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# THE TOPOLOGY OF NORMAL SINGULARITIES OF AN ALGEBRAIC SURFACE

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(d'après un article de D. MUMFORD [4])

We shall study MUMFORD's results in the complex-analytic case.

## 1. Regular graphs of curves.

Let $X$ be a complex manifold of complex dimension $2$. A regular graph $\Gamma$ of curves on $X$ is defined as follows.

i. $\Gamma = \{E_1, E_2, \dots, E_n\}$.

ii. Each $E_i$ is a compact connected complex submanifold of $X$ of complex dimension $1$.

iii. Each point of $X$ lies on at most two of the $E_i$.

iv. If $x \in E_i \cap E_j$ and $i \neq j$, then $E_i$, $E_j$ intersect regularly in $x$ and $E_i \cap E_j = \{x\}$.

$\Gamma$ defines a graph $\Gamma'$ in the usual sense (i. e. a one-dimensional finite simplicial complex) by associating to each $E_i$ a vertex $e_i$ and by joining $e_i$ and $e_j$ by an edge if and only if $E_i \cap E_j$ intersect. $\Gamma'$ becomes a "weighted graph" by attaching to each $e_i$ the self-intersection number $E_i \cdot E_i$, i. e. the Euler number of the normal bundle of $E_i$ in $X$. We have the symmetric matrix

$$S(\Gamma) = ((E_i \cdot E_j))$$

where $E_i \cdot E_j$ $(i \neq j)$ equals $1$ if $E_i \cap E_j \neq \emptyset$ and equals $0$ if $E_i \cap E_j = \emptyset$. This matrix is called the intersection matrix of $\Gamma$ and defines a bilinear symmetric form $S$ over the $\mathbb{Z}$-module $V = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \dots + \mathbb{Z}e_n$. The matrix $S(\Gamma)$ depends (up to the ordering of the $e_i$) only on the weighted tree and may be denoted by $S(\Gamma')$. The subset $A$ of $X$ is called a tubular neighbourhood of $\Gamma$ if

i. $A = \bigcup_{i=1}^{n} A_i$,

where $A_i$ is a (compact) tubular neighbourhood of $E_i$,

ii. $E_i \cap E_j = \emptyset$ implies $A_i \cap A_j = \emptyset$

iii. $E_i \cap E_j = \{x\}$ implies the existence of a local coordinate system $(z_1, z_2)$ with center $x$ and a positive number $\varepsilon$ such that the open neighbourhood

$$U = \{p| \quad p \in X \wedge |z_1(p)| < 2\varepsilon \wedge |z_2(p)| < 2\varepsilon\}$$

is defined in this coordinate system and

$$A_i \cap U = \{p| \quad p \in U \cap |z_2(p)| \leq \varepsilon\} \quad ,$$

$$A_j \cap U = \{p| \quad p \in U \cap |z_1(p)| \leq \varepsilon\} \quad ,$$

$$A_i \cap A_j \subset U \quad .$$

Such tubular neighbourhoods always exist.

$A$ is a compact 4-dimensional manifold (differentiable except "corners") whose boundary $M$ is a 3-dimensional manifold (without boundary). It is easy to see that $A$ has $E = \bigcup_{i=1}^{n} E_i$ as deformation retract. Thus

(1) $$H_i(A) \sim H_i(E) \quad .$$

Suppose that the graph $\Gamma'$ is connected. This is the case if $M$ is connected. If, moreover, $\Gamma'$ has no cycles, then $E$ is homotopically equivalent to a wedge of $n$ compact oriented topological surfaces with the genera $g_i = \text{genus}(E_i)$. If $\Gamma'$ has $p$ linearly independent cycles, then the homotopy type of $E$ is the wedge of $n$ surfaces as above and $p$ one-dimensional spheres. The first Betti number of $E$ is given by the formula

(2) $$b_1(E) = 2 \sum_{i=1}^{n} g_i + p \quad .$$

We have the exact sequence (rational cohomology)

(3) $$H^1(A, M) \to H^1(A) \to H^1(M) \quad .$$

By Poincaré duality $H^1(A, M) \cong H_3(A)$ which vanishes by (1).

