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Operator algebras

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OPERATOR ALGEBRAS

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A Banach algebra which is isomorphic to a closed subalgebra of the linear operators on some Hilbert space is called an operator algebra and it is these algebras which I wish to discuss today. In particular, I shall consider the extent to which these algebras can be characterized as those for which the multiplication maps

$$A \otimes \dots \otimes A \longrightarrow A \quad ; \quad a_1 \otimes \dots \otimes a_n \longmapsto a_1 \cdot a_2 \cdot \dots \cdot a_n$$

are continuous relative to some norm on the tensor product. I shall begin by defining tensor products of Banach spaces. Then I shall describe the construction of universal algebras used to study classes of Banach algebras. Finally I shall turn to those results which are specific to operator algebras.

All Banach spaces E will be complex with unit ball B(E) and dual E^*. To avoid irrelevant complications with Russel's paradox we will assume that all of the spaces considered lie in some fixed universe. A Banach algebra A is a Banach space which is given an algebra structure for which the multiplication

$$A \times A \longrightarrow A$$

is continuous. We shall not always demand that this has unit norm, although this could always be achieved by renorming A suitably.

A tensor product α (of rank r) gives a norm on every r-fold tensor product of Banach spaces :

$$E_1 \otimes \dots \otimes E_r \quad .$$

We demand that whenever $T_n : E_n \to F_n$ are bounded linear maps then

$$T_1 \otimes \dots \otimes T_r \; : \; E_1 \otimes \dots \otimes E_r \longrightarrow F_1 \otimes \dots \otimes F_r$$

has bound $\|T_1\| \cdot \dots \cdot \|T_r\|$ relative to the α-norms ; and we normalize α by demanding the α-norm on

$$\mathbb{C} \otimes \dots \otimes \mathbb{C} = \mathbb{C}$$

is simply the usual modulus. It is clear that each of Grothendieck's tensor norms is an example of such a tensor product, with rank 2. Given a tensor product α we shall denote by

$$\alpha(E_1,\ldots,E_r) \qquad (\text{or } E_1 \alpha E_2 \text{ when } r=2)$$

the completion if $E_1 \otimes \cdots \otimes E_r$ relative to the α-norm, and by

$$\alpha(T_1,\ldots,T_r) : \alpha(E_1,\ldots,E_r) \longrightarrow \alpha(F_1,\ldots,F_r)$$

the continuous extension of $T_1 \otimes \cdots \otimes T_r$. A Banach algebra A is an <u>α-algebra</u> if the multiplication map

$$m(A) : A \otimes \cdots \otimes A \longrightarrow A$$

is continuous when the tensor product is given the α-norm.

A collection $\mathcal{A}$ of Banach algebras will be called a <u>class</u> if it satisfies the following conditions.

(i) Every $A \in \mathcal{A}$ has $\|a_1 \cdot a_2\| \leq \|a_1\| \cdot \|a_2\|$ for $a_1,\ a_2 \in A$.

(ii) $\mathbb{C} \in \mathcal{A}$.

(iii) If B is a closed subalgebra of $A \in \mathcal{A}$, then $B \in \mathcal{A}$.

(iv) If $A_i \in \mathcal{A}$ for each $i \in I$ then $\oplus_\infty (A_i : i \in I) \in \mathcal{A}$.

There are many examples of such classes. The largest contains all Banach algebras which satisfy condition (i), while the smallest consists only of uniform algebras. Furthermore, if α is any tensor product, then the collection of Banach algebras A for which the multiplication map

$$\alpha(A,\ldots,A) \longrightarrow A$$

is a contraction form a class. For classes of Banach algebras one can construct universal algebras analogous to the universal tensor algebras. Let E be a Banach space. Then T(E) is the vector space

$$E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus \cdots \quad .$$

This becomes an algebra for the multiplication

$$E^{\otimes r} \times E^{\otimes s} \longrightarrow E^{\otimes(r+s)} \quad ; \quad (u,v) \longmapsto u \otimes v$$

and is called the <u>universal tensor algebra</u> over E. It has the universal property that any linear map $R : E \to A$ into an algebra extends uniquely to an algebra homomorphism

$$\tilde{R} : T(E) \longrightarrow A \quad .$$

For a class $\mathcal{A}$ of Banach algebras we can define a semi-norm on T(E) by

$$\|u\| = \sup(\|\tilde{R}u\| : R : E \to A \in \mathcal{A} \text{ is a linear contraction}) \quad .$$

Condition (i) ensures that this is finite, so we can define $T_{\mathcal{A}}(E)$ to be the Hausdorff completion relative to this semi-norm. This will be called the <u>$\mathcal{A}$-universal algebra</u> over E. Since $\mathbb{C} \in \mathcal{A}$, the Hahn-Banach theorem implies that the natural map

$$E \longrightarrow T_{\mathcal{A}}(E)$$

is a metric embedding. Also, $T_{\mathcal{A}}(E)$ is a closed subalgebra of $\oplus_\infty(A : \|R : E \to A\| \leq 1)$ and hence lies in $\mathcal{A}$. The very construction of $T_{\mathcal{A}}(E)$ ensures that any linear contraction

$$R : E \longrightarrow A \in \mathcal{A}$$

extends uniquely to an algebra homomorphism

$$\tilde{R} \quad : \quad T_{\mathcal{A}}(E) \longrightarrow A$$

which is also a contraction.

