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S E M I N A I R E  
D ' A N A L Y S E F O N C T I O N N E L L E  
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BANACH SPACES ALL OF WHOSE SUBSPACES HAVE  
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THE APPROXIMATION PROPERTY  
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## INTRODUCTION

Let  $X$  be a Banach space all of whose subspaces have the approximation property. A consequence of a recent result of Szankowski [19] (cf. also Theorem 1.g.6. in [16]) is that  $X$  is of type  $p$  for all  $p < 2$  and of cotype  $q$  for all  $q > 2$ . In terms of inequalities, this means that for each  $1 \leq p < 2 < q < \infty$  there are constants  $0 < A_q < B_p < \infty$  so that

$$(*) \quad A_q (\sum \|x_i\|^q)^{1/q} \leq \text{Average}_{\pm} \|\sum_{\pm} x_i\| \leq B_p (\sum \|x_i\|^p)^{1/p}$$

holds for all finite sequences  $\{x_i\}$  of vectors in  $X$ . If  $A_q^{-1}$  and  $B_p$  remain bounded as  $p \rightarrow 2$ ,  $q \rightarrow 2$ ; i.e., if  $(*)$  holds for  $p = 2 = q$ , then  $X$  is isomorphic to a Hilbert space by a result of Kwapien [11]. Szankowski's result thus gave support to the conjecture that  $X$  must be isomorphic to a Hilbert space.

In this note we give a simple criterion which guarantees that every subspace of every quotient of a given space has the approximation property - even a much stronger property called the uniform projection approximation property (u.p.a.p., in short). The interest in such a criterion stems from the fact that there are Banach spaces which satisfy it and are not isomorphic to Hilbert spaces. One such example; namely,  $(\sum_{n=1}^{\infty} \ell_{p_n}^{k_n})_2$  for appropriately chosen  $k_n \rightarrow \infty$  and  $p_n \rightarrow 2$ ;

has already appeared as Example 1.g.7 in [16] with essentially our original proof that every subspace has the u.p.a.p. It takes some additional work to check that every subspace of every quotient has the u.p.a.p., so we briefly comment on this kind of space at the beginning of section 2. However, the emphasis in section 2 is on a more interesting class of examples which do not contain isomorphic copies of  $\ell_2$  and which show that the characterization of Hilbert space given by Lindenstrauss and Tzafriri [13] cannot be improved. Moreover, these examples have the property that every subspace of every quotient has a Schauder basis.

We use standard Banach space theory notation, as may be found in [15] and [16]. Let us just mention that  $B_E$  is the closed unit ball of the Banach space  $E$ .

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## 1. CRITERION FOR THE U.P.A.P.

A Banach space  $X$  is said to have the uniform approximation property -u.a.p.- (respectively, uniform projection approximation property - u.p.a.p.) provided there is a uniformity function  $f: \mathbb{N}^+ \rightarrow \mathbb{R}^+$  and a constant  $K < \infty$  so that for every finite dimensional subspace  $E$  of  $X$ , there is an operator (respectively, projection)  $T$  on  $X$  so that  $\|T\| < K$ ,  $Te = e$  for all  $e \in E$ , and  $\dim TX \leq f(\dim E)$ .

The u.a.p., introduced in [18], is a stronger and quantitative version of the bounded approximation property. There are some interesting classes of spaces which possess even the u.p.a.p., such as  $L_p$  spaces [18] and reflexive Orlicz spaces [14]. Heinrich [6], [7] used ultraproduct techniques to check that the u.a.p. is a self-dual property. From this result and the technique in [9] it follows easily (see Proposition 1.1) that the u.p.a.p. is also self-dual.