Therefore $H^1(A)$ maps injectively into $H^1(M)$ which proves in virtue of (1) and (2) :

LEMMA. – <u>If the regular graph of curves</u> $\Gamma := \{E_1, \dots, E_n\}$ <u>has a tubular neighbourhood</u> $A$ <u>whose boundary</u> $M$ <u>is a rational homology sphere, then the graph</u> $\Gamma'$ <u>is a tree (i. e.</u> $\Gamma'$ <u>is connected and has no cycles). Furthermore, the genera of the curves are all</u> $0$, <u>thus all the</u> $E_i$ <u>are</u> 2-<u>spheres</u>.

## 2. The fundamental group of the "tree manifold" $M$.

Suppose $M$ is obtained as in Section 1, assume that $\Gamma'$ is a tree and all the $E_i$ are 2-spheres. By the lemma of Section 1 this is true if $M$ is a rational homology sphere. The fundamental group $\pi_1(M)$ is presented by the following theorem.

THEOREM. – Put $S(\Gamma) = ((E_i \cdot E_j)) = s_{ij}$. Then, with the above assumptions, $\pi_1(M)$ is isomorphic with the free group generated by the vertices $e_1, \dots, e_n$ of $\Gamma'$ modulo the relations

(a) $$e_i e_j^{s_{ij}} = e_j^{s_{ij}} e_i$$

(b) $$1 = \prod_{1 \leq j \leq n} e_j^{s_{ij}} ,$$

the product in (b) being ordered from left to right by increasing $j$. Recall that the exponents $s_{ij}$ are all 1 or 0 (for $i \neq j$).

Remark. – Each weighted tree with a numbering of its vertices defines by this recipe a group. A change of the numbering gives an isomorphic group. This is not difficult to prove. Thus it makes sense to speak (up to an isomorphism) of $\pi_1(\Gamma')$ where $\Gamma'$ is any weighted tree.

We sketch a proof of the theorem. The boundary of $A_i$, denoted by $\partial A_i$, is a circle bundle over $S^2$ with Euler number $s_{ii}$. A generator $e_i$ of $\pi_1(\partial A_i)$ is represented by a fibre. The only relation is

$$e_i^{s_{ii}} = 1 .$$

Recall $M = \partial A$ and put $B_i = \partial A \cap A_i$ which is a 3-dimensional manifold obtained from $\partial A_i$ by removing for each $j$ with $j \neq i$ and $s_{ij} \neq 0$ a fibre preserving neighbourhood of some fibre This neighbourhood to be removed has in local coordinates (Section 1, (iii)) the description $(|z_1| < \varepsilon , \ |z_2| = \varepsilon)$ and thus is of the type $D^2 \times S^1$. The boundary of $B_i$ consists of a certain number of 2-dimensional tori (one for each $j$ with $j \neq i$ and $s_{ij} \neq 0$). The fundamental group $\pi_1(B_i)$ has generators $e_j$ $(j = i$ or $s_{ij} \neq 0)$ with the only relations

(a) $$e_i e_j = e_j e_i$$

(b) $$e_i^{-s_{ii}} = \prod e_j ,$$

the product is in increasing order of $j$ (over those $e_j$ with $j \neq i$ and $s_{ij} \neq 0$). Here $e_i$ is representable by any fibre, thus also by a fibre on the

$j^{th}$ torus. $e_j$ is represented on the $j^{th}$ torus by ( $z_1 = \varepsilon^{2\pi it}$ , $z_2 =$ constant of absolute value $1$ ). It becomes a fibre in $B_j$ . Since $M = \bigcup B_i$ , we can use van Kampen's theorem to present $\pi_1(M)$ as the free product of the $\pi_1(B_i)$ modulo amalgamation of certain subgroups $\pi_1(S^1 \times S^1)$ . This gives the theorem. Our notation takes automatically care of the amalgamation because for $s_{ij} \neq 0$ and $i \neq j$ the symbols $e_i$ , $e_j$ denote elements of $\pi_1(B_i)$ and of $\pi_1(B_j)$ . Of course, there is all the trouble with the base point which we have neglected in this sketch. The trouble is not serious, mainly because $\Gamma'$ is a tree. A further remark to visualize the relations : $B_i$ , as a circle bundle over $S^2$ – (disjoint union of small disks), is trivial. Thus $e_i$ lies in the center of $\pi_1(B_i)$ . There is a section of $\partial A_i$ over the oriented $S^2$ with one singular point. This gives an "oriented disk-like 2-chain" in $\partial A_i$ with $e_i^{-s_{ii}}$ as boundary (characteristic class = negative transgression!). The small disks lift to disks in that 2-chain. They have to be removed and have the $e_j$ $(j \neq i$ , $s_{ij} \neq 0)$ as boundary. Knowledge of the fundamental group of a disk with small disks removed gives (b).

