Séminaire Brelot-Choquet-Deny.
Théorie du potentiel

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Quasi topology and fine topology

Séminaire Brelot-Choquet-Deny. Théorie du potentiel, tome 10, n° 2 (1965-1966), exp. n° 12, p. 1-14
<http://www.numdam.org/item?id=SBCD_1965-1966__10_2_A6_0>

QUASI TOPOLOGY AND FINE TOPOLOGY

by Bent FUGLEDE

Introduction. - When a subadditive capacity is given on a space, one may study quasi topological notions such as quasi closed sets, quasi continuous functions, etc. (cf. [8]). The results obtained are analogous to those for the fine topology in classical cases. Under suitable assumptions the quasi topology is shown to be equivalent to the fine topology, as is well known in classical potential theory. In case of the usual capacity with respect to a kernel, we establish this key result under the principal hypothesis of a dilated domination principle which is fulfilled, e. g. by the kernels of order α for all α . Some of the results of the present report were announced in [9].

1. Quasi topology. - Let X denote a Hausdorff space, and c a capacity on X in the sense of CHOQUET [3], defined for all compact subsets of X and with values in $[0 , + \infty)$. Let c^* be the associated outer capacity, defined for arbitrary subsets of X . We assume that $c(\emptyset) = 0$ that c is subadditive :

$$c(K_1 \cup K_2) \leqslant c(K_1) + c(K_2) .$$

It follows that c^* is countably subadditive on arbitrary sets :

$$c^*(\cup A_n) \leqslant \sum c^*(A_n) .$$

A capacity c will be called true if, for any increasing sequence of sets,

$$c^*(\cup A_n) = \sup c^*(A_n) .$$

A mapping $f : X \longrightarrow Y$ of X into a topological space Y with a countable base is called quasi continuous if there corresponds to any number $\varepsilon > 0$ an open set $\omega \subset X$ with $c^*(\omega) < \varepsilon$ such that the restriction of f to $X \setminus \omega$ is continuous. Quasi closed, quasi open, and quasi compact subsets of X are defined similarly (cf. [8]).

For two sets A_1 , A_2 , we say that A_2 quasi contains A_1 , or that A_1 is quasi contained in A_2 , if $c^*(A_1 \cap \complement A_2) = 0$. Equivalent sets are sets which quasi contain one another. A property valid except in some set equivalent to $\emptyset$ is said to hold quasi everywhere (q. e.).

LEMMA 1. - A quasi closed set A_1 is quasi contained in a quasi closed set A_2 if, and only if, $c^*(A_1 \cap \omega) \leqslant c^*(A_2 \cap \omega)$ for every open set ω.

Proof. - The necessity is obvious. As to the sufficiency, the stated inequality immediately extends to quasi open sets ω, and we merely have to take $\omega = \complement A_2$.

The intersection of any countable family of quasi closed sets is quasi closed. Although the restriction to countable families is indispensable, we have the following main result :

THEOREM 1. - Suppose that X has a countable base of open sets, and that the subadditive capacity c is a true capacity. Then every non-void set $\mathcal{H}$ of quasi closed sets which is stable under countable intersection has a quasi minimal element, i. e. a set $H_0 \in \mathcal{H}$ quasi contained in any other set $H \in \mathcal{H}$.

Another way of formulating the conclusion of this theorem is that any non-void family $\mathcal{H}$ of quasi closed sets contains a countable subfamily whose intersection is quasi contained in each of the sets of $\mathcal{H}$.

To prove the theorem, one applies the method in Choquet's proof [6] of a theorem of Getoor [10]. Moreover, lemma 1 is used under observation of the fact that, since c is true, it suffices, in verifying the inequality of that lemma, to consider sets ω from a countable base for X so chosen that it is stable under finite union, and consequently that any open set ω is the union of an increasing sequence of open sets from the base.

It follows, in particular, from theorem 1 that every set $A \subset X$ has an equivalence class of quasi closures, i. e. quasi minimal elements in the set of all quasi closed sets quasi containing A. There are, of course, dual results concerning quasi open sets.

