_o \overset{\subset}{\neq} \mathfrak{a}_1$ because $\mathcal{O}_{S,z} \to \mathcal{O}_{F',z}/\mathfrak{m}_{F',z}^{n_o+1}$ is surjective (F' is unramified over S', and we have now equal residue fields in z because $Z' \to S'$ is closed immersion) ; they are $\mathfrak{p}$-primary ideals, where $\mathfrak{p}$ denotes the prime ideal of Λ corresponding with the point z. Finally, we note that $\Lambda/\mathfrak{a}_o$ is of dimension 1 (z is <u>closed</u> point in Z'), and $\mathfrak{p}/\mathfrak{a}_o \cdot \mathfrak{a}_1/\mathfrak{a}_o = (o)$. We can therefore apply (F_7) to the commutative diagram :

$$\begin{array}{ccccc} F & \leftarrow & Z & \leftarrow & \mathrm{Spec}(\Lambda/\mathfrak{a}_1) \\ \downarrow & \swarrow & & & \downarrow \\ S & & \leftarrow & & \mathrm{Spec}(\Lambda/\mathfrak{a}_o) \end{array} .$$

The morphism $\xi' : \mathrm{Spec}(\Lambda/\mathfrak{a}_1) \to F$ can <u>not</u> be lifted to a morphism $\mathrm{Spec}(\Lambda/\mathfrak{a}_o) \to F$, but $\mathrm{Spec}(\Lambda/\mathfrak{a}_o)_z \to F'$ exists, which contradicts (F_7). Therefore $J = (o)$; i. e. τ' is open immersion.

We construct (by "recollement") $X = X' \coprod_{Y'} Y$, with $Y' = Y_{S'}$, and using (F_1), we obtain $\rho : X \to F$. This X represents F by lemma 1 .

<u>Step</u> 2b. - $S = \mathrm{Spec}(\Lambda)$ with Λ noetherian <u>local</u> ring $\dim S = n$. Consider $S' = \mathrm{Spec}(\hat{\Lambda})$ where $\hat{\Lambda}$ is the completion of Λ, and $F' = F \times_S S'$. By step 2a, this $F' = X'$ representable and X' has by construction a descent date with respect to $S' \to S$. We conclude by means of the following "descent theorem" :

PROPOSITION 1. - <u>Let</u> $S' \to S$ <u>be faithfully flat and quasi-compact, with</u> S <u>and</u> S' <u>locally noetherian. Let</u> $X' \to S'$ <u>be locally quasi-finite and separated over</u> S'. <u>Then every descent date on</u> X' <u>relative to</u> $S' \to S$ <u>is effective</u>.

For a sketch of the proof, see the appendix.

<u>Step</u> 2c. - $\dim S = n$, S otherwise arbitrary.

CASE II. - $\dim S = \infty$.

Both follow from :

PROPOSITION 2. - *Let* F *satisfy the conditions of theorem* 1. *Suppose*

$$F \times_S \operatorname{Spec}(\mathcal{O}_{S,s}) = F_{(s)}$$

is representable for all $s \in S$. *Then* F *itself is representable*.

Remark. - Some of the conditions follow from the representability of $F_{(s)}$. On the other hand, the conditions imply that the $X_{(s)}$ which represent the $F_{(s)}$ are locally of finite type, unramified and separated over $\operatorname{Spec}(\mathcal{O}_{S,s})$.

We first prove some lemmas :

LEMMA 2. - *Let* F *be as in proposition* 2. *Assume moreover that* F *is representable on all closed subschemes of* S (*strictly contained in* S).
Let $\xi : Y \to F$ *be given with* Y *non-ramified, separated and of finite type over* S . *Then the set of points* U *where* ξ *is étale is open*.

Remark. - Let $y \in Y$, $\tau : Y \to S$ the structure map, and $s = \tau(y)$. We say that ξ is étale in y if

$$\xi_{(s)} : Y_{(s)} = Y \times_S \operatorname{Spec}(\mathcal{O}_{S,s}) \to F_{(s)}$$

is étale in the point y . We note that U is stable under generalization, i. e. if y' is a specialization of y and if ξ is étale in y' , then ξ is étale in y . (We also note that it is possible to define the notion étale in a point entirely in terms of the functor (see [3], SGAD, XI, remarques 1.6 and 1.7).)