The main result of [13] is that if  $X$  is a Banach which has the u.p.a.p. with uniformity function  $f(n) = n$  (or, equivalently,  $f(n) = n + m$  for some constant  $m$ ), then  $X$  is isomorphic to a Hilbert space. In section 2 we show that for any function  $g: \mathbb{N}^+ \rightarrow \mathbb{R}^+$  for which  $0 \leq g(n) - n \rightarrow \infty$ , there is a Banach space  $X$  such that every subspace of every quotient of  $X$  and  $X^*$  has the u.p.a.p. with uniformity function  $g$ , and yet  $\ell_2$  is not isomorphic to a subspace of any quotient of  $X$ .

Proposition 1.1. A Banach space  $X$  has the u.p.a.p. if and only if  $X^*$  has the u.p.a.p.

Proof: assume that  $X$  has the u.p.a.p. Then by [7],  $X^*$  has the u.a.p., so we can suppose that  $X$  (respectively,  $X^*$ ) has the u.p.a.p. (respectively, u.a.p.) with the same uniformity function  $f$  and for a certain constant  $K < \infty$ .

Now suppose  $E \subseteq X$ ,  $G \subseteq X^*$ ,  $\dim E = \dim G = n$ . We use the technique of [9] to construct a projection  $P$  on  $X$  so that  $P|_G = I_G$ ,  $\|P\|$  depends only on  $K$ , and  $\dim PX$  is a function only of  $n$ . To do this, choose an operator  $T$  on  $X^*$  so that  $T|_G = I_G$ ,  $\|T\| < K$ , and  $\dim TX^* \leq f(n)$ . By the version of the principle of local reflexivity proved in [9], we can assume that  $T = S^*$  for some operator  $S$  on  $X$ . Now choose a projection  $Q$  on  $X$  so that  $Qx = x$  for  $x \in E \cup SX$ ,  $\|Q\| < K$ , and  $\dim QX \leq f(n + f(n))$ . Using the identities  $Q^2 = Q$  and  $QS = S$ , one easily checks that  $P = S + Q - SQ$  is a projection onto  $QX$ . Since  $SP = S$ , it also follows that  $P^*|_G = I_G$ .

Conversely, if  $X^*$  has the u.p.a.p., then so does  $X^{**}$  by the first part of the proof, hence  $X$  does also by the original version [12] (or see Lemma 1.e.6. in [15]) of the principle of local reflexivity.  $\square$

We come now to the definition which yields a criterion for all subspaces of a given space to have the u.p.a.p.

Definition 1.2. Given a Banach space  $X$ , positive integers  $n$  and  $m$ , and a constant  $K$ , we say that  $X$  satisfies  $C(n,m,K)$  provided that there is an  $n$ -codimensional subspace  $Y$  of  $X$  so that every subspace  $E$  of  $Y$  with  $\dim E \leq m$  is the range of a projection  $P$  from  $X$  with  $\|P\| < K$ .

Proposition 1.3. Suppose  $X$  satisfies  $C(n,m,K)$  and  $Z$  is a subspace of  $X$ . If  $F$  is a subspace of  $Z$  with  $\dim F \leq m - 5^n$ , then there is a subspace  $G$  of  $Z$  so that  $F \subseteq G$ ,  $\dim G < \dim F + 5^n$ , and  $G$  is the range of a projection  $R$  from  $Z$  with  $\|R\| \leq 4K + 3$ .

Proof: We make use of the following well-known (see, e.g., p. 112 in [16]) fact:

Fact 1.4: Let  $Q: X \rightarrow W$  be a quotient mapping and suppose  $E$  is a subspace of  $W$  with  $\dim E = n$ . Then there is a subspace  $G$  of  $X$  with  $\dim G \leq 5^n$  so that  $B_E \subseteq 3QB_G \subseteq E$ .

