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COMPLEMENTED SUBSPACES OF L_p WHICH EMBED INTO $\ell_p \oplus \ell_2$

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In this seminar we report on joint work with Ted Odell [5] concerning the isomorphic classification of complemented subspaces of L_p, $1 < p \neq 2 < \infty$. There are now known to exist uncountably many mutually non-isomorphic complemented subspaces of L_p for each $1 < p \neq 2 < \infty$ [1]. However, there probably are only finitely many which are "small". For example, the only complemented subspace of L_p which embeds into ℓ_p is ℓ_p itself [6]. The question studied in [5] is "what are the complemented subspaces of L_p which embed into $\ell_p \oplus \ell_2$?" For $1 < p < 2$, the following partial answer is given:

__Theorem A__: __If__ X __is a complemented subspace of__ L_q $(1 < q < 2)$ __which has an unconditional basis and__ X __embeds into__ $\ell_q \oplus \ell_2$, __then__ X __is isomorphic to__ ℓ_q, ℓ_2, __or__ $\ell_q \oplus \ell_2$.

It is of course a major unsolved problem whether every complemented subspace of L_p $(1 < p \neq 2 < \infty)$ has an unconditional basis.

Theorem A is an immediate consequence of the result of [6] mentioned above and:

__Proposition B__: __Let__ X __be a subspace of__ L_p $(2 < p < \infty)$ __which has an unconditional basis and which is isomorphic to a quotient of__ $\ell_p \oplus \ell_2$. __Then there is a subspace__ U __of__ ℓ_p __(possibly__ $U = \{0\}$) __so that__ X __is isomorphic to__ U, ℓ_2, __or__ $U \oplus \ell_2$.

The classification of complemented subspaces of L_p which embed into $\ell_p \oplus \ell_2$ is more complicated for $2 < p < \infty$ because of the presence of Rosenthal's space X_p [11]. However, in [5] the following is proved:

Theorem C: If X is a complemented subspace of L_p $(2 < p < \infty)$ which has an unconditional basis and which embeds into $\ell_p \oplus \ell_2$, then X is isomorphic to ℓ_p, ℓ_2, $\ell_p \oplus \ell_2$, or X_p.

Below we give a more-or-less complete proof of Proposition B and outline the proof of Theorem C. Actually, Theorem A is also a consequence of Theorem C and the following result from [5] which will not be discussed in this seminar:

Theorem D: If X is a subspace of L_p $(2 < p < \infty)$ which is isomorphic to a quotient of a subspace of $\ell_p \oplus \ell_2$, then X embeds into $\ell_p \oplus \ell_2$.

Proof of Proposition B: Let (x_n) be a normalized unconditional basis for X and let Q be a norm one operator from $\ell_p \oplus \ell_2$ onto X.

Claim: There exists $\epsilon > 0$ so that for all $0 < \delta < \epsilon$, $\{i: \delta \le \|x_i\|_2 \le \epsilon\}$ is finite. (Here $\|x\|_r = (\int_0^1 |x(t)|^r dt)^{1/r}$ for $1 \le r < \infty$.)

If the claim is false, then there are $\epsilon_1 > \epsilon_2 > \dots > 0$ and infinite sets M_n of integers so that $\epsilon_{n+1} < \|x_i\|_2 \le \epsilon_n$ for $i \in M_n$ and $n = 1,2,\dots$. Since (x_i) is unconditional, it follows from the classical results of Kadec and Pelczynski [7] that $(x_i)_{i \in M_n}$ is equivalent to the unit vector basis for ℓ_2 for each $n = 1,2,\dots,$ hence so is

$(f_i)_{i \in M_n}$, if (f_i) is the sequence of biorthogonal functionals to (x_i). But this means that for each $n = 1,2,\ldots$ the ℓ_q - contribution to the norm of (Q^*f_i) tends to zero as $i \to \infty$ in M_n, because every operator from ℓ_2 into ℓ_q is compact. Consequently, since Q^* is an isomorphism, we can select $i_n \in M_n$ so that $(Q^*f_{i_n})_{n=1}^{\infty}$ is equivalent to the unit vector basis of ℓ_2, hence the same is true of $(x_{i_n})_{n=1}^{\infty}$. But $(x_{i_n})_{n=1}^{\infty}$ has a subsequence equivalent to the unit vector basis of ℓ_p because $\lim_{n\to\infty} \|x_{i_n}\|_2 = 0$. This completes the proof of the claim.