Proof of lemma 2. - First we note that we may assume $\tau : Y \to S$ is a closed immersion. Because if $y \in U$, then we have (after replacing, if necessary, Y by a neighbourhood of y) a factorization $Y \xrightarrow{\tau'} S' \xrightarrow{\varphi} S$, with τ' closed immersion and φ étale. From $\xi : Y \to F$, we get a ξ' in the following commutative diagram :

$$\begin{tikzcd}
F \arrow[d] & F' = F \times_S S' \arrow[l, "\varphi'"'] \arrow[d] & \\
S & S' \arrow[l, "\varphi"'] & Y \arrow[l, "\tau'"'] \arrow[ul, "\xi'"']
\end{tikzcd}$$

and since φ' is étale (always after base extensions $\operatorname{Spec}(\mathcal{O}_{S,s}) \to S$), it suffices to prove the lemma for ξ' . We may have lost our assumption on the closed subschemes, but we still have that F' is representable above closed subschemes which are strictly contained in an irreducible component of S', and this we shall use. Therefore assume $\tau : Y \to S$ closed immersion.

According to EGA (III, chap. 0, cor. 9.2.6) it suffices to show : if $y \in Y$, then $\{\bar{y}\} \cap U$ either is empty or contains an open set of $\{\bar{y}\}$. We distinguish between :

(a) If $s = \tau(y)$, then s is not a generic point of an irreducible component of S. If $\mathcal{J}$ is the Ideal defining the (reduced) subscheme $\{\bar{s}\}$, then consider the subschemes $S_n = \left(\{s\} ; \mathcal{O}_S/\mathcal{J}^{n+1}\right)$ of S ; put $F_n = F \times_S S_n$ and $Y_n = Y \times_S S_n$. By our modified assumption, F_n is representable and clearly

$$F_n = F_{n+1} \times_{S_{n+1}} S_n .$$

The F_n (resp. Y_n) determine a formal scheme $\hat{F}$ (resp. $\hat{Y}$), and we have a morphism $\hat{\xi} : \hat{Y} \to \hat{F}$ obtained from the

$$\xi_n = \xi \times_S 1_{S_n} : Y_n \to F_n .$$

Now it is easily seen that ξ is étale in a point of $\hat{Y}$, if and only if $\hat{\xi}$ is étale, and furthermore $\hat{\xi}$ is étale in an open set.

(b) s is generic point of an irreducible component of S which we clearly can assume to be S itself. Then Y (being a closed subscheme) is defined by a nilpotent Ideal $\mathfrak{J}$ (write $Y = V(\mathfrak{J})$) ; let $\mathfrak{N}$ be the Nilradical of S. Consider the commutative diagram

$$\begin{array}{lcl} F & \xleftarrow{\quad\xi\quad} & \\ \downarrow & & \\ S \leftarrow V(\mathfrak{J}.\mathfrak{N}) & \leftarrow & V(\mathfrak{J}) = Y . \end{array}$$

We can assume $y \in U$ (otherwise $U \cap \{\bar{y}\} = \emptyset$ because U is stable under generalization). Since $\xi_{(y)} : Y_{(y)} \to F_{(y)}$ is a closed immersion, we have that the étale homomorphism $\mathcal{O}_{F_{(y)},\xi(y)} \to \mathcal{O}_{Y,y}$ is an isomorphism, hence $\xi_{(y)}$ can not be lifted to a subscheme of $\mathrm{Spec}\left(\mathcal{O}_{S,y}\right)$ strictly larger than $\mathrm{Spec}\left(\mathcal{O}_{Y,y}\right)$. Therefore, by (F_8), there exists a neighbourhood W of y, and, for simplicity, we may assume $W = S$, such that in $W = S$ for all open subschemes W_1 of S, the $\xi/\left(Y/W_1\right)$ can not be lifted to a subscheme between $V(\mathfrak{J}.\mathfrak{N})/W_1$ and $V(\mathfrak{J})/W_1$. It follows easily that $\xi/\left(Y/W_1\right)$ can not be lifted in W_1 to a subscheme between W_1 and Y/W_1. As we have noted before, it follows then (using (F_2)) that for every $z \in Y$ the $\xi_z : \mathrm{Spec}\left(\mathcal{O}_{Y,z}\right) \to F$ can not be lifted to a subscheme between $\mathrm{Spec}\left(\mathcal{O}_{S,z}\right)$ and $\mathrm{Spec}\left(\mathcal{O}_{Y,z}\right)$ (and strictly larger than $\mathrm{Spec}\left(\mathcal{O}_{Y,z}\right)$).