Certain examples of $\mathcal{A}$-universal algebras can be described explicitely. For example, when $\mathcal{A}$ consists of all Banach algebras with $\|a_1 \cdot a_2\| \leq \|a_1\| \cdot \|a_2\|$, then $T_{\mathcal{A}}(E)$ is the ℓ_1-direct sum of the projective powers of E :

$$T_{\mathcal{A}}(E) = E \underset{1}{\oplus} E^{\hat{\otimes} 2} \underset{1}{\oplus} E^{\hat{\otimes} 3} \underset{1}{\oplus} \ldots \quad .$$

When $\mathcal{A}$ is the class of uniform algebras, then $T_{\mathcal{A}}(E)$ is the closed subalgebra of $C(B(E^*), \sigma(E^*, E))$ generated by E. However, in most cases we have to be content with less exact information about $\mathcal{A}$-universal algebras.

The algebra $T_{\mathcal{A}}(E)$ can be decomposed into a direct sum of subspaces $T_{\mathcal{A},r}(E)$ corresponding to the decomposition of T(E) as $\oplus(E^{\otimes r})$. To see this observe that, for each $z \in \mathbb{C}$ with $|z| \leq 1$, the contraction $zI : E \to E$ induces an algebra homomorphism

$$T_{\mathcal{A}}(zI) : T_{\mathcal{A}}(E) \longrightarrow T_{\mathcal{A}}(E) \quad .$$

Then

$$P_r = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{T_{\mathcal{A}}(zI)}{z^r} \, dz$$

is a contractive projection onto a subspace $T_{\mathcal{A},r}(E)$ of $T_{\mathcal{A}}(E)$. The natural map

$$T(E) \longrightarrow T_{\mathcal{A}}(E)$$

sends $E^{\otimes r}$ into a dense subspace of $T_{\mathcal{A},r}(E)$ so the decomposition given by these projections is the one we require. The space $T_{\mathcal{A},r}(E)$ need not be a tensor product as the example of uniform algebras shows, however we can associate a tensor product of rank r with $T_{\mathcal{A},r}(E)$ in a natural way. Let $E_1,\ldots,E_r$ be Banach spaces and E their ℓ_1-direct sum. Then the map

$$E^{\otimes r} \longrightarrow T_{\mathcal{A},r}(E)$$

induces a norm on the subspace $E_1 \otimes \ldots \otimes E_r$ of $E^{\otimes r}$ and this is readily seen to define a tensor product of rank r, which we call α_r. In fact, α_r is the smallest tensor product β such that the multiplication map

$$\beta(A,\ldots,A) \longrightarrow A$$

is a contraction for every $A \in \mathcal{A}$. In particular, every algebra in $\mathcal{A}$ is an α_r-algebra.

From now on we shall consider only the class $\mathcal{A}$ of closed subalgebras of the linear operators on Hilbert spaces. Then a Banach algebra A is an operator algebra if, and only if, it is isomorphic to an element of $\mathcal{A}$. Thus, if A is an operator algebra then there is a constant C such that the map

$$\frac{1}{C} I : A \longrightarrow A$$

extends to a contractive algebra homomorphism

$$\Phi : T_{\mathcal{A}}(A) \longrightarrow A \quad .$$

Conversely, if such a C exists then A is isomorphic to a quotient of the operator algebra $T_{\mathfrak{a}}(E)$. It is known that any quotient of an operator algebra is itself an operator algebra so it follows that A is an operator algebra. Using the decomposition of $T_{\mathfrak{a}}(A)$ we see that A is an operator algebra if, and only if, there is a constant C' such that the multiplication map

$$T_{\mathfrak{a},r}(A) \longrightarrow A$$

has norm $\leq C'^{r}$ for $r = 2,3,\ldots$. For the class of operator algebras one can show that the Banach-Mazur distance of $T_{\mathfrak{a},r}(E)$ from $\alpha_r(E,\ldots,E)$ is at most K^r for some constant K. Thus we obtain the following criterion :

> A is an operator algebra if, and only if, there exists a constant C'' such that the multiplication map
>
> $$\alpha_r(A,\ldots,A) \longrightarrow A$$
>
> has norm $\leq C''^{r}$ for $r = 2,3,\ldots$.