Let  $Y$  be an  $n$ -codimensional subspace of  $X$  which satisfies the conditions in the definition of  $C(n,m,K)$ . Let  $Q: Z \rightarrow Z/Z \cap Y$  be the quotient mapping and select a subspace  $G$  of  $Z$  with  $\dim G \leq 5^n$  so that

$$B_{Z/Z \cap Y} \subseteq 3QB_G.$$

By replacing  $G$  with  $G + F$ , we can assume that  $F \subseteq G$  (but now we know only that  $\dim G \leq 5^n + \dim F \leq m$ ). By the  $C(n,m,K)$  condition on  $X$ , there is a projection  $P$  from  $Z$  (even from  $X$ ) onto  $G \cap Y$  with  $\|P\| < K$ . The restriction of  $Q$  to  $(I-P)G$  is one-to-one and onto, since  $\ker Q \cap G = G \cap Y$ , and thus

$$S = (Q|_{(I-P)G})^{-1}$$

is well-defined. Now

$$QB_{(I-P)G} = \|I-P\|^{-1}QB_G = 3^{-1}\|I-P\|^{-1}B_{Z/Z \cap Y} = 3^{-1}(K+1)^{-1}B_{Z/Z \cap Y}$$

so that  $\|S\| < 3(K+1)$  and  $R = SQ + P$  is a projection from  $Z$  onto  $G$  with  $\|R\| \leq 3(K+1) + K = 4K + 3$ .  $\square$

Next we mention a property which is closely related to  $C(\cdot)$  but is easier to check and work with.

Definition 1.5: Given a Banach space  $X$ , positive integers  $n$  and  $m$ , and a constant  $K$ , we say that  $X$  satisfies  $H(n,m,K)$  provided that there is an  $n$ -codimensional subspace  $Y$  of  $X$  so that every subspace  $E$  of  $Y$  with  $\dim E \leq m$  is  $K$ -Euclidean; i.e., is  $K$ -isomorphic to a Hilbert space.

If  $X$  satisfies  $H(n,m,K)$  and is of type 2 with constant  $\lambda$ , then by Maurey's extension theorem [17]  $X$  also satisfies  $C(n,m,\lambda K)$ . On the other hand, if  $X$  satisfies  $C(n,m,K)$  then Theorem 6.7 of [4] yields that there is a constant  $M = M(K)$  so that  $X$  satisfies  $H(n,m,M)$ . Now the main examples in section 2 are type 2 spaces, so that  $C(\cdot)$  and  $H(\cdot)$  are essentially equivalent properties for them. However, in order to investigate quotients of our examples we need to dualize the  $C(\cdot)$  property. The substance of Proposition 1.8 is that the  $H(\cdot)$  property dualizes and implies the  $C(\cdot)$  property even for spaces which are not of type 2.

Recall that a subspace  $Y$  of  $X$  is said to be  $\lambda$ -norming over a subspace  $Z$  of  $X^*$  provided

$$\|z\| \leq \lambda \sup \{z(y) : y \in B_Y\}$$

for each  $z \in Z$ . This is equivalent to saying that the natural restriction mapping  $R$  from  $Z$  to  $Y^*$  (defined by  $(Rz)y = z(y)$  for  $z \in Z, y \in Y$ ) is a  $\lambda$ -isomorphism, or that the natural evaluation mapping  $T: Y \rightarrow Z^*$  (defined by  $(Ty)z = z(y)$  for  $y \in Y, z \in Z$ ) satisfies

$$\text{weak } * \text{ cl } TB_Y \supseteq \lambda^{-1} B_{Z^*}.$$

(The weak\* closure can of course be eliminated if  $Y$  is reflexive, or if  $\dim Z < \infty$  and  $\lambda^{-1}$  is replaced by any strictly smaller number.)

Lemma 1.6: Suppose  $Y$  is a subspace of  $X, Z$  is a subspace of  $X^*$ , and  $Y$  is  $\lambda$ -norming over  $Z$ . If every  $5^n$ -dimensional subspace of  $Y$  is  $K$ -Euclidean, then every subspace  $E$  of  $Z$  with  $\dim E \leq n$  is  $3\lambda K$ -Euclidean and  $3\lambda K$ -complemented in  $X^*$ .