From the commutative diagram of preschemes above we get a commutative diagram of local rings :

$$\begin{tikzcd} \mathcal{O}_{S,z} \arrow[rr, "\varphi"] \arrow[dr, "\psi"'] & & \mathcal{O}_{Y,z} \\ & \mathcal{O}_{F_{(z)},\xi(z)} \arrow[ur, "\alpha"'] & \end{tikzcd}$$

(In the following, we write $\mathcal{O}_S$, $\mathcal{O}_Y$, and $\mathcal{O}_F$.) Here, φ and α are surjective (coming from closed immersions), ψ is unramified. We want to show that α is an isomorphism. Let $J = \mathfrak{J}_z$ (resp. K) be the kernel of φ (resp. α). We claim $J.\mathcal{O}_F = K$. It suffices to show this for the corresponding completions and there it follows from the fact that $\hat{\psi}$ is surjective (unramified and residue fields the same). Since J is nilpotent, K is nilpotent. Then, using the fact that $\psi(\mathcal{O}_S)$ and K generate $\mathcal{O}_F$, we obtain easily $K \subset \psi(\mathcal{O}_S)$. Hence ψ is surjective, however then $K = (0)$ for otherwise, we could lift ξ_z .

LEMMA 3. - Let F be as in proposition 2. Suppose that for every point $x \in F$ (i. e. for every point x of every $F_{(s)}$) there exists $\tau_x : U_x \to S$ and $\xi_x : U_x \to F$ such that τ_x is of finite type, ξ_x a monomorphism and étale. Then F is representable (follows from SGAD, XI, prop. 3.5).

LEMMA 4. - Let T be of finite type over S and $\xi_1 , \xi_2 : T \rightrightarrows F$. Let $G = \mathrm{Ker}(\xi_1 , \xi_2)$. If F satisfies the conditions of proposition 2, then G is representable and in fact is representable as an open immersion of T .

Proof. - From the first three properties of F , follow the same properties of G . Moreover, from the fact that F is unramified, follows that G satisfies (F_5'), i. e. G is étale (we need for F only the weak form of (F_5), the strong form follows from the representability of the $F_{(s)}$). Furthermore, $G \to T$ is a monomorphism. Then apply SGAD ([3], XI, prop. 3.1).

Proof of proposition 2. - We proceed by noetherian induction. Starting with $x \in F$, i. e. $x \in F_{(s)}$ for some $s \in S$, we take an open affine neighbourhood $U_{(s)}$ of x in $F_{(s)}$ and we take $U_{(s)}$ to be of finite type over $\mathrm{Spec}(\mathcal{O}_{S,s})$. Then we obtain ([1], EGA, IV, 8.8.2) a $\tau : U \to S$ of finite type such that $U \times_S \mathrm{Spec}(\mathcal{O}_{S,s}) = U_{(s)}$; we can take U so small that U is non-ramified and separated over S and that there exists a $\xi : U \to F$ (use (F_2)) such that $\xi \times 1_{\mathrm{Spec}(\mathcal{O}_{S,s})} = \xi_{(s)}$ (= canonical injection). Moreover, using lemma 1, we can assume that ξ is étale. We want to show that we can take U so small that ξ is a monomorphism. Let η_1 , η_2 be obtained by composition :

$$U \times_S U \underset{p_2}{\overset{p_1}{\rightrightarrows}} U \xrightarrow{\xi} F .$$

By lemma 4, $R = \mathrm{Ker}(\eta_1 , \eta_2) \to U \times_S U$ is representable as an open immersion ; $U \times_S U$ is noetherian (for U small) and hence R is of finite type over S . Since $\xi_{(s)}$ is a monomorphism, we have $R_{(s)} = \Delta_{(s)}$ with Δ the diagonal. Then ([1], EGA, IV, 8.8.2) $R = \Delta$ above a neighbourhood of s , i. e. we may assume ξ to be a monomorphism. Finally we do this for every point $x \in F$, and we apply lemma 3.

3. Applications.

(A) The flattening functor and the construction of quotients.

THEOREM 2. - *Let S be locally noetherian,* $f : X \to S$ *of finite type, and* $\mathcal{E}$ *a coherent* $\mathcal{O}_X$*-Module. Consider the following functor* $F : (\mathrm{Sch}/S)^o \to (\mathrm{Sets})$:

$$F(T) = \begin{cases} \text{Consists out of one point if } \mathcal{E}_T \text{ is flat over } T \text{ (resp. faithfully} \\ \quad \text{flat over } T \text{).} \\ \emptyset \text{ if } \mathcal{E}_T \text{ is not flat over } T \text{ (resp. not faithfully flat).} \end{cases}$$

(*Here,* $\mathcal{E}_T$ *means the inverse image of* $\mathcal{E}$ *on* $X_T = X \times_S T$.)