This is a restatement of a result of Varopoulos [7] and it shows that operator algebras can be characterized by the sequence of tensor products α_r. The question arises whether α_2 alone suffices. Charpentier [3] showed that every operator algebra is an H'-algebra for the tensor norm H' introduced by Grothendieck [4]. Tonge [5] [6] complemented this by showing that, for the closely related tensor norm /H', every /H'-algebra is an operator algebra. However, we shall see below that $\alpha_2 = H'$ and not every H'-algebra is an operator algebra. Indeed, the operator algebras cannot be characterized as the β-algebras for any single tensor product β. (See [1] and [2] where this is explained in greater detail.)

<u>Lemma</u> : A linear functional $\varphi : E_1 \otimes \ldots \otimes E_r \to \mathbb{C}$ is a contraction for the α_r-norm if, and only if, there exist Hilbert spaces

$$\mathbb{C} = H_o,\ H_1,\ \ldots\ H_{r-1}, H_r = \mathbb{C}$$

and linear contractions

$$T_n : E_n \longrightarrow \mathrm{Hom}(H_{n-1}, H_n)$$

such that

$$\varphi(e_1 \otimes \ldots \otimes e_n) = T_r(e_r) \circ \ldots \circ T_1(e_1) \quad .$$

<u>Proof</u> : Suppose first that φ is a contraction for the α_r-norm. With $E = E_1 \oplus_1 \cdots \oplus_1 E_r$ we have embeddings

$$\alpha_r(E_1,\ldots,E_r) \hookrightarrow T_{\mathfrak{a}}(E) \hookrightarrow \mathrm{Hom}(K,K)$$

for some Hilbert space K. The Hahn-Banach theorem yields a contraction $\psi : \mathrm{Hom}(K,K) \to \mathbb{C}$ extending φ. Since $\mathrm{Hom}(K,K)$ is a C^*-algebra, there exists a representation $\pi : \mathrm{Hom}(K,K) \to \mathrm{Hom}(H,H)$ and elements $x \in B(H)$, $y \in B(H^*)$ with

$$\psi(a) = \langle y \, , \, \pi(a)x \rangle \quad \text{for } a \in \mathrm{Hom}(K,K) \quad .$$

Then we obtain the desired factorization by setting :

$$H_1 = H_2 = \cdots = H_{r-1} = H \quad \text{and}$$

$$T_1 : E_1 \longrightarrow \mathrm{Hom}(\mathbb{C},H) = H \quad ; \quad e_1 \longmapsto \pi(e_1)x$$

$$T_n : E_n \longrightarrow \mathrm{Hom}(H,H) \quad ; \quad e_n \longmapsto \pi(e_n)$$

$$T_r : E_r \longrightarrow \mathrm{Hom}(H,\mathbb{C}) = H \quad ; \quad e_r \longmapsto y \circ \pi(e_r) \quad .$$

Conversely, if φ factorizes as in the lemma, then we may set $K = H_0 \oplus_2 H_1 \oplus_2 \cdots \oplus_2 H_r$ and consider

$$S_n : E_n \longrightarrow \mathrm{Hom}(K,K) : e_n \longmapsto \begin{pmatrix} \bigcirc & \cdots & \bigcirc \\ \vdots & T_n(e_n) & \vdots \\ \bigcirc & \cdots & \bigcirc \end{pmatrix} .$$

These are contractions, and $\mathrm{Hom}(K,K)$ is an operator algebra, so

$$\alpha_r(E_1,\ldots,E_r) \xrightarrow{\alpha_r(S_1,\ldots,S_r)} \alpha_r(\mathrm{Hom}(K,K),\ldots,\mathrm{Hom}(K,K)) \longrightarrow \mathrm{Hom}(K,K)$$

$$e_1 \otimes \cdots \otimes e_r \longmapsto \begin{pmatrix} & \varphi(e_1 \otimes \cdots \otimes e_r) \\ \bigcirc & \end{pmatrix}$$

is also a contraction. Hence φ has norm ≤ 1. ■

Note especially the case $r = 2$. Then φ is a contraction for the α_2-norm precisely when it factorizes as

$$E_1 \otimes E_2 \xrightarrow{T_1 \otimes T_2} H \otimes H^* \xrightarrow{\text{scalar product}} \mathbb{C} \quad .$$

Grothendieck defined the tensor norm H' by this property, so $\alpha_2 = H'$. This shows that every operator algebra is an H'-algebra. We shall show that this does not characterize the operator algebras by constructing an H'-algebra which is not an α_3-algebra and so certainly not an operator algebra.