Exercise: Where was unconditionality of (x_n) used in the proof of the claim?

Since for any $\epsilon > 0$, the closed linear span of $\{x_i : \|x_i\|_2 > \epsilon\}$ is either finite dimensional or isomorphic to ℓ_2, we can, in view of the claim, assume that $\|x_n\|_2 \to 0$ and hence [7] that no subsequence of (x_n) is equivalent to the unit vector basis for ℓ_2. We will show that this condition implies that X embeds into ℓ_p.

Let $f_i = g_i \oplus e_i \in \ell_q \oplus \ell_2$ $(1/p + 1/q = 1)$ be a normalized sequence which is equivalent to the biorthogonal functionals to (x_i). In view of Lemma 1 below, we can assume that (g_i) is a monotonely unconditional basic sequence in ℓ_q, and (h_i) is orthogonal in ℓ_2. Since no subsequence of (f_i) is equivalent to the unit vector basis of ℓ_2, there exists $\delta > 0$ and n so that $\|g_i\| \geq \delta$ for all $i \geq n$. Letting P denote the natural projection of $\ell_q \oplus \ell_2$ onto ℓ_q, we complete the proof by observing that P is an isomorphism when restricted to $[(f_i)_{i=n}^{\infty}]$, the closed linear span of $(f_i)_{i=n}^{\infty}$. Indeed, since (g_i) is monotonely unconditional, we have for all scalars (a_i) that $(\Sigma|a_i|^2)^{1/2} \leq K_q\delta^{-1} \|\Sigma a_i g_i\|$ where K_q is

Khintchine's constant for L_q. Hence for any $f = \sum_{i=n}^{\infty} a_i f_i \in [(f_i)_{i=n}^{\infty}]$,

$$\|Pf\| \leq \|f\| = \max\left(\|\Sigma a_i g_i\|,\ \|\Sigma a_i h_i\|\right) \leq \max\left(\|Pf\|,\ \left(\sum_{i=1}^{\infty} |a_i|^2\right)^{1/2}\right) \leq$$

$K_q \delta^{-1} \|Pf\|$. □

In the proof of Proposition B, we used:

Lemma 1: *Let (x_i) be an unconditional basic sequence in $\ell_p \oplus \ell_2$ $(1 < p < \infty)$. Then there is a monotonely unconditional basic sequence (u_i) in ℓ_p and an orthogonal sequence (v_i) in ℓ_2 so that (x_i) is equivalent to $(u_i \oplus v_i)$ in $\ell_p \oplus \ell_2$.*

Proof. The proof uses an idea of Schechtman's [13]. Note that by a perturbation argument we can assume that, if (e_n) denotes the natural basis for $\ell_p \oplus \ell_2$, then for any $n = 1,2,\dots$, only finitely many of the x_i's have a non-zero nth coordinate when x_i is expanded in terms of (e_n). We can represent (e_n) in $L_p[-1,1]$ by having $(e_{2n})_{n=1}^{\infty}$ be a sequence of L_p-normalized indicator functions of disjoint subsets of $[-1,0)$ and letting $(e_{2n-1})_{n=1}^{\infty}$ be the Rademacher functions on $[0,1]$. Write $x_i = y_i + z_i$ with $y_i \in [(e_{2n})_{n=1}^{\infty}]$ and $z_i \in [(e_{2n-1})_{n=1}^{\infty}]$. The sequence (x_i) is easily seen to be equivalent to the sequence $(r_i \otimes y_i + r_i \otimes z_i)$ in $L_p([0,1] \times [-1,1])$, where (r_i) is the usual sequence of Rademacher functions. Of course, $(r_i \otimes z_i)$ is equivalent to an orthogonal sequence; the point is that the terms of the monotonely unconditional sequence $(r_i \otimes y_i)$ are measurable with respect to a purely atomic sub-sigma field of $[0,1] \times [-1,0]$ so that $[(r_i \otimes y_i)]$ embeds isometrically into ℓ_p. □