Proof: Given  $E \subseteq Z$  with  $\dim E \leq n$ , define  $Q: X \rightarrow E^*$  by  $(Qx)e = e(x)$ . Since  $Y$  is  $\lambda$ -norming over  $Z$  and  $\dim E < \infty$ , the restriction of  $Q$  to  $Y$  is a quotient mapping up to constant  $\lambda$ . Therefore, by Fact 1.4, there is a subspace  $G$  of  $Y$  with  $\dim G \leq 5^n$  so that

$$B_{E^*} \subseteq 3\lambda QB_G$$

Thus  $E^*$ , whence also  $E$ , is  $3\lambda K$ -Euclidean.

The complementation follows from the next lemma:

Lemma 1.7: Suppose that  $G$  is a  $K$ -Euclidean subspace of  $X$  and  $G$  is  $\lambda$ -norming over a subspace  $E$  of  $X^*$ . Then  $E$  is  $\lambda K$ -complemented in  $X^*$ .

Proof: Define  $T: X^* \rightarrow G^*$  by  $Tx^* = x^*|_G$ . Then  $T_E$  has an inverse,  $S$ , with  $\|S\| \leq \lambda$ . Since  $G^*$  is  $K$ -Euclidean, there is a projection  $P$  from  $G^*$  onto  $TE$  with  $\|P\| \leq K$ . Hence  $Q = SPT$  is a projection from  $X^*$  onto  $E$ .  $\square$

Proposition 1.8: Suppose that  $X$  satisfies  $H(n,m,K)$  and  $m \geq 5^k$ . Then  $X^*$  satisfies  $H(5^n, k, 12K)$  and  $C(5^n, k, 12K)$ .

Proof: In view of Lemma 1.6, we only need to observe that if  $Y$  is an  $n$ -codimensional subspace of  $X$ , then  $Y$  is 4-norming over some  $5^n$ -codimensional subspace of  $X^*$ . This is a consequence of Fact 1.4. To see this, let  $Q$  be the quotient mapping from  $X$  onto  $X/Y$ , and choose a subspace  $E$  of  $X$  with  $\dim E \leq 5^n$  so that

$$3Q B_E \supseteq B_{X/Y}.$$

We claim that  $Y$  is 4-norming over  $E^\perp$ . Indeed, if  $f \in E^\perp$ ,  $\|f\| > 1$ , and  $x \in B_X$  with  $f(x) > 1$ , then we can choose  $e \in 3B_E$  so that  $Qe = Qx$ . Thus  $x - e \in 4B_Y$  and  $f(x-e) = f(x) > 1$ .  $\square$

## 2. THE EXAMPLES

Since property  $H(n,m,K)$  is clearly a hereditary property, Propositions 1.8 and 1.3 imply that if  $m(n)$  is sufficiently large relative to  $n$  and there is a constant  $K$  so that  $X$  satisfies  $H(n,m(n),K)$  for infinitely many  $n$ , then every subspace of every quotient of  $X$  and  $X^*$  has the u.p.a.p. The space

$$X = \left( \sum_{n=1}^{\infty} \ell_{p_n}^{k_n} \right)_2$$

has this property if  $p_n \rightarrow 2$  and  $k_n \rightarrow \infty$  are chosen appropriately. The only restriction is that, having chosen  $p_i, k_i$  for  $1 \leq i \leq n$ , the  $p_i$ 's for  $i > n$  must be chosen sufficiently close to 2 so that every subspace  $E$  of

$$\left( \sum_{i=n+1}^{\infty} \ell_{p_i} \right)_2 \quad \text{with} \quad \dim E = m = m\left( \sum_{i=1}^n k_i \right)$$

is 2-Euclidean. If, having chosen  $p_{n+1} \neq 2$ , one chooses  $k_{n+1}$  so that

$$d\left( \ell_2^{k_{n+1}}, \ell_{p_{n+1}}^{k_{n+1}} \right) = k_{n+1} \left| 2^{-1} - p_{n+1}^{-1} \right| > n,$$

then the resulting space

$$X = \left( \sum_{n=1}^{\infty} \ell_{p_n}^{k_n} \right)_2$$

is not isomorphic to  $\ell_2$ . (See p.112 in [16] for one way of carrying out this construction.)