The natural place to seek such a counter-example is from the universal tensor algebras. We are only concerned with triple products so let us take three Banach spaces E_1, E_2 and E_3 and consider $T_\alpha(E_1 \oplus E_2 \oplus E_3)$. (Here $\oplus$ can be any direct sum, eg. the ℓ_1-direct sum.) Even in this algebra we can quotient out everything which is not involved in the products $e_1 \cdot e_2 \cdot e_3$ for $e_n \in E_n$. Thus we are led to consider the following situation : let

$$\varphi \; : \; E_1 \otimes E_2 \otimes E_3 \longrightarrow \mathbb{C}$$

be a linear functional which has continuous extensions to both $(E_1 H' E_2) H' E_3$ and $E_1 H' (E_2 H' E_3)$. Then A is the algebra

$$[E_1 \oplus E_2 \oplus E_3] \oplus [(E_1 H' E_2) \oplus (E_2 H' E_3)] \oplus \mathbb{C}$$

with the multiplication

$$(e_1, e_2, e_3 \, ; \, u_{12}, u_{23} \, ; \, \lambda) \cdot (\bar{e}_1, \bar{e}_2, \bar{e}_3 \, ; \, \bar{u}_{12}, \bar{u}_{23} \, ; \, \bar{\lambda}) =$$

$$= (0,0,0 \, ; \, e_1 \otimes \bar{e}_2 \, , \, e_2 \otimes \bar{e}_3 \, ; \, \varphi(e_1 \otimes \bar{u}_{23}) + \varphi(u_{12} \otimes \bar{e}_3)) \; .$$

Our hypotheses ensure that this is an H'-algebra. If it were an operator algebra then the lemma would show that φ factorizes as

$$E_1 \otimes E_2 \otimes E_3 \longrightarrow H_1 \otimes \mathrm{Hom}(H_1, H_2) \otimes H_2^* \xrightarrow{\text{composition}} \mathbb{C} \quad . \qquad (*)$$

For an appropriate choice of φ we shall show that this is impossible.

Set $E_1 = E_3 = \ell_1(\mathbb{Z})$, $E_2 = \ell_2(\mathbb{Z})$ and let $J : \ell_1(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ be the natural injection. The convolution map

$$\Phi : \ell_1(\mathbb{Z}) \otimes \ell_1(\mathbb{Z}) \longrightarrow \ell_2(\mathbb{Z}) \quad ; \quad x \otimes y \longmapsto J(x * y)$$

induces

$$\varphi : \ell_1(\mathbb{Z}) \otimes \ell_2(\mathbb{Z}) \otimes \ell_1(\mathbb{Z}) \longrightarrow \mathbb{C} \quad ; \quad x \otimes z \otimes y \longmapsto \langle z, \Phi(x \otimes y)\rangle$$

and one can readily check that φ has continuous extensions to $\ell_1(\mathbb{Z})\, H'\, (\ell_2(\mathbb{Z})\, H'\, \ell_1(\mathbb{Z}))$ and $(\ell_1(\mathbb{Z})\, H'\, \ell_2(\mathbb{Z}))\, H'\, \ell_1(\mathbb{Z})$ as required. We must show that φ does not factorize as in (*). If it did, then Φ would factorize as

$$\Phi : \ell_1(\mathbb{Z}) \otimes \ell_1(\mathbb{Z}) \xrightarrow{R_1 \otimes R_2} H_1 \hat{\otimes} H_2 \xrightarrow{S} \ell_2(\mathbb{Z})$$

for some continuous linear maps R_1, R_2 and S with $\|R_1\|$, $\|R_2\| \le 1$. In other words, there would be positive Hermitian forms ρ_n on $\ell_1(\mathbb{Z})$ given by $\rho_n(x,y) = \langle R_n x, R_n y\rangle$ with

$$\|\Phi(x \otimes y)\|^2 \le \|S\|^2 \cdot \rho_1(x,x) \cdot \rho_2(y,y) \quad .$$

The symmetry of the convolution operator Φ now enables us to obtain a contradiction.

Let $T : \ell_p(\mathbb{Z}) \to \ell_p(\mathbb{Z})$ be the shift operator, then

$$\Phi(T^a x \otimes y) = T^a \cdot \Phi(x \otimes y) \qquad \text{for each } a \in \mathbb{Z} \quad .$$

So

$$\|\Phi(x \otimes y)\|^2 \le \|S\|^2 \cdot \rho_1(T^a x, T^a x) \cdot \rho_2(y,y) \quad .$$

If $\mathcal{U}$ is a non-trivial ultrafilter on $\mathbb{N}$ then

$$\tilde{\rho}_n(x,y) = \lim_