It follows also from theorem 1 that every outer measure μ^* (or outer, subadditive capacity) on X which does not charge the sets equivalent to $\emptyset$ has an equivalence class of quasi closed supports, i. e. quasi minimal quasi closed sets carrying μ^*.

LEMMA 2. - For any decreasing sequence of quasi closed sets H_n of finite capacity, one has $\inf c^*(H_n) = c^*(\cap H_n)$.

This holds for any subadditive capacity on any Hausdorff space. The proof is elementary, based on the special case where the sets H_n are compact.

2. A key property of the fine topologies of potential theory ([1]).

In this section, we consider a set X on which a topology called the "fine" topology is given. All topological notions referring to this topology will be qualified by the term "fine(ly)".

For any set $A \subset X$, we denote by $\bar{A}$ the fine closure of A, by $i(A)$ the set of finely isolated points of A, and by

$$b(A) = \bar{A} \cap \complement\, i(A)$$

the finely derived set. Following BRELOT, we call $b(A)$ the base of A ([2]). Any set B of the form $B = b(A)$ for some $A \subset X$ is called a base.

We denote by $\mathcal{E}$ the class of all finely isolated sets $E \subset X$, i. e. sets E such that $i(E) = E$. The fine topology of classical potential theory has the property forming the hypothesis of the following theorem.

THEOREM 2. - Suppose that every finely isolated set E is finely closed. Then we have for any sets A, A_1, $A_2 \subset X$:

(a) $(A_1 \cap \complement A_2 \in \mathcal{E}) \Longrightarrow (b(A_1) \subset b(A_2))$. The converse implication holds if A_1 and A_2 are finely closed.

(b) $A \cap \complement\, b(A) = i(A) = i(\bar{A}) \in \mathcal{E}$.

(c) $b(b(A)) = b(A) = b(\bar{A})$. Thus any base $B = b(A)$ is finely perfect, $B = b(B)$.

The proof is elementary and based on the fact that, since every point forms a finely closed set, we have $i(\bar{A}) = i(A)$ for any set $A \subset X$.

3. Quasi continuity implies fine continuity q. e. (quasi everywhere).

Let X denote a Hausdorff space and c a subadditive capacity on X with $c(\emptyset) = 0$. In addition to the given topology, we consider another topology on X, finer than the given one. We shall refer to this second topology as the "fine" topology, and we denote by $\bar{A}$ the fine closure of a set $A \subset X$.

([1]) For a similar study covering also the probabilistic cases, see DOOB [7].

([2]) In Brelot's definition, the so-called non-polar points of A are included in $b(A)$. There would be no difficulty in modifying the subsequent definitions and results accordingly. We have not adopted this convention here because, in the applications we have in mind, all points are polar.

THEOREM 3. - Suppose just that $c^*(\tilde{A}) = c^*(A)$ for every set $A \subset X$. Then any quasi closed (quasi open) set is equivalent to its fine closure (fine interior). If Y is a topological space with a countable base, then every quasi continuous function is finely continuous q. e.

Proof. - If A is quasi closed, there corresponds (cf. [8]) to $\varepsilon > 0$ a closed set $F \subset A$ such that $c^*(A \setminus F) < \varepsilon$. Writing $A \setminus F = B$, we have

$$\tilde{A} \subset (F \cup B)^{\sim} \subset \tilde{F} \cup \tilde{B} = F \cup \tilde{B} \subset A \cup \tilde{B} ,$$

$$c^*(\tilde{A} \setminus A) \leqslant c^*(\tilde{B}) = c^*(B) < \varepsilon .$$

If $f : X \longrightarrow Y$ is quasi continuous, and if (Ω_n) is a countable base of open subsets of Y, we write

$$A_n = f^{-1}(\Omega_n) , \qquad B_n = \text{fine interior } A_n .$$

Then A_n is quasi open, and hence $\bigcup_n (A_n \setminus B_n) = E$ has $c^*(E) = 0$; and f is continuous in the fine topology on X at any point of $X \setminus E$.