By the results of [10], every subspace of every quotient of such a space  $X$  is isomorphic to a space of the form  $(\sum E_n)_2$  with  $\dim E_n < \infty$  for every  $n$ . In particular, every subspace of every quotient of such an  $X$  has an unconditional decomposition into finite dimensional subspaces. However not every subspace of a space of the form

$$X = \left( \sum_{n=1}^{\infty} \ell_{p_n}^{k_n} \right)_2 \quad \text{with } p_n \downarrow 2$$

can have an unconditional basis unless  $X$  is isomorphic to  $\ell_2$ . Indeed, in [3] (or see [2]) it is shown that there is a subspace  $E_n$  of  $\ell_{p_n}^{k_n}$  with  $\dim E_n = [k_n/2]$  so that

$$gl E_n \geq c n^{1/2-1/p_n} = cd(\ell_{p_n}^{k_n}, \ell_2^{k_n}),$$

where  $c > 0$  is an absolute constant and  $gl E$  is the Gordon-Lewis constant of  $E$  (see [2]). From [5] it follows that  $(\sum E_n)_2$  cannot have an unconditional basis if

$$\sup_n n^{1/2-1/p_n} = \infty;$$

that is, if  $X$  is not isomorphic to  $\ell_2$ .

By using Kwapien's characterization of Hilbert space [11] and the same reasoning as above, one obtains the following:

Proposition 2.1: Suppose that  $\{X_n\}_{n=1}^{\infty}$  is a sequence of Banach spaces, none of which is isomorphic to a Hilbert space.  $X_n$  has type  $p_n$  and cotype  $q_n$  with constant  $K$  ( $K$  independent of  $n$ ), and  $q_n - p_n \rightarrow 0$  as  $n \rightarrow \infty$ . Then there are finite dimensional subspaces  $E_n$  of  $X_n$  so that every subspace of every quotient of  $(\sum E_n)_2$  has the u.p.a.p., but  $(\sum E_n)_2$  is not isomorphic to  $\ell_2$ .

We turn now to a more interesting class of examples.

Example 2.2: There is a constant  $M < \infty$  so that if  $g(n)$  is a positive integer valued function for which  $5 \leq g(n) \uparrow \infty$ , then there is a Banach space  $X = X(g)$  which has the following properties.

(2.3)  $X$  has a monotonely unconditional basis  $\{e_n\}_{n=1}^{\infty}$ .

(2.4)  $X$  is of type 2 and of cotype  $q$  for all  $q > 2$ .

(2.5)  $\ell_2$  is not isomorphic to a subspace of a quotient of  $X$ .

(2.6) Let  $Y$  be any subspace of a quotient of  $X$  or  $X^*$ . If  $F$  is a subspace of  $Y$  with  $\dim F = m$ , there are  $E \subseteq F \subseteq G \subseteq$  with  $\dim E \geq m - g(m)$ ,

$\dim G \leq m + g(m)$ , so that  $E$  is  $M$ -Euclidean and both  $E$  and  $G$  are  $M$ -complemented in  $Z$ .

(2.7) Every subspace of every quotient of  $X$  has a Schauder basis.