Throughout the rest of this seminar, we let $2 < p < \infty$ and let (e_n) (respectively, (δ_n)) denote the unit vector basis for ℓ_p (respectively, ℓ_2). Given $z = y \oplus z \in \ell_p \oplus \ell_2$, we let $|x|_p = \|y\|$ and $|x|_2 = \|z\|$. Given a sequence $w = (w_n)$ of non-negative weights, the space $X_{p,w}$ is defined to be the subspace $[e_n \oplus w_n\delta_n]$ of $\ell_p \oplus \ell_2$. We use (b_n) to denote the natural basis $(e_n \oplus w_n\delta_n)$ for a generic $X_{p,w}$ space; if confusion is likely to result, we use $|\cdot|_{2,w}$ to denote the ℓ_2- part of the norm in $X_{p,w}$, so that for $x = \Sigma a_n b_n \in X_{p,w}$, $|x|_{2,w} = (\Sigma\ |a_n w_n|^2)^{1/2}$.

No matter what the weight sequence w is, the space $X_{p,w}$ is isomorphic to ℓ_2, ℓ_p, $\ell_p \oplus \ell_2$ or the space X_p introduced by Rosenthal [11]. Rosenthal showed that $X_{p,w}$ is isomorphic to X_p if and only if for each $\epsilon > 0$,

$$\sum_{w_n < \epsilon} w_n^{2p/(p-2)} = \infty.$$

X_p is isomorphic to a complemented subspace of L_p but is not isomorphic to a complemented subspace of $\ell_p \oplus \ell_2$. It has become clear during the last ten years that, rather than being a pathological example, X_p plays a fundamental role in the study of L_p (cf., e.g. [2], [4], and [12]).

There are three important steps in the proof of Theorem C:

<u>Proposition 2</u>: <u>Let</u> X <u>be a subspace of</u> $\ell_p \oplus \ell_2$ $(2 < p < \infty)$ <u>and let</u> T <u>be an operator from</u> L_p <u>into</u> X. <u>Then</u> T <u>factors through</u> X_p.

<u>Proposition 3</u>: <u>If</u> X <u>is isomorphic to a complemented subspace of</u> X_p <u>and</u> X_p <u>is isomorphic to a complemented subspace of</u> X, <u>then</u> X <u>is isomorphic to</u> X_p.

Proposition 4: Let X be a subspace of $\ell_p \oplus \ell_2$ $(2 < p < \infty)$ with a normalized basis $x_n = y_n \oplus z_n$, where (y_n) (respectively. (z_n)) is a basic sequence in ℓ_p (respectively, ℓ_2). Assume that $|z_n|_2 \to 0$ as $n \to \infty$. Then either X embeds into ℓ_p or X_p is isomorphic to a complemented subspace of X.

Notice that Proposition 2 implies that a complemented subspace of L_p which embeds into $\ell_p \oplus \ell_2$ is isomorphic to a complemented subspace of X_p. Suppose now that X is a complemented subspace of L_p which embeds into $\ell_p \oplus \ell_2$ and X has normalized unconditional basis which in $\ell_p \oplus \ell_2$ can be represented as $x_n = y_n \oplus z_n$, where by Lemma 1 we can assume that (y_n) is unconditional in ℓ_p and (z_n) is orthogonal in ℓ_2. Suppose that

$$(*)\quad \begin{cases} \text{There are } 1 > \epsilon_1 > \epsilon_2 > \dots > 0 \text{ so that for } n = 1,2,\dots, \\ M_n = \{i: \ \epsilon_{n+1} \le |z_i|_2 < \epsilon_n\} \quad \text{is infinite.} \end{cases}$$

We can then use a standard gliding hump and perturbation argument to find infinite $M_n' \subseteq M_n$ so that, setting $M = \bigcup_{n=1}^{\infty} M_n'$, we have that $(y_i)_{i \in M}$ is equivalent to the unit vector basis of ℓ_p and $(z_i)_{i \in M}$ is equivalent to an orthogonal sequence in ℓ_2. Thus by Rosenthal's characterization of X_p mentioned earlier, $[(x_i)_{i \in M}]$ is isomorphic to X_p and is complemented in X because (x_i) is unconditional, hence by Propositions 2 and 3, X is isomorphic to X_p.

If $(*)$ is false, then there is $\epsilon > 0$ and $A \subseteq \mathbb{N}$ so that $|z_i|_2 \ge \epsilon$ for $i \notin A$ and $\lim_{\substack{i \to \infty \\ i \in A}} |z_i|_2 = 0$.