4. Thinness and fine topology with respect to a cone.

Let X be a Hausdorff space with a countable base, and let c denote a subadditive, true capacity on X. Further let $\mathcal{U}$ denote a convex cone of lower semicontinuous functions on X with values in $[0, +\infty]$.

Following BRELOT [1], we associate with any set $A \subset X$ the reduced function R_A defined by

$$R_A(x) = \inf\{u(x) \mid u \in \mathcal{U}, \ u \geqslant 1 \text{ on } A\}$$

(interpreted as $+\infty$ if no such u exists).

A is called thin of order α $(0 < \alpha \leqslant 1)$ at $x \in X$, if there is a function $u \in \mathcal{U}$ such that

$$u(x) < \alpha \liminf u(y) \quad \text{as} \quad y \longrightarrow x, \qquad y \in A \cap \complement\{x\} ,$$

or equivalently if there is a neighbourhood ω of x such that

$$R_{A \cap \omega \cap \complement\{x\}}(x) < \alpha .$$

We say that A is thin at x if A is thin of order 1 at x, and that A is strongly thin at x if A is thin of every order $\alpha > 0$ at x. For any set $A \subset X$, we denote by $e(A)$ the set of points of X at which A is thin.

It was discovered by CARTAN that the complements of the sets thin at x and not containing x form the neighbourhoods of x in a topology on X called the fine topology (with respect to $\mathcal{U}$), and this topology is the coarsest among all topologies on X finer than the given topology and such that the functions of class $\mathcal{U}$ are all continuous.

In the sequel, the qualification "fine(ly)" always refers to the fine topology with respect to $\mathcal{U}$. The finely isolated sets, forming the class $\mathcal{E}$, considered in § 2, are the sets E thin at each of their points, $E \subset e(E)$. The base of a set $A \subset X$ is $b(A) = \complement e(A)$.

We proceed to study the consequences of the following axioms [3] :

(I) $c^*(\tilde{A}) = c^*(A)$ for every set $A \subset X$.

(II) $(e(E) \supset E) \implies (c^*(E) = 0)$.

(III) $(c^*(E) = 0) \implies (e(E) = X)$.

(IV) Every function $u \in \mathcal{U}$ is quasi continuous.

Note that axiom (I) forms the hypothesis of theorem 3 above, and that axioms (II) and (III) together imply the hypothesis of theorem 2 because any set E thin at all points of $\complement E$ is finely closed. Consequently, we obtain the following lemma :

LEMMA 3. - Suppose the axioms (II) and (III). The sets E of class $\mathcal{E}$ (the finely isolated sets) are characterized by each of the following equivalent properties :

$$(e(E) \supset E) \iff (e(E) = X) \iff (c^*(E) = 0) .$$

For any set A , the fine closure $\tilde{A}$ is equivalent to the base $b(A)$.

THEOREM 4. - Suppose axioms (II), (III), and (IV). Then [4] :

(a) For any set $A \subset X$ and any number $\varepsilon > 0$, there is an open set $\omega \subset X$ such that $e(A) \subset \omega$ and $c^*(A \cap \omega) < \varepsilon$.

(b) Any finely closed (finely open) set $A \subset X$ is quasi closed (quasi open).

(c) If Y is a topological space with a countable base, then any finely continuous function $f : X \longrightarrow Y$ is quasi continuous.

Proof. - Suppose first axioms (II), (III), and (IV). We begin by proving that, for any set $A \subset X$, R_A is quasi upper semi-continuous, i. e. the set

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(3) The present axioms are, as a whole, somewhat weaker than the axioms in [9], partly because we do not assume anything about the polar sets.