Proof:  $X = X(g)$  is the 2-convexification (see p.53 of [16]) of the space spanned by a suitable subsequence of the basis constructed in [8]. The construction of this space is given in Example 5.3 of [4].  $X$  can be described as the completion of the space  $X_0$  of finitely non-zero sequences of scalars under a certain norm  $\|\cdot\|$ . The unit vectors form a monotonely unconditional basis for  $X$ . For a certain increasing sequence  $\{k_n\}_{n=1}^\infty$  of positive integers which depends on  $g$ , the norm  $\|\cdot\|$  satisfies property (2.8) below; in fact,  $\|\cdot\|$  is the unique norm on  $X_0$  which satisfies (2.8). (For  $x \in X_0$  and  $A \subseteq \mathbb{N}^+$ , let  $Ax$  denote the sequence which agrees with  $x$  for coordinates in  $A$  and is 0 elsewhere.)

$$(2.8) \quad \|x\| = \max \left\{ \|x\|_{c_0}, \frac{1}{2} \sup_{i=1}^{k_n} \left( \sum_{i=1}^{k_n} \|A_i x\|^2 \right)^{1/2} \right\},$$

where the sup is over all  $n$  and all pairwise disjoint sequences  $\{A_i\}_{i=1}^k$  of subsets of  $\mathbb{N}^+$  for which

$$\bigcup_{i=1}^{k_n} A_i \subseteq \{n + j\}_{j=1}^\infty.$$

It follows from [8] (see section 4 of [1]) that  $\ell_2$  does not embed into  $X$ .  $X$  is 2-convex with constant 1 since it is the 2-convexification of the space constructed in [8]. It is also clear that if  $\{x_i\}_{i=1}^k$  are disjointly supported vectors in  $\text{span}(e_i)_{i=n+1}^\infty$ , then

$$(2.9) \quad \frac{1}{2} \left( \sum_{i=1}^{k_n} \|x_i\|^2 \right)^{1/2} \leq \left\| \sum_{i=1}^{k_n} x_i \right\| \leq \left( \sum_{i=1}^{k_n} \|x_i\|^2 \right)^{1/2}.$$

Thus if  $k_n \uparrow \infty$  sufficiently quickly,  $(X, \|\cdot\|)$  will satisfy a lower  $\ell_q$ -estimate for every  $q > 2$ ; that is,  $\left\| \sum x_i \right\| \geq c \left( \sum \|x_i\|^q \right)^{1/q}$  for all disjoint vectors  $\{x_i\}$  in  $X$  and some constant  $c = c(q) > 0$ . Thus (see section 1.f of [16])  $X$  has type 2 and cotype  $q$  for all  $q > 2$ .

To see that (2.5) is true it is enough to observe that  $\ell_2$  does not embed into  $X^*$ . Indeed, if  $\ell_2$  embeds into a quotient  $Y$  of  $X$ , then  $\ell_2$  is complemented in  $Y$  because  $Y$  is of type 2 (cf. [17]); hence  $\ell_2$  is isomorphic to a quotient of  $X$ , whence to a subspace of  $X^*$ . But  $X$  is reflexive (since (2.9) precludes its containing a copy of  $c_0$  or  $\ell_1$ , so that James' theorem applies; cf. Theorem 1.c.12 in [15]) so  $X^*$  has a 2-concave basis. Now if  $\ell_2$  embeds into  $X^*$ , then some block basis of the 2-concave basis for  $X^*$  is equivalent to the unit vector basis for  $\ell_2$  and thus (see, e.g., the argument for Theorem 3.1 in [18]) spans a

complemented subspace. This would imply that  $\ell_2$  embeds into  $X^{**} = X$ , which is false.

We now turn to the proof of (2.6).

From the argument for Theorem 2.1 of [18] (see Proposition 3 of [14] for a sketch of the proof in the generality we need) it follows that if  $H$  is an  $n$ -dimensional subspace of a space which has a monotonely unconditional basis, then  $H$  is 2-isomorphic to a subspace spanned by  $\ell(n)$  disjointly supported vectors, where  $\ell(n)$  depends only on  $n$ . ( $\ell(n)$  is in fact of order  $\leq n^{2n}$ .) Therefore, if  $H$  is a subspace of the subspace of  $X$  spanned by  $\{e_i\}_{i=n+1}^{\infty}$ ,  $\dim H = s$  and  $\ell(s) \leq k_n$ , then by (2.9)  $H$  is 4-Euclidean. This means that for arbitrary  $d_n$ ,  $X$  satisfies  $H(n, d_n, 4)$  if  $k_n$  is sufficiently large. Thus by Proposition 1.8 and obvious duality arguments we have:

There is a constant  $K$  so that for any sequence  $d_n \uparrow \infty$  there is a sequence  $k_n \uparrow \infty$  so that if  $Y$  is a subspace of a quotient of  $X$  or  $X^*$  (2.10) (where  $X = X(k_n)$  satisfies (2.8)), then for each  $n = 1, 2, \dots$ ,  $Y$  satisfies  $H(n, d_n, K)$  and  $C(n, d_n, K)$  for the same  $n$ -codimensional subspace  $Y_n$  of  $Y$ .

Suppose now that for each  $n = 1, 2, \dots$ ,  $Y$  satisfies  $H(n, d_n, K)$  and  $C(n, d_n, K)$  for the same  $n$ -codimensional subspace  $Y_n$  of  $Y$ . Let  $F$  be any  $m$ -dimensional subspace of  $Y$  and suppose that  $n$  satisfies

$$(2.11) \quad m \leq d_n.$$

If also

$$(2.12) \quad n \leq g(m)$$

then  $E \equiv F \cap Y_n$  fulfills the conditions in (2.6) for  $M = K$ .

On the other hand, if

$$(2.13) \quad m \leq d_n - 5^n$$

then by Proposition 1.3 there is a subspace  $G$  of  $Y$  so that  $F \subseteq G$ ,  $\dim G \leq m + 5^n$ , and  $G$  is  $4K + 3$ -complemented in  $Y$ . This  $G$  fulfills the conditions in (2.6) for  $M = 4K + 3$  as long as

$$(2.14) \quad 5^n \leq g(m).$$

This completes the proof of (2.6), because for any sequence  $g(m) \uparrow \infty$  we can select  $d_n \uparrow \infty$  so that for every  $m$  there is an  $n$  for which (2.11)-(2.14) are satisfied.

To prove (2.7), we use the technique of [9]. By the proof of (2.6), we can assume that the space  $X$  has the property that there is a constant  $M$  so that if  $Z$  is any subspace of a quotient of  $X$  or of  $X^*$ , then for all  $n = 1, 2, \dots$  we have:

$$(2.15) \quad Z \text{ satisfies } H(n, 3^{n+1}, M)$$

$$(2.16) \quad \text{If } E \text{ is a subspace of } Z \text{ then there is a projection } P \text{ on } Z \text{ so that } Pe = e \text{ for } e \in E, \|P\| \leq M, \text{ and } \dim PZ \leq 1.01 \dim E.$$

Let  $Y$  be a subspace of a quotient of  $X$ , and for  $n = 1, 2, \dots$  let  $Y_n$  be a codimensional  $n$  subspace of  $Y$  as in the definition of  $H(n, 3^{n+1}, M)$ . We will use (2.15) and (2.16) to construct a sequence  $\{P_n\}_{n=1}^\infty$  of projections on  $Y$  to satisfy the following conditions for all  $1 \leq n, m < \infty$ :

$$(2.17) \quad P_n P_m = P_{\min(n,m)}$$

$$(2.18) \quad \bigcup_{n=1}^\infty P_n Y \text{ is dense in } Y$$

$$(2.19) \quad \dim P_n Y \leq 3^n$$

$$(2.20) \quad \|P_n\| \leq 2M + M^2$$

$$(2.21) \quad (I - P_n)Y \subseteq Y_n$$

Conditions (2.17)-(2.20) are standard conditions which guarantee that  $E_n \equiv (P_n - P_{n-1})Y$  (where  $P_0 \equiv 0$ ) forms a finite dimensional decomposition for  $Y$  with  $\dim E_n \leq 3^n$ . Now (2.21) implies that  $E_{n+1}$  is  $M$ -Euclidean by (2.15).