By Proposition 4, either X_p is complemented in $[(x_i)_{i \in A}]$ and hence in X, so that, by Proposition 3, X and X_p are isomorphic, or $[(x_i)_{i \in A}]$ embeds into ℓ_p, and so is finite dimensional or isomorphic to ℓ_p since it embeds into L_p as a complemented subspace. Of course, $[(x_i)_{i \notin A}]$ is isomorphic to a Hilbert space and so if $[(x_i)_{i \in A}]$ embeds into ℓ_p, then X is isomorphic to ℓ_p, $\ell_p \oplus \ell_2$, or ℓ_2 if, respectively, $\mathbb{N} \sim A$ is finite, A and $\mathbb{N} \sim A$ are infinite, or A is finite.

To indicate how to prove Proposition 2, we need to recall the concept of a blocking of a finite dimensional decomposition (f.d.d., in short). Given an f.d.d. (E_n) for some space Z, a <u>blocking</u> of (E_n) is an f.d.d. for Z of the form (E_n'), where for $k = 1,2,\ldots$, $E_k' = [(E_i)_{i=n(k)}^{n(k+1)-1}]$ for some sequence $1 = n(1) < n(2) < \ldots$ of integers. The simplest version of the <u>blocking method</u>, introduced in [6] (cf. also Proposition 1.g.4 in [8]) can be stated qualitatively as follows: If Z has a shrinking f.d.d. (E_n), Y has an f.d.d. (F_n), and $T: Z \to Y$ is an operator, then there are blockings (E_n') of (E_n) and (F_n') of (F_n) so that for all $n = 1,2,\ldots$, $T E_n'$ is "essentially" contained in $F_n' + F_{n+1}'$. ("Essentially" means: given any $\epsilon_n \downarrow 0$, (E_n') and (F_n') may be chosen so that for $x \in E_n'$, $d(Tx, F_n' + F_{n+1}') \leq \epsilon_n \|x\|$.) An easy consequence of this blocking principle is:

<u>Lemma 5</u>: <u>If (E_n) is a shrinking f.d.d. for Z, (F_n) is an f.d.d. for Y, and $T: Z \to Y$ is an operator, then there are blockings (E_n') of (E_n) and (F_n') of (F_n) so that $T: (\sum_{n=1}^{\infty} E_n')_p \to (\sum_{n=1}^{\infty} F_n')_p$ is bounded.</u>

We are now ready to prove Proposition 2. By a change of density on the underlying measure space, we can by one of Maurey's theorems [9] assume that T is bounded as an operator from L_2 into $(X, |\cdot|_2)$, i.e., for all $x \in L_p$, $|Tx|_2 \leq K \|x\|_2$ for some constant K. Secondly, by Lemma 5, we can find a blocking (H_n) of the Haar basis so that T is bounded as an operator from $(\sum_{n=1}^{\infty} (H_n, \|\cdot\|_p))_p$ into $(X, |\cdot|_p)$. (To see this, embed $(X, |\cdot|_p)$ into ℓ_p and block the unit vector basis for ℓ_p.) Consequently, if for $x \in L_p$, $x = \Sigma x_n$ $(x_n \in E_n)$, we define $|||x||| = \max((\Sigma \|x_n\|_p^p)^{1/p}, \|x\|_2)$ then we have that T is bounded as an operator from $(L_p, |||\cdot|||)$ into X. The identity mapping from L_p into $(L_p, |||\cdot|||)$ is bounded because the Haar basis, being unconditional, admits a lower ℓ_p-estimate. Thus the operator $T: L_p \to X$ factors through $(L_p, |||\cdot|||)$.

To complete the proof of Proposition 2 we only need to observe that the completion of $(L_p, |||\cdot|||)$ is isomorphic to a complemented subspace of $X_{p,w}$ for some weight sequence w. This is done by seeing that the completion of $(L_p, |||\cdot|||) = (\Sigma H_n, |||\cdot|||)$ is norm one complemented in $(\Sigma E_n, |||\cdot|||)$ by the orthogonal projection, where for $n = 1,2,\ldots$, $E_n = [(h_i)_{i=1}^{2^{k(n)}}]$ and $k(n)$ is chosen so that $H_n \subseteq E_n$. If $f_i^n \in E_n$ denotes the L_p-normalized indicator function of the interval $[(i-1)2^{-k(n)}, i\, 2^{-k(n)})$ for $1 \leq i \leq 2^{k(n)}$; $n = 1,2,\ldots$, then one can easily see that $((f_i^n)_{i=1}^{2^{k(n)}})_{n=1}^{\infty}$ in $(\Sigma E_n, |||\cdot|||)$ is equivalent to the natural basis of $X_{p,w}$ for the weight sequence $w = ((\|f_i^n\|_2)_{i=1}^{2^{k(n)}})_{n=1}^{\infty}$.