(4) Property (a), which implies (b) and (c), was discovered by CHOQUET [5] in the classical cases. This property also implies the axioms (II) and (IV).

$$B = \{x \in X \mid R_A(x) \geqslant t\}$$

is quasi closed for every real t. Let $\mathcal{H}$ denote the class of all sets

$$E_u = \{x \in X \mid u(x) \geqslant t\} ,$$

where $u \in \mathcal{U}$ and $u \geqslant 1$ on A. According to the axiom (IV) the sets of class $\mathcal{H}$ are all quasi closed, and it follows therefore from theorem 1 that there exists a sequence of sets $H_n \in \mathcal{H}$ whose intersection H is quasi contained in every set from $\mathcal{H}$. In view of theorem 2 (a),

$$b(H) \subset b(E_u) \subset E_u ,$$

and hence

$$b(H) \subset \bigcap_u E_u = B \subset H .$$

Since $b(H)$ is equivalent to H (lemma 3), so is B, and consequently B is quasi closed.

In the rest of the proof, we do not use the axioms (directly), execpt when stated. The quasi upper semi-continuity of R_A for all sets A is equivalent to (b). In fact, for any finely closed set A, we have $A \supset b(A)$, i. e. $\complement A \subset e(A)$, and hence

$$\complement A = (\complement A) \cap e(A) = \bigcup_n (\omega_n \cap \{x \in X \mid R_{A \cap \omega_n}(x) < 1\}) .$$

Being thus a countable union of quasi open sets, $\complement A$ is quasi open, i. e. A is quasi closed. Conversely, (b) implies that any finely upper semi-continuous function $f : X \longrightarrow [0 , +\infty]$ (in particular $f = R_A$ for any set A) is quasi upper semi-continuous because the sets $\{x \in X \mid f(x) \geqslant t\}$ are finely closed and hence quasi closed.

Next, we observe that the Choquet property (a) is equivalent to the conjunction of (b) and the axiom (II). In fact, (b) amounts to stating that, for every set $A \subset X$, $\tilde{A}$ is quasi closed, i. e. to every $\varepsilon > 0$ there corresponds a closed set $F \subset \tilde{A}$ with $c^*(\tilde{A} \setminus F) < \varepsilon$, or in other words an open set ω_1 $(= \complement F) \supset \complement \tilde{A}$ with $c^*(\tilde{A} \cap \omega_1) < \varepsilon$. And the axiom (II) amounts to stating that, for every set $A \subset X$,

$$c^*(A \cap e(A)) = c^*(i(A)) = 0 ,$$

i. e. to every $\varepsilon > 0$ corresponds an open set $\omega_2 \supset i(A)$ with $c^*(\omega_2) < \varepsilon$. Consequently, (b) and the axiom (II) are together equivalent to the existence, for every set A and every $\varepsilon > 0$, of an open set ω $(= \omega_1 \cup \omega_2)$ with

$$\omega \supset (\complement \tilde{A}) \cup i(A) = e(A)$$

such that

$$c^*(A \cap \omega) \leqslant c^*(A \cap \omega_1) + c^*(\omega_2) \leqslant c^*(\tilde{A} \cap \omega_1) + c^*(\omega_2) < 2\varepsilon .$$

Finally, we show that (b) $\Longrightarrow$ (c) $\Longrightarrow$ axiom (IV). Suppose (b), and let $f : F \longrightarrow Y$ be finely continuous. If (Ω_n) denotes a countable base of open subsets of Y, the sets $f^{-1}(\Omega_n)$ are finely open, hence quasi open, and it follows easily that f is quasi continuous. Clearly, (c) implies axiom (IV) since the functions of class $\mathcal{U}$ are finely continuous.

Scholium. - Suppose all four axioms. Then the quasi topology is equivalent to the fine topology in the following sense : Quasi continuity is the same as fine continuity quasi everywhere. A set A is quasi closed if, and only if, it is equivalent to a finely closed set, e. g. $\tilde{A}$ or $b(A)$. A quasi closed set A_1 is quasi contained in a quasi closed set A_2 if, and only if, $b(A_1) \subset b(A_2)$. Every equivalence class of quasi closed sets contains precisely one base, the base of all sets of the class.