Therefore we can select a sequence  $\{y_i\}_{i=s_n+1}^{s_{n+1}}$  in  $E_{n+1}$  (where  $s_0 = 0$  and  $s_n = \dim E_1 + E_2 + \dots + E_n$  for  $n \geq 1$ ) which is  $M$ -equivalent to an orthonormal sequence in a Hilbert space. The sequence  $\{y_i\}_{i=1}^\infty$  is thus a basis for  $Y$ .

It remains to construct the sequence  $\{P_n\}_{n=1}^\infty$ . Let  $\{x_i\}_{i=1}^\infty$  be dense in  $Y$  with  $x_1 = 0$ . Choose  $f \in Y_1^\perp$  with  $\|f\| = 1$ , and  $y \in Y$  with  $\|y\| = \|f\| = f(y)$ , and set  $P_1 = f \otimes y$ .  $Y_1$  has codimension one in  $Y$ , so  $\text{span}\{f\} = Y_1^\perp$  or  $\ker f = Y_1$  and hence  $(I - P_1)X \subseteq Y_1$ . Thus (2.17) and (2.19)-(2.21) are satisfied for  $n = 1 = m$  and  $P_{x_1} = x_1$ .

Having constructed  $P_1, \dots, P_k$  to satisfy (2.17) and (2.19)-(2.21) for all  $1 \leq n, m \leq k$  and  $P_n x_i = x_i$  for  $1 \leq i \leq n \leq k$ , we construct  $P_{k+1}$  as follows: By (2.16) and the reflexivity of  $Y$ , there is a projection  $P$  on  $Y$  so that  $P^*g = g$  for  $g \in Y_{k+1}^\perp \cup P_k^*Y^*$ ,  $\|P\| \leq M$ , and  $\dim P Y \leq 1.01(k+3^k)$ . Again by (2.16), there is a projection  $Q$  on  $Y$  so that  $Qy = y$  for  $y \in PY \cup P_k Y \cup \{x_{k+1}\}$ ,  $\|Q\| \leq M$ , and

$$\dim QY \leq 1.01[1.01(k+3^k) + 3^k + 1] \leq 3^{k+1}$$

Set  $P_{k+1} = Q + P - PQ$ . Using the identities  $Q P_{k+1} = P_{k+1}$ ,  $P_{k+1} Q = Q$ , and  $P_{k+1}^2 = P_{k+1}$ , we have that  $P_{k+1}$  is a projection onto  $QY$ , and thus  $P_{k+1} P_m = P_m$  for  $m \leq k+1$ , and  $P_{k+1} x_i = x_i$  for  $1 \leq x_i \leq k+1$ . Since  $P_{k+1}^* P^* = P^*$ , we have that  $P_{k+1}^* f = f$  for  $f \in P_k^* Y \cup Y_{k+1}^\perp$ . Therefore  $P_m P_{k+1} = P_m$  for  $m \leq k+1$  and  $(I - P_{k+1})Y \subseteq Y_{k+1}$ . Finally,  $\|P_{k+1}\| \leq 2M + M^2$ , so (2.17) and (2.19)-(2.21) are satisfied for  $1 \leq n, m \leq k+1$  and  $P_n x_i = x_i$  for  $1 \leq i \leq n \leq k+1$ .

It is clear that the constructed sequence  $\{P_n\}_{n=1}^{\infty}$  satisfies (2.17) - (2.21).  $\square$

Remarks: 1. We do not know whether the  $X$  of Example 2.2 can be constructed so that each of its subspaces has an unconditional basis.

2. We do not know whether there is a space with a symmetric basis (other than  $\ell_2$ ) or a non-reflexive space such that every subspace has the approximation property. If the space  $Z$  is not isomorphic to  $\ell_2$  but satisfies  $C(n, d_n, K)$  for infinitely many  $n$  and some  $d_n \uparrow \infty$ , then certainly  $Z$  cannot have a symmetric or even subsymmetric basis, and  $Z$  must be super-reflexive.

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