COROLLARY. – The determinant of the matrix $(s_{ij})$ is different from $0$ if and only if $H_1(M ; \mathbb{Z})$ is finite. If this is so, then $|\det (s_{ij})|$ equals the order of $H_1(M ; \mathbb{Z})$ .

Proof. – Recall that $H_1(M ; \mathbb{Z})$ is the abelianized $\pi_1(M)$ . The corollary follows from relation (b) of the theorem. The result can also be obtained directly from the exact homology sequence of the pair $(A , M)$ which identifies $H_1(M ; \mathbb{Z})$ with the cokernel of the homomorphism $V \to V^*$ defined by the quadratic form $S$ (for the notation see Section $1$ ). $H_2(A ; \mathbb{Z})$ may be identified with $V$ and $H_2(A , M ; \mathbb{Z})$ by Poincaré duality with $V^* = \mathrm{Hom}(V , \mathbb{Z})$ .

## 3. Elementary trees.

In this section we shall prove a purely algebraic result.

A weighted tree is a finite tree with an integer associated to each vertex.

An elementary transformation (of the first kind) of a weighted tree adds a new vertex $x$ , joins it to an old vertex $y$ by a new edge, gives $x$ the weight $-1$ and $y$ the old weight diminished by $1$ . Everything else remains unchanged.

An elementary transformation (of the second kind) adds a new vertex $x$ , joins it to the two vertices $y_1$ , $y_2$ of an edge $k$ by edges $k_1$ , $k_2$ , removes $k$ , gives x the weight $-1$ and $y_i$ $(i = 1, 2)$ the old weight of $y_i$ diminished by $1$. The following proposition is easy to prove.

PROPOSITION. – If $\Gamma'$ is a weighted tree and $\Gamma''$ obtainable from $\Gamma$ by an elementary transformation, then $S(\Gamma'')$ is negative definite if and only if $S(\Gamma')$ is. Furthermore $\pi_1(\Gamma') \sim \pi_1(\Gamma'')$ (for the notation see Section 1 and the Remark in Section 2).

An elementary tree is a weighted tree obtainable from the one-vertex-tree with weight $-1$ by a finite number of elementary transformations.

THEOREM. – Let $\Gamma'$ be a weighted tree. Suppose that $\pi_1(\Gamma')$ is trivial and that the matrix (integral quadratic form) $S(\Gamma')$ is negativ definite. Then $\Gamma'$ is an elementary tree.

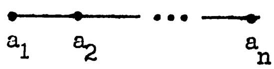
For the proof a group theoretical lemma is essential whose proof we omit.

LEMMA. – Let $G_1$, $G_2$, $G_3$ be non-trivial groups, and $a_i \in G_i$. Then the free product $G_1 * G_2 * G_3$ modulo the relation $a_1 a_2 a_3 = 1$ is a non-trivial group.

Inductive proof of the theorem. Suppose it is proved if the number of vertices in the weighted tree is less than $n$. Let $\Gamma'$ have $n$ vertices $e_1, \dots, e_n$.

First case. – There is no vertex in $\Gamma'$ which is joined by edges with at least three vertices.

Then $\Gamma'$ is linear




where $a_i$ is the associated weight. It follows that one of the $a_i$ must be $-1$, if not $\det S(\Gamma')$ would be up to sign the numerator of the continued fraction

$$|a_1| - \cfrac{1}{|a_2| - \ddots - \cfrac{1}{|a_n|}} \qquad (a_i \leqslant -2)$$

which is not $1$. This contradicts the corollary in Section 2. Thus $\Gamma'$ is an elementary transform of a tree $\Gamma''$ with $n-1$ vertices. By the proposition and the induction assumption $\Gamma'$ is elementary.

Second case. – There is a vertex $e_1$, say, joined with $e_2, \dots, e_m$ $(m \geqslant 4)$.

We may choose this notation since the numbering plays no rôle for the fundamental group (see the Remark in Section 2).

Take $\Gamma'$ remove $e_1$ and the edges joining it to $e_2, \dots, e_m$. The remaining one-dimensional complex is a union of $m-1$ trees $T_2, \dots, T_m$ where $T_i$ has $e_i$ as edge. The free product of the $\pi_1(T_i)$, $i = 2, \dots, m$, modulo the relation $e_2 e_3 \dots e_m = 1$ gives obviously (see Section 2) the group $\pi_1(\Gamma')$ modulo $e_1 = 1$. By assumption $\pi_1(\Gamma')$ is trivial. By the lemma at least one of the groups $\pi_1(T_i)$, say $\pi_1(T_2)$, is trivial. By induction assumption $T_2$ is elementary and thus can be reduced by removing a vertex $x$ with weight $-1$ to give a weighted tree $T_2'$ of which $T_2$ is an elementary transform of first or second kind. If $x \neq e_2$ or if $x = e_2$ and joined only with one vertex in $T_2$, then $\Gamma'$ is elementary transform of the tree consisting of the $T_i$, $(i = 3, \dots, m)$, $T_2'$, and $e_1$ (with the weight unchanged or increased by 1 respectively). By induction and the proposition, $\Gamma'$ would be elementary. In the remaining case $x = e_2$ and $e_2$ is joined with exactly three vertices in $\Gamma'$, namely $e_1$ and, say, $e_{m+1}$, $e_{m+2}$ of $T_2$. Again, either $\Gamma'$ would be elementary transform of a smaller tree, or the weight of $e_1$ or $e_{m+1}$ or $e_{m+2}$ would be $-1$. But the latter case cannot occur, since the quadratic form takes on $e_r + e_s \in V$ (see Section 1) the value $0$, if $e_r$, $e_s$ have weight $-1$ and are joined by an edge, and this would be true for $r = 2$ and $s = 1$, $m+1$ or $m+2$ and contradict the negative definiteness of $S(\Gamma')$.

## 4. A blowing-down theorem.

THEOREM. - Let $X$ be a complex manifold of complex dimension $2$ and $\Gamma = \{E_1, E_2, \dots, E_n\}$ a regular graph of curves on $X$. Suppose the boundary of some tubular neighbourhood of $\Gamma$ be simply-connected and the matrix $S(\Gamma')$ negative-definite. Then the topological space $X/E$ (i. e. $X$ with $E = \bigcup_{i=1}^{n} E_i$ collapsed to a point) is a complex manifold in a natural way : The projection $X \to X/E$ is holomorphic and the bijection $X - E \to X/E - E/E$ is biholomorphic.

Proof. - By the lemma in Section 1 and the theorem in Section 3 all curves $E_i$ are 2-spheres and $\Gamma'$ is an elementary tree. If $\Gamma'$ has only one vertex, then the above theorem is due to GRAUERT or, in the classical algebraic geometric case, to CASTELNUOVO-ENRIQUES. By the very definition of an elementary tree and easy properties of "quadratic transformations" the result follows.

## 5. Resolution of singularities.

Let $Y$ be a complex space of complex dimension $2$ in which all points are non-singular except possibly the point $y_0$ which is supposed to be normal. The theorem on desingularization states that there exist a complex manifold $X$ , a regular (see Section 1) graph $\Gamma$ of curves $E_1, \dots, E_n$ on $X$ , a holomorphic map $\pi : X \to Y$ with

$$\pi(E) = \{y_0\}\ ,\ \text{where}\ E = \bigcup_{i=1}^{n} E_i\ ,$$

$$\pi | X - E : X - E \to Y - \{y_0\}\ \text{biholomorphic}\ .$$

Thus the topological investigation of $A$ and $M$ (Section 1) which we have carried through so far contains as special case the investigation of singularities. A theorem, which we do not prove here, states that $S(\Gamma)$ is negative-definite if $\Gamma$ comes from desingularizing a singularity.

## 6. The Main theorem of Mumford.

THEOREM. - Let $Y$ , $y_0$ be as in Section 5. Suppose that $y_0$ has in $Y$ a neighbourhood $U$ homeomorphic to $R^4$ by local coordinates $t_1, \dots, t_4$ . Then $y_0$ is non-singular.

"Desingularize" $y_0$ as in Section 5. Take a tubular neighbourhood $A$ of $\Gamma$ . We can find a positive number $\delta$ such that $K = \pi^{-1}\{p \mid p \in U \wedge \sum t_i^2(p) < \delta\} \subset A$. There exists a tubular $A'$ with

$$A' \subset K \subset A$$

and such that $A'$ is obtained from $A$ just by multiplying the "normal distances" by a fixed positive number $r < 1$ . Any path in $A - E$ is homotopic to a path in $A' - E$ which is nullhomotopic in $A - E$ because $\pi_1(K - E) = \pi_1(R^4 - \{0\})$ is trivial. The theorem in Section 4 together with the theorem mentioned at the end of Section 5 completes the proof.

## 7. Further remarks.

For any weighted tree $\Gamma'$ the construction in Section 1 can be topologized (assume genus $(E_i) = 0$ ). In this way we may attach to each weighted tree $\Gamma'$ a 3-dimensional manifold $M(\Gamma')$ (see von RANDOW [5]) which, as can be shown, depends only on $\Gamma'$ (up to a homeomorphism).

We have $\pi_1(M(\Gamma')) = \pi_1(\Gamma')$ (See Section 2). Von RANDOW [5] has investigated the tree manifold $M(\Gamma')$ and shown in analogy to Mumford's theorem (Section 6) that $M(\Gamma')$ is homeomorphic $S^3$ if $\pi_1(\Gamma')$ is trivial. Thus there is no counter-example to Poincaré's conjecture in the class of tree manifolds $M(\Gamma')$ . Von Randow's investigations and also the topological part of Mumford's paper are in close connection to the classical paper of SEIFERT [6]. The oriented Seifert manifolds (fibred in circles over $S^2$ with a finite number of exceptional fibres) are special tree manifolds [5].

Interesting trees (always with genus $(E_i) = 0$ ) occur when desingularizing the singularities

$$(z_1^2 + z_2^n)^{1/2}, \quad (n \geq 2), \qquad (z_1(z_2^2 + z_1^n))^{1/2}, \quad (n \geq 2),$$
$$(z_1^3 + z_2^4)^{1/2}, \qquad (z_1(z_1^2 + z_2^3))^{1/2}, \qquad (z_1^3 + z_2^5)^{1/2}.$$

Each of these algebroid function elements generates a complex space with a singular point at the origin.

These singularities give rise to the well known trees $A_{n-1}$, $D_{n+2}$, $E_6$, $E_7$, $E_8$ of Lie group theory (all vertices weighted by $-2$ ). The corresponding manifolds $M$ are homeomorphic to $S^3/G$ where $G$ is a finite subgroup of $S^3$ (cyclic, binary dihedral, binary tetrahedral, binary octahedral, binary pentagondodecahedral). Up to inner automorphisms these are the only finite subgroups of $S^3$. The manifold $M(E_8)$ is specially interesting. Since $\det S(E_8) = 1$, it is by the corollary in Section 2 a Poincaré manifold, i. e. a 3-dimensional manifold with non-trivial fundamental group and trivial abelianized fundamental group. $M(E_8)$ was constructed by "plumbing" 8-copies of the circle bundle over $S^2$ with Euler number $-2$. By replacing this basic constituent by the tangent bundle of $S^{2k}$ one obtains a manifold $M^{4k-1}(E_8)$ of dimension $4k-1$. This carries a natural differentiable structure. For $k \geq 2$ it is homeomorphic to $S^{4k-1}$, but not diffeomorphic (Milnor sphere).

The above mentioned singularities are classical (e. g. DU VAL [1]). For the preceding remarks see also [3].

For quadratic transformations, desingularization, etc. see the papers of ZARISKI and also [2]. We have only been able to sketch some aspects of Mumford's paper, leaving others aside, e. g. the local Picard variety, etc.

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