5. A countability property of the fine topology.

Keeping the notations and assumptions stated at the beginning of the preceding section, we infer from theorem 1 in view of the above scholium that every family (H_α) of bases, $b(H_\alpha) = H_\alpha$, contains a countable subfamily (H_{α_n}) such that $b(\bigcap_n H_{\alpha_n}) \subset b(H_\alpha)$ for every α, i. e.

$$b(\bigcap_\alpha H_\alpha) = b(\bigcap_n H_{\alpha_n}) .$$

Clearly this base is the greatest minorant for the given family (H_α) within the set of all bases, ordered under inclusion. Thus, we obtain the following result (cf. DOOB [7]) which serves as a substitute for the absence of a countable base of finely open sets.

THEOREM 5. - Suppose all four axioms. Then the set $\mathcal{B}$ of all bases in X is a lower semi-complete lattice under inclusion. For any subset $\mathcal{A} \subset \mathcal{B}$, the greatest minorant is the base of the intersection of the sets of $\mathcal{A}$:

$$\inf \mathcal{A} = b(\bigcap \mathcal{A}) ,$$

and there is always a countable subset with the same greatest minorant.

COROLLARY. - Any outer measure μ^* (or outer, subadditive capacity) on X which does not charge the sets E with $c^*(E) = 0$, has a smallest finely closed support, and it is a base.

In classical cases, this is Getoor's theorem [10] (cf. also CHOQUET [6]).

THEOREM 6. - Suppose all four axioms. For any decreasing directed family of sets A_α contained in a quasi compact set,

$$\inf_\alpha c^*(A_\alpha) = c^*(\bigcap_\alpha \tilde{A}_\alpha) = c^*(\bigcap_\alpha b(A_\alpha)) \quad .$$

This result is due to BRELOT [2] in classical cases and in his axiomatic theory of superharmonic functions.

Proof. - For any $A \subset X$, we have

$$c^*(A) = c^*(\tilde{A}) = c^*(b(A))$$

in view of axiom (I) and lemma 3. It remains therefore only to be proved that

$$c^*(\bigcap_\alpha b(A_\alpha)) \geqslant \inf_\alpha c^*(A_\alpha) \quad ,$$

and for this, we may assume that the sets A_α are bases, $A_\alpha \in \mathcal{B}$. According to theorem 5 there is a sequence (α_n) such that

$$b(\bigcap_\alpha A_\alpha) = b(\bigcap_n A_{\alpha_n}) \quad .$$

We may suppose the sequence (A_{α_n}) is decreasing. Note that

$$c^*(\bigcap_\alpha A_\alpha) = c^*(b(\bigcap_\alpha A_\alpha)) = c^*(b(\bigcap_n A_{\alpha_n})) = c^*(\bigcap_n A_{\alpha_n}) \quad .$$

Since the sets A_{α_n} are finely closed, they are quasi closed. By assumption they are contained in a quasi compact set, and they are therefore themselves quasi compact. Since c does not charge the points, and hence not either the finite sets, every compact set has finite capacity. Consequently it follows from lemma 2 that

$$c^*(\bigcap_n A_{\alpha_n}) = \inf_n c^*(A_{\alpha_n}) \geqslant \inf_\alpha c^*(A_\alpha) \quad ,$$

which completes the proof.

6. On the verification of the axioms.

The axiom (I) is obviously fulfilled if $c^*(A)$ depends only on

$$\{u \in \mathfrak{U} \mid u \geqslant 1 \text{ in } A\} \quad ;$$

in fact, any $u \in \mathfrak{U}$ is finely continuous. Usually, in applications to potential theory, the sets E with $c^*(E) = 0$ are precisely the polar sets, i. e. subsets of sets of the form $u^{-1}(\{+\infty\})$, $u \in \mathcal{U}$. The axiom (III) is then fulfilled if, for every polar set E and every point $x \in \complement E$, there exists $u \in \mathcal{U}$ such that $u = +\infty$ in E whereas $u(x) < +\infty$ (cf. [9]).