This "flattening functor" is representable as an unramified S-*scheme, locally of finite type over* S , *if and only if* F *satisfies* (F_3).

COROLLARY 1. - *The above functor is representable if* X *is proper over* S .

F is a subfunctor of S . (F_1) , (F_5) , and (F_6) are easy ; the other points follow from [1], EGA, IV, namely (F_2) from 11.2.6, (F_4) and (F_7) from 11.4.1, and (F_8) from 11.4.4.

As to (F_3), let A be the complete local noetherian ring, and $T = \mathrm{Spec}(A)$. It follows from [2], SGA, IV, théorème 5.6 and théorème 6.10 that $\mathcal{E}_T$ is T-flat in an open piece containing the fibre of the closed point, hence corollary 1.

COROLLARY 2. - S *locally noetherian,* $f : X \to S$ *of finite type. The following conditions are equivalent* :

(i) f factors in $X \xrightarrow{f'} S' \xrightarrow{f''} S$ with f' faithfully flat. f'' monomorphism.

(ii) f factors in $X \xrightarrow{f'} S' \xrightarrow{f''} S$ with f' flat, f'' monomorphism.

(iii) $X \times_S X \xrightarrow[p_1]{} X$ flat.

The factorization in (i) is unique up to isomorphisms.

Proof. - (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) trivial.

Uniqueness in (i) because f' is quasi-compact and hence S' is the quotient of X under the equivalence relation $X \times_S X \rightrightarrows X$.

(iii) $\Rightarrow$ (i) . Apply theorem to $\mathcal{E} = \mathcal{O}_X$. There remains only to be checked (F_3) ; we want X_A faithfully flat over A . There exists an open set U in X_A containing the fibre of the closed point such that U is flat over $\mathrm{Spec}(A)$, but $U \to \mathrm{Spec}(A)$ is open, hence this is faithfully flat and quasi-compact. Look to :

$$\begin{array}{ccccc}
X_A & \longleftarrow & X_A \times_A X_A & \longleftarrow & X_A \times_A U \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(A) & \longleftarrow & X_A & \longleftarrow & U
\end{array} .$$

The middle vertical arrow is flat, hence the right one, hence the left one.

As an application of corollary 2, we mention : let $u : G \to H$ be an homomorphism of S-preschemes of groups locally of finite type and u of finite type. Assume that $N = \mathrm{Ker}(u)$ is flat over the base. Then G/N exists. A further application of corollary 2 to the construction of quotients has been made by RAYNAUD.

Finally, we mention another application of the flattening functor of a different type namely to the Picard functor.

COROLLARY 3. - Let X be proper over a noetherian integral prescheme S . Then there exists a non empty open set U on S such that the Picard scheme of X/U exists.

One reduces to the projective case using Chow's lemma. It is not possible to give the details here. This generalizes a theorem in [7]. We mention also that assuming that the generic fibre of X is geometrically integral, that there is a section, and that X is flat (and always proper) over S , one can drop the assumption that S is integral ; in order to obtain this result, one starts with corollary 3, but one has also to use quite different techniques.

(B) Functors of type $\prod\limits_{X/S} Y/X$.

THEOREM 3. - S locally noetherian. Let $f : X \to S$ be flat and of finite type, and $Y \to X$ a closed immersion. Then the functor $F = \prod\limits_{X/S} Y/X$ is representable by a closed subscheme of S, if and only if F satisfies (F_3). (In particular : representable in case X is proper over S.)

Some indications : (F_1), (F_5), (F_6) clear, (F_2) easy (see [1], EGA, IV, 8.8.2). For (F_4), one has to show if $F(A') \neq \emptyset \implies F(A) \neq \emptyset$ (obvious notations), i. e. $\mathcal{O}_{X_A} \to \mathcal{O}_{Y_A}$ an isomorphism. Since we have surjectivity (Y closed subscheme), one has only to look for injectivity ; since X is flat over S, one has $\mathcal{O}_{X_A} \to \mathcal{O}_{X_{A'}}$, injective, and since $\mathcal{O}_{X_{A'}} \xrightarrow{\sim} \mathcal{O}_{Y_{A'}}$, we are done. For (F_7), one remarks that the open set U consisting out of the generic point of $T = \mathrm{Spec}(A)$ is "schematically dense" in T ([3], SGAD, IX, § 4), X being flat the same is true for X_U in X_T. From this follows $X_U = Y_U$ implies $X_T = Y_T$ (this is stronger than (F_7) namely the valuative criterium for properness). For (F_8), it suffices to show that F is "generic representable" on T, i. e. representable above an open set U of T. This follows since X_U is "essentially free" over U (from the flatness, see also [2], SGA, IV, lemme 6.7) (see [3], SGAD, VIII, § 7). For applications of this type of functors, see [3], SGAD, VIII and IX.