To prove Proposition 3 we need the following:

<u>Lemma 6</u>: <u>There exists</u> $M_p < \infty$ <u>so that if</u> T <u>is an operator on</u> $X_{p,w}$ <u>for some weight sequence</u> $w = (w_n)_{n=1}^{\infty}$, <u>then there exists a weight sequence</u>

v <u>so that</u> $|T|_{2,v} \leq M_p \|T\|$ <u>and</u> $|||\, x \,||| \equiv \max\,(|x|_p, |x|_{2,v})$ <u>is</u> M_p-<u>equivalent to the usual norm on</u> $X_{p,w}$.

The lemma can be proved by embedding X_p into $L_p[-1,1]$ by identifying the nth-unit vector of $X_{p,w}$ with the function $f_n = g_n + w_n r_n$, where (g_n) are disjointly supported unit vectors in $L_p[-1,0]$, $\|g_n\|_2 \leq w_n$, and (r_n) are the Rademacher functions on $[0,1]$. Note that $|\cdot|_{2,w}$ on $X_{p,w}$ is equivalent to $\|\cdot\|_2$ under this identification. Now one uses [3] to get a change of density $\phi \geq \frac{1}{2}$ on $[-1,1]$ so that T is bounded when considered as an operator from $([f_n], \|\cdot\|_{L_2(\phi dm)})$ into itself. One can check that the weight sequence $v = (v_n)$ defined by $v_n^2 = w_n^2 + \|\phi^{-1/p} g_n\|^2_{L_2(\phi dm)}$ does the job.

We are now ready to prove Proposition 3. The idea is to use Pelczynski's classical proof [10] that every complemented subspace of ℓ_p is isomorphic to ℓ_p. We need to write X_p as a symmetric sum $(X_p \oplus X_p \oplus \ldots)$ in such a way that $(X \oplus X \oplus \ldots)$ is complemented in $(X_p \oplus X_p \oplus \ldots)$. The problem is that X_p is <u>not</u> isomorphic to $(X_p \oplus X_p \oplus \ldots)_p$. However, if we represent X_p as $X_{p,w}$, then X_p is isomorphic to $(X_{p,w} \oplus X_{p,w} \oplus \ldots)_{p,2}$ where for $x_n \in X_{p,w}$, the norm in $(X_{p,w} \oplus X_{p,w} \oplus \ldots)_{p,2}$ of $y = (x_n)_{n=1}^{\infty}$ is given by $\|y\| = \max\,((\Sigma\, |x_n|_p^p)^{1/p}, (\Sigma\, |x_n|^2_{2,w})^{1/2})$. (One checks the isomorphism of X_p with $(X_{p,w} \oplus X_{p,w} \oplus \ldots)_{p,2}$ by observing that $(X_{p,w} \oplus X_{p,w} \oplus \ldots)_{p,2}$ is isometric to $X_{p,v}$, where the weight sequence v consists of all terms of the weight sequence w, each repeated infinitely many times.) Unfortunately, it is not true that $(X \oplus X \oplus \ldots)$ must be complemented in $(X_{p,w} \oplus X_{p,w} \oplus \ldots)_{p,2}$ if X is complemented in $X_{p,w}$, so Pelczynski's argument does not apply. However, if the projection

$P: X_p \to X$ is bounded in both the $|\cdot|_p$ and the $|\cdot|_{2,w}$ norms on X, then $(X \oplus X \oplus \dots)$ is complemented in $(X_{p,w} \oplus X_{p,w} \oplus \dots)_{p,2}$ by the projection $P \oplus P \oplus \dots$. The point of Lemma 6 is that we can assume, without loss of generality, that $|P|_{2,w} < \infty$. Of course, $|P|_p$ might be infinite, but there is by Lemma 5 a blocking (E_n) of the natural basis for $X_{p,w}$ so that P is bounded as an operator from $(\Sigma E_n)_p$ into itself, where each space E_n has the $X_{p,w}$ norm, $\|\cdot\|$, on it. If we define $|\ |_p'$ on $X_{p,w}$ by $|x|_p' = (\Sigma \|x_n\|_p^p)^{1/p}$ $(x = \Sigma x_n,\ x_n \in E_n)$ then it is easy to check that the $X_{p,w}$ norm is equivalent to the norm $|||\, x \,||| = \max(|x|_p', |x|_{2,w})$. Since $|P|_p'$ and $|P|_2$ are both finite, $(X \oplus X \oplus \dots)$ is complemented in $((X_{p,w}, |||\cdot|||) \oplus (X_{p,w}, |||\cdot|||) \oplus \dots)_{p,2}$ and this letter space is easily seen to be isomorphic to X_p. This completes the sketch of the proof of Proposition 3.