The most difficult axiom to verify in applications is (II). For any function $f : X \longrightarrow [0, +\infty]$, denote by $\hat{f}$ the greatest lower semi-continuous function majorized by f.

LEMMA 4. - Suppose there is a constant α, $0 < \alpha \leqslant 1$, such that, for every sequence of functions $u_n \in \mathcal{U}$,

$$\widehat{\inf_n u_n} \geqslant \alpha \inf_n u_n \quad \text{q. e.} .$$

Suppose further that c does not charge the points of X. Then any set E, thin of order α at each of its points, has $c^*(E) = 0$.

In the applications to potential theory the thinness is always strong, i. e. thinness (of order 1) implies thinness of any order $\alpha > 0$. In that case, the lemma gives a sufficient condition for the validity of axiom (II). In the classical cases $\widehat{\inf u_n}$ is itself a function from $\mathcal{U}$ and equal q. e. to $\inf u_n$. This follows essentially from the lower envelope principle. The above lemma indicates that it suffices to assume a certain dilated lower envelope principle (cf. § 7).

Proof of lemma 4. - Let (ω_n) denote a countable base of open subsets of X, and let E be a set thin of order α at each of its points. Writing

$$E_{n,p} = \{x \in E \cap \omega_n \mid R_{E\cap\omega_n \setminus \{x\}}(x) < \alpha - \frac{1}{p}\} ,$$

we have $E = \bigcup E_{n,p}$. It suffices therefore to verify that $c^*(E_{n,p}) = 0$ for each n and p. Let S denote a countable, dense subset of such a set $E_{n,p}$. For each $s \in S$, there is $u_s \in \mathcal{U}$ such that $u_s(s) < \alpha - 1/p$, and $u_s \geqslant 1$ in $E \cap \omega_n \setminus \{s\}$, in particular in $E_{n,p} \setminus S$. Writing

$$h = \inf_{s\in S} u_s ,$$

we thus have $h < \alpha - 1/p$ in S, and $h \geqslant 1$ in $E_{n,p} \setminus S$. By hypothesis, it follows from the latter inequality that $\hat{h} \geqslant \alpha$ q. e. in $E_{n,p} \setminus S$. On the other hand, $\hat{h} \leqslant h < \alpha - 1/p$ in S implies $\hat{h} \leqslant \alpha - 1/p$ in $\overline{S} \supset E_{n,p}$ because $\hat{h}$ is lower semi-continuous. This leads to a contradiction unless $c^*(E_{n,p} \setminus S) = 0$. Since S is countable, $c^*(S) = 0$, and we conclude that $c^*(E) = 0$.

7. <u>The usual capacity with respect to a kernel.</u>

From now on, let X be a locally compact space with a countable base of open sets, and let G denote a kernel on X, i. e. a lower semi-continuous function on $X \times X$ with values in $[0, +\infty]$. We denote by c the usual capacity (cf. [8]) with respect to G, and by $\mathcal{U}$ the class of all potentials $\check{G}\lambda$ of bounded measures $\lambda \geqslant 0$ on X with respect to the adjoint kernel $\check{G}(x, y) = G(y, x)$.

THEOREM 7. - <u>Suppose that</u> :

(i) G <u>and</u> $\check{G}$ <u>satisfy the continuity principle,</u>

(ii) G <u>is finite and continuous off the diagonal, infinite on the diagonal</u> $(x = y)$, <u>and</u>

(iii) $G(x, y) \to 0$ <u>as one of the variables tends to the point at infinity, the convergence being uniform with respect to the other variable on any compact set.</u>

<u>Then</u> :

(a) c <u>is a subadditive, true capacity which does not charge the points.</u>

(b) $c^*(A) = \inf\{\lambda(X) \mid \check{G}\lambda \geqslant 1$ <u>in</u> $A\}$.

(c) $(c^*(A) = 0) \iff (A$ <u>is polar</u>$)$.