GROTHENDIECK has informed me that he has also results on functors of this type without assuming $Y \to X$ to be a monomorphism, namely by using theorem 1 together with a method of MATSUMURA of reduction to the case where X is projective over S. In this way, one obtains generalizations of the theorem of Matsumura and Oort on the representability of $\underline{\mathrm{Aut}}_k(X)$ for X a proper scheme over a field k.

(C) The functor of correspondence classes.

Let $f : X \to S$, and $g : Y \to S$; consider the map

$$\underline{\mathrm{Pic}}_{X/S} \times \underline{\mathrm{Pic}}_{Y/S} \xrightarrow{u} \underline{\mathrm{Pic}}_{X \times_S Y/S} ,$$

and let $\underline{\mathrm{Corr}}_S(X, Y)$ be the sheaf (in the faithfully flat quasi-compact topology) determined by the cokernel of u (this is the functor of divisorial correspondence classes between X and Y over S). Suppose that $f_*(\mathcal{O}_X) = \mathcal{O}_S$, $g_*(\mathcal{O}_Y) = \mathcal{O}_S$ universally, and that X and Y have sections $e_X : S \to X$, $e_Y : S \to Y$, then it is possible to give the following description for this functor : $\underline{\mathrm{Corr}}_S(X, Y)(S')$ is the set of isomorphism classes of the invertible

$\left(\mathcal{O}_{X' \times_{S'} Y'}\right)$-Modules (putting $X' = X \times_S S'$, etc.) with trivialisations given on $e_{X'} \times_{S'} Y'$ and $X' \times_{S'} e_{Y'}$ which must agree on $e_{X'} \times_{S'} e_{Y'}$.

One has the following result :

THEOREM 4. - S locally noetherian, $f : X \to S$ and $g : Y \to S$ proper and flat. Let $k(s) \xrightarrow{\sim} H^0\left(X_s , \mathcal{O}_{X_s}\right)$ and $k(s) \xrightarrow{\sim} H^0\left(Y_s , \mathcal{O}_{Y_s}\right)$ for all $s \in S$. If $\underset{\sim}{\mathrm{Corr}}_S(X , Y)$ is separated (problem : probably always true under the previous assumptions!), then it is representable by an unramified S-scheme which is locally of finite type over S . Moreover $\underset{\sim}{\mathrm{Corr}}_S(X , Y)$ is separated if $\underset{\sim}{\mathrm{Pic}}_{X/S}$ or $\underset{\sim}{\mathrm{Pic}}_{Y/S}$ is separated, for instance if X or Y has geometrical integral fibres.

COROLLARY 1. - $\underset{\sim}{\mathrm{Corr}}_S(A , B)$ is representable for $A \to S$, $B \to S$ abelian schemes.

Some indications for the proof : making, if necessary, the faithfully flat quasi-compact base change $S \leftarrow X \times_S Y$, one can assume that there are sections. (F_1) , ... , (F_4) follow then easily from the same properties of $\underset{\sim}{\mathrm{Pic}}_{X \times_S Y/S}$. For the non-ramification, one uses the following lemma :

LEMMA (Assumptions as in theorem 4). - If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_S$-Module, then the natural map

$$R^1 f_*\left(\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{F}\right) \oplus R^1 g_*\left(\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{F}\right) \xrightarrow{\varphi} R^1\left(f \times_S g\right)_* \left(\mathcal{O}_{X \times_S Y} \otimes_{\mathcal{O}_S} \mathcal{F}\right)$$

is bijective.

One constructs, using the sections, a map ψ in the other direction. Then $\psi \circ \varphi =$ identity : there remains to be shown $\mathrm{Ker}(\psi) = 0$ (in case $F = \mathcal{O}_S$ this is the infinitesimal form of the theorem of the square). First one proves this for a field k (Künneth), next for an artin ring Λ (induction on n with $\Lambda_n = \Lambda/\mathfrak{m}^n$), for arbitrary S use the comparison theorem.