We complete this seminar by giving a proof of Proposition 4.

If ℓ_2 does not embed into X, then X embeds into ℓ_p by a result of Johnson and Odell (or see [2]). Thus we may assume X contains a copy of ℓ_2.

Since $|z_n|_2 \to 0$, we can assume without loss of generality that $|z_n|_2 < 1$ for each n. For a subspace Y of X, let $\delta(Y) = \sup\{|y|_2 : \|y\| = 1\}$. Note that since X contains ℓ_2, if $\dim X/Y < \infty$, then $\delta(Y) = 1$. By the blocking technique [6] there exists $0 = k(1) < k(2) < \dots$ such that if $E_n = [(y_i)_{k(n)+1}^{k(n+1)}]$ and $F_n = [(z_i)_{k(n)+1}^{k(n+1)}]$, then (E_n) is an ℓ_p-f.d.d. for $[(y_n)]$ and (F_n) is an ℓ_2-f.d.d. for $[(z_n)]$. Thus if $u_n \in E_n$, then $|\Sigma u_n|_p \sim (\Sigma |u_n|_p^p)^{1/p}$ and a similar statement holds for (F_n). Also by our above remark we can insure that

$\delta([x_i]_{k(n)+1}^{k(n+1)}) \geq 1/2$ for each n. Since $|z_n|_2 \to 0$, we can find $q(n)$ $k(n) < q(n) < k(n+1)$ such that if $H_n = [(x_i)]_{k(n)+1}^{q(n)}$ then

$$1 > \delta(H_n) > 0 \quad \text{for each } n,$$

$$\sum_{n=1}^{\infty} \delta(H_n)^{2p/(p-2)} = \infty, \quad \text{and} \quad \lim_{n\to\infty} \delta(H_n) = 0.$$

Let $e_n \in H_n$ so that $\|e_n\| = 1$ and $|e_n|_2 = \delta(H_n)$. Clearly $[(e_n)]$ is isomorphic to X_p. We must show it is also complemented in X. Thus we wish to find $\tilde{f}_n \in X^*$ so that $(\tilde{f}_n)$ is biorthogonal to (e_n) and $P(x) = \Sigma\, \tilde{f}_n\,(x)\, e_n$ is a bounded operator, and hence a projection onto $[(e_n)]$.

Let f_n be the functional on H_n defined by $f_n(h) = \langle h,\, e_n\, |e_n|_2^{-2}\rangle$. Then

$$|f_n|_p = \max_{\substack{|h|_p=1 \\ h\in H_n}} \langle h,\, e_n\, |e_n|_2^{-2}\rangle$$

$$\leq \max_{\substack{|h|_p=1 \\ h\in H_n}} |h|_2\, |e_n|_2^{-1} = 1,$$

since $|e_n|_2 = \delta(H_n)$ and $\|\cdot\| = |\cdot|_p$ on H_n. Thus f_n is a norm 1 functional on H_n in the ℓ_p norm. Extend f_n to a functional $\tilde{f}_n$ on X by letting $\tilde{f}_n(x_i) = 0$ if $i < k(n)$ or $i > q(n)$. Since (y_i) and (z_i) are basic, we have

$$|\tilde{f}_n|_p \leq K \quad \text{and} \quad |\tilde{f}_n|_2 \leq K|f_n|_2 = K|e_n|_2^{-1}$$

where K is twice the larger basis constant of (y_i) and (z_i). Moreover, since (E_n) and (F_n) are p- and 2-f.d.d.'s, respectively, and $|e_n|_p \le 1$, we see that $P(x) = \Sigma \tilde{f}_n(x)\, e_n$ is bounded. □

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