(d) $(c^*(A) < \infty) \iff (A$ <u>is contained in a quasi compact set</u>$)$.

(e) <u>The thinness is always strong.</u>

(f) <u>The axioms</u> (I), (III), <u>and</u> (IV) <u>are valid.</u>

<u>Indication of proof.</u> - The assertions (a), (d) and the axiom (IV) were established in [8] (the fact that c does not charge the points being equivalent to the hypothesis $G(x, x) = +\infty$). It was also proved in [8] that (b) holds for compact A, and that, for arbitrary A,

$$c^*(A) = \inf\{\lambda(X) \mid \check{G}\lambda \geqslant 1 \text{ q. e. in } A\} .$$

It follows easily that (b) holds for any K_σ, in particular for any open set, and therefore for arbitrary A. Next, (c) and the axiom (I) are easily derived from (b), and so is the axiom (III) because each transposed potential $\check{G}\lambda$ with $\lambda(X) < +\infty$ is finite and continuous off the support of λ (cf. also § 6). Finally (e) was established by BRELOT [1].

I do not know whether the remaining axiom (II) holds under the present rather weak assumptions on the kernel. It suffices, however, to add the following hypothesis, called the <u>dilated domination principle</u> :

(D_{dil}) There exists a constant k such that, for any two positive measures μ and ν, of which μ has finite energy $\int G\mu \, d\mu$,

$$(\check{G}\mu \leqslant \check{G}\nu \text{ on the support of } \mu) \implies (\check{G}\mu \leqslant k\,\check{G}\nu \text{ in } X) .$$

Since the assertion in axiom (II) is of a local character, and since an exceptional set equivalent to $\emptyset$ is involved, it would actually suffice to assume that quasi every point x of X has a neighbourhood on which (D_{dil}) holds (the constant k being allowed to depend on x). Whether this weaker form of (D_{dil}) holds for all kernels satisfying the hypotheses of theorem 7 does not seem to be known. It might be added that the analogous weaker form of the dilated maximum principle is known to hold under the assumptions of theorem 7 (cf. CHOQUET [4], OHTSUKA [12]), and the same is easily shown to apply to the corresponding weak form of the following dilated principle of positivity of masses :

(P_{dil}) There exists a constant m such that, for any two positive measures μ and ν,

$$(\check{G}\mu \leqslant \check{G}\nu \text{ everywhere}) \implies (\int d\mu \leqslant m \int d\nu)$$

In proving that axiom (II) follows from (D_{dil}) under the hypothesis of theorem 7, we may therefore assume that (P_{dil}) holds.

We begin by deriving from (P_{dil}) and (D_{dil}) the following dilated lower envelope principle :

(L_{dil}) There exists a constant k such that, for any finite set of potentials $u_1, \dots, u_p$ from the class $\mathcal{U}$, there is a potential $u \in \mathcal{U}$ with the following properties :

$$u \geqslant \inf(u_1, \dots, u_p) \quad \text{quasi everywhere} ,$$

$$u \leqslant k \inf(u_1, \dots, u_p) \quad \text{everywhere} .$$

To prove this, write $u_j = \check{G}\lambda_j$ with bounded measures λ_j, $j = 1, \dots, p$, and put $f = \inf(u_1, \dots, u_p)$.

Since f is lower semi-continuous, there is an increasing sequence of continuous functions $f_n \geqslant 0$ of compact supports such that $f = \sup_n f_n$. Applying Kishi's existence theorem [11] (or, if G is symmetric, the Gauss variational principle), we find measures $\mu_n \geqslant 0$ of compact supports such that

$$\check{G}\mu_n \geqslant f_n \quad \text{quasi everywhere } (^5) ,$$
$$\check{G}\mu_n \leqslant f_n \quad \text{on the support of } \mu_n .$$