For the non-ramification assume (for simplicity) $S = k[\varepsilon]/(\varepsilon^2)$; let

$$\xi \in \underset{\sim}{\mathrm{Corr}}_S(X , Y)\ (S)$$

be represented by an invertible $\left(\mathcal{O}_{X \times_S Y}\right)$-Module $\mathcal{L}$. This $\mathcal{L}$ becomes trivial on $X_o \times_k Y_o$ (with $X_o = X \times_S k$, etc.). Then $\mathcal{L}$ corresponds with an element ζ of $H^1\left(X \times_S Y , \mathcal{O}_{X_o \times_k Y_o}\right)$; the fact that $\zeta = 0$ follows, since $\mathcal{L}$ is trivial on the sections $e_X \times_S Y$ and $X \times_S e_Y$ and from the lemma which tells us that

$$H^1\left(X \times_S Y\ ,\ \mathcal{O}_{X_o \times_k Y_o}\right) = H^1\left(X\ ,\ \mathcal{O}_{X_o}\right) \oplus H^1\left(Y\ ,\ \mathcal{O}_{Y_o}\right)\ .$$

The difficult condition is (F_7). Assume for simplicity that $S = \mathrm{Spec}(A)$ (the A from (F_7)). There is given an element $\xi \in \underset{\sim}{\mathrm{Corr}}_S(X\ ,\ Y)(\mathrm{Spec}\ A/I)$, represented by an invertible $\mathcal{L} \in \underset{\sim}{\mathrm{Pic}}_{X\times_S Y/S}(A/I)$; the obstruction to lift $\mathcal{L}$ to $\underset{\sim}{\mathrm{Pic}}_{X\times_S Y/S}(A)$ is an element ζ of $H^2\left(X \times_S Y\ ,\ \mathcal{O}_{X\times_S Y} \otimes_A I\right)$. The $\mathcal{L}$ is trivial on the sections and therefore the obstruction ζ is in the kernel of

$$H^2\left(X \times_S Y\ ,\ \mathcal{O}_{X\times_S Y} \otimes_A I\right) \to H^2\left(X\ ,\ \mathcal{O}_X \otimes_A I\right) \qquad \text{(and similar for } Y\text{)}\ .$$

Take a parameter $u \in A$ which is zero in the closed point of $\mathrm{Spec}(A)$. From the exact sequence $0 \longrightarrow I \xrightarrow{u} I \longrightarrow I/uI \longrightarrow 0$ (multiplication by u), one gets an exact sequence

$$\cdots \to H^1\left(X \times_S Y\ ,\ \mathcal{O}_{X\times_S Y} \otimes_A I\right) \to H^1\left(X \times_S Y\ ,\ \mathcal{O}_{X\times_S Y} \otimes_A I/uI\right)$$
$$\to H^2\left(X \times_S Y\ ,\ \mathcal{O}_{X\times_S Y} \otimes_A I\right) \xrightarrow{u} H^2\left(X \times_S Y\ ,\ \mathcal{O}_{X\times_S Y} \otimes_A I\right) \to \cdots$$

and for a suitable u, the ζ is in the kernel of u by the assumption of (F_7). There are similar sequences for X and Y, and since ζ is also in the kernel of

$$H^2\left(X \times_S Y\ ,\ \mathcal{O}_{X\times_S Y} \otimes_A I\right) \to H^2\left(X\ ,\ \mathcal{O}_X \otimes_A I\right) \qquad \text{(and similar for } Y\text{)}\ ,$$

it follows, using the above lemma, that $\zeta = 0$. Hence $\mathcal{L}$ can be lifted ; the indeterminancy of the lifting is in $H^1\left(X \times_S Y\ ,\ \mathcal{O}_{X\times_S Y} \otimes_A I\right)$, and again, using the lemma, it follows that ξ can be lifted.

For (F_8), one can assume (by taking a small neighbourhood) that I is A_o-free (with $A_o = A_{red}$) and $H^2\left(X \times_S Y\ ,\ \mathcal{O}_{X_o \times_{S_o} Y_o}\right)$ is A_o-free. Take a base $\{e_\alpha\}$ for the latter, and start with an element ξ represented by an invertible $\mathcal{L}$ as above. The obstruction for the lifting of L can be written as $\zeta = \sum_\alpha i_\alpha e_\alpha$ with $i_\alpha \in I$. The assumption of (F_8) is that the ideal J, generated by these i_α, is such that for the generic point t of $\mathrm{Spec}(A)$ the $J_t = I_t$. However then $J = I$ in a small neighbourhood of t, and we are done.

Let for instance $\underset{\sim}{\mathrm{Pic}}_{X/S}$ be separated, then $\underset{\sim}{\mathrm{Corr}}_S(X\ ,\ Y)$ is separated, and $\underset{\sim}{\mathrm{Pic}}_{X/S}$ is separated, if X has geometrical integral fibres (one can assume

$S = \mathrm{Spec}(V)$ with V a discrete valuation ring) ; one has an invertible $\mathcal{L}$ on X , take a Cartier divisor D for $\mathcal{L}$, then one finds that D must be a multiple of the special fibre and therefore D lineairly equivalent to zero.

Moreover if $X \to S$ and $Y \to S$ are smooth, proper and have integral fibres, one sees that $\underset{\sim}{\mathrm{Corr}}_S(X , Y)$ fulfills the valuative criterium for properness (i. e. is formally proper ; cf. [4], n° 236, théorème 2.1). This is the case in particular if $X = A$ and $Y = B$ are abelian schemes.

Let $X = Y = A$ (abelian scheme), consider the map

$$\delta : \underset{\sim}{\mathrm{Pic}}_{A/S} \to \underset{\sim}{\mathrm{Corr}}_S(A , A)$$

defined by

$$\delta(\mathcal{L}) = \pi^*(\mathcal{L}) - p_1^*(\mathcal{L}) - p_2^*(\mathcal{L})$$

(where $\pi : A \times_S A \to A$ is the group multiplication). Then,

(i) $\mathrm{Ker}(\delta) = \underset{\sim}{\mathrm{Pic}}^{\tau}_{A/S}$ (see [5], p. 90 theorem 2, and p. 100 corollary 3 ; also note that, since $\underset{\sim}{\mathrm{Corr}}_S(A , A)$ is unramified, it suffices to prove (i) over fields).

(ii) δ is formally smooth (this follows from the cohomology theory of abelian varieties, see [8], p. 191, theorem 10 ; again it suffices to work over fields). From this one finds that the Neron-Severi scheme $\underset{\sim}{\mathrm{NS}}_{A/S} = \underset{\sim}{\mathrm{Pic}}_{A/S}/\underset{\sim}{\mathrm{Pic}}^{\tau}_{A/S}$ is representable as an open piece of $\underset{\sim}{\mathrm{Corr}}_S(A , A)$ (Note : $\underset{\sim}{\mathrm{Pic}}^{\tau}_{A/S} = \underset{\sim}{\mathrm{Pic}}^{o}_{A/S}$).

An application of this is the following : <u>an abelian scheme</u> A <u>over a normal variety is projective</u>. One starts with an ample sheaf $\mathcal{L}_t$ for the generic fibre A_t (theorem of Weil). This is a <u>rational section</u> in $\underset{\sim}{\mathrm{NS}}_{A/S}(S)$. But $\underset{\sim}{\mathrm{NS}}_{A/S}$ being an unramified, separated and formally proper scheme over S , one shows that for a normal S such a rational section is a section ; i. e. we have $\sigma \in \underset{\sim}{\mathrm{NS}}_{A/S}(S)$. Now $\delta(\sigma) \in \underset{\sim}{\mathrm{Corr}}_S(A , A)(S)$ determines an invertible sheaf on $A \times_S A$; from the diagonal $\Delta : A \to A \times_S A$, we obtain an invertible sheaf $\mathcal{L}'$ on A and $\mathcal{L}'_t$ is algebraic equivalent with $L_t^{\otimes 2}$. The set of points t' on S such that $\mathcal{L}'_{t'}$ is ample is both open ([1], EGA, III, 4.7.1) and closed (we can reduce to A over the spectrum of a discrete valuation ring ; one takes a positive Cartier divisor D , and $D_{t'}$ is ample if and only if $D_{t'}^{(n)} \neq 0$, where $n = \dim A$; this being so in the open point of $\mathrm{Spec}(V)$, it remains so in the closed point).

Finally, we mention the following variant : S <u>locally noetherian</u>, $f : X \to S$ <u>proper and flat with a section</u> σ , <u>and assume</u> $\mathcal{O}_S \xrightarrow{\sim} f_*(\mathcal{O}_X)$ <u>universally. Let</u> G <u>be a group scheme locally of finite type over</u> S . <u>Then</u> $\underset{\sim}{\mathrm{Hom}}_{S\text{-pointed}}\big((X,\sigma) , (G,e)\big)$

is representable by an unramified S-scheme locally of finite type over S (compare with the preceding with $G = \underline{Pic}_{Y/S}$). There is a similar theorem where one gives some properties of G , but where one does not assume G to be representable.