Since $f_n \leqslant f \leqslant u_j = \check{G}\lambda_j$, we infer from (D_{dil}) that,

$$\check{G}\mu_n \leqslant k\ \check{G}\lambda_j \quad \text{everywhere} .$$

By virtue of (P_{dil}) this shows, in particular, that $\mu_n(X) \leqslant mk\ \lambda_1(X)$, and so the total masses $\mu_n(X)$ remain bounded. Hence, the Brelot-Choquet convergence theorem is applicable (cf. [8]). Let μ be any vague cluster point for the sequence (μ_n) . Then

$$\check{G}\mu \geqslant \liminf_n \check{G}\mu_n \geqslant \lim_n f_n = f \quad \text{quasi everywhere} .$$

On the other hand, for any $j = 1 , \dots , p$,

$$\check{G}\mu \leqslant \limsup_n \check{G}\mu_n \leqslant k\ \check{G}\lambda_j \quad \text{everywhere} ,$$

and consequently $\check{G}\mu \leqslant k\ f$ everywhere.

Having thus established (L_{dil}), we next extend this principle to the case of an infinite sequence (u_n) of potentials $u_n = \check{G}\lambda_n \in \mathcal{U}$, again under application of the convergence theorem. Writing this time

$$f_n = \inf(u_1 , \dots , u_n) , \qquad f = \lim_n f_n = \inf_n u_n ,$$

we obtain from (L_{dil}) a bounded measure $\mu_n \geqslant 0$ such that

$$\check{G}\mu_n \geqslant f_n \quad \text{quasi everywhere} ,$$
$$\check{G}\mu_n \leqslant k\ f_n \quad \text{everywhere} .$$

Since $\check{G}\mu_n \leqslant k\ f_n \leqslant k\ \check{G}\lambda_1$, it follows from (P_{dil}) that the total masses $\mu_n(X)$ remain bounded. Let μ denote a vague cluster point for (μ_n) . Then

$$\check{G}\mu \geqslant \liminf_n \check{G}\mu_n \geqslant \lim_n f_n = f \quad \text{quasi everywhere} ,$$

$$\check{G}\mu \leqslant \limsup_n \check{G}\mu_n \leqslant k \lim_n f_n = kf \quad \text{everywhere} .$$

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(5) In the first instance, we obtain $\check{G}\mu_p \geqslant f_p$ except in some set of inner capacity 0 . Since $\check{G}\mu_p$ is quasi continuous (axiom (IV)), and f_p is continuous, the exceptional set is quasi open, hence capacitable (cf. [8]), and consequently also of outer capacity 0 .

It is now clear that also the principal hypothesis of lemma 4 is fulfilled with $\alpha = 1/k$. In fact, since $u = \check{G}\mu$ is lower semi-continuous, and $\alpha u = k^{-1} u \leqslant f$ everywhere, we have $\alpha u \leqslant \hat{f}$, and consequently

$$\hat{f} \geqslant \alpha u \geqslant \alpha f \quad \text{quasi everywhere} \quad .$$

This completes the proof of the remaining axiom (II) under the hypotheses of theorem 7 and the additional assumption (D_{dil}) (or just the above mentioned weaker form of this principle).

It is well known that the kernels of order α on R^n ,

$$G(x , y) = |x - y|^{\alpha - n} \qquad (0 < \alpha < n) \quad ,$$

satisfy the actual domination principle for $\alpha \leqslant 2$ only, but the dilated domination principle (D_{dil}) for all α . Hence the preceding theory is applicable to these kernels.

In conclusion, let us remark that results similar to those described above for the usual capacity with respect to a kernel G can be obtained for the energy capacity with respect to a symmetric kernel of positive type, taking for $\mathcal{U}$ the set of all potentials $G\lambda$ of measures $\lambda \geqslant 0$ of finite energy. The continuity principle and the hypothesis (iii) of theorem 7 are now replaced by the principle of consistency and the hypothesis that $G(x , y_1)/G(x , y_2)$ remains bounded as x tends to infinity in X while y_1 and y_2 remain in a compact set. In order to secure that any set of outer energy capacity 0 be polar (with respect to the present cone $\mathcal{U}$), it is assumed moreover that X is not thin at any point.

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