4. Appendix (proof of proposition 1).

THEOREM. - Let S be a prescheme, $f : S' \to S$ faithfully flat and quasi-compact, and $g : X' \to S'$ locally of finite presentation, locally quasi-finite and separated. Every descent date on X' relative to $S' \to S$ is effective in the following cases :

(a) g quasi-compact (hence of finite presentation) ;

(b) f universally open (for instance ; f locally of finite presentation) ;

(c) S discrete ;

(d) S' locally noetherian ;

Proof. - The descent date is effective if g is quasi-affine ([2], SGA, VIII, 7.9), and g is quasi-affine under the above assumptions if g is of finite presentation ([2], SGA, VIII, 6.2 and 6.6). This settles case (a), and in the other cases it suffices to show that every affine open set U' of X' is contained in a quasi-compact open set V' , which is saturated for the equivalence relation belonging to the descent date. Consider the "standard diagram" :

$$\begin{tikzcd} & X' \arrow[d, "g"'] & X'' \arrow[l, shift right, "q_1"'] \arrow[l, shift left, "q_2"] \arrow[d] \\ S & S' \arrow[l, "f"'] & S'' = S' \times_S S' \arrow[l, shift right] \arrow[l, shift left] \end{tikzcd}$$

Let $U_1' = q_1 . q_2^{-1}(U')$, this is a quasi-compact and saturated set (q_2 is quasi-compact because f is such) which is open in case (b) and (c). We are ready in these cases, and in case (d), we take for V' the interior of the closure of U_1'. The formation of the closure commutes with flat base extension (U_1' is pro-constructible, see [1], EGA, IV, 2.3.10) and similarly for the formation of the interior ($\overline{U_1'}$ is ind-constructible). Therefore V' is saturated ; furthermore it contains U' , and we want to show that it is quasi-compact. We need the following lemma.

LEMMA 1. - Let Z be separated and locally quasi-finite over a T , which is the spectrum of a universally japanese noetherian ring. If U is a noetherian open set in Z , then the closure $\overline{U}$ is quasi-compact.

Proof. – It suffices to prove the following : Z integral implies Z noetherian (look to the irreducible components of U). We may assume : $Z \to T$ dominating, and T integral. Moreover we can assume Z normal because (T being universally japanese) the normalization of Z in its function-field K is finite over Z . Let T' be the normalization of T in K , again this is finite over T . From the main theorem follows easily that every noetherian open piece of Z can be identified with an open piece of T' ; hence Z can be identified with an open piece of T' and since T' is noetherian, Z is quasi-compact.

Returning to case (d) ; apply the lemma to a quasi-compact open set containing U'_1 , we see that the theorem is true if S' is "universally japanese". In particular it follows that we are done in case S is the spectrum of a local ring, because we may in that case assume that S' is the spectrum of a complete local noetherian ring, and then ([1], EGA, IV, 7.7.4) S' is universally japanese (replace otherwise S' by $\mathrm{Spec}(\hat{\mathcal{O}}_{S',s'})$, where s' is a point above the closed point of S). The general case follows from the following lemma :

LEMMA 2. – Let $f : S' \to S$ be faithfully flat and quasi-compact, and $g : X' \to S'$ quasi-separated and locally of finite presentation. A descent date on X' relative to $S' \to S$ is effective if it is effective after the base changes $\mathrm{Spec}(\mathcal{O}_{S,s}) \to S$ for all $s \in S$.

Proof. – We can assume that S and S' are affine. For $s \in S$, put

$$S^{(s)} = \mathrm{Spec}(\mathcal{O}_{S,s}) \ .$$

From the descent date on $X'^{(s)} = X' \times_S S^{(s)}$, we get a prescheme $X^{(s)}$ which is union of affine open pieces $V_i^{(s)}$ which are of finite presentation over $S^{(s)}$. By [1], EGA, IV, théorème 8.8.2, we can find a W_i of finite presentation over a neighbourhood U_s of s in S such that $W_i \times_S S^{(s)} = V_i^{(s)}$, and we may assume U_s so small that $W'_i = W_i \times_S S' \to X'$ is an open immersion, that this open set is stable for the descent date on X' , and that the induced descent date on W'_i corresponds with the natural one coming from the fact that we have already W_i over S (because all of this is true after base change $U_s \leftarrow S^{(s)}$). These W'_i cover X' , and this completes (see [2], SGA, VIII, 7.2) the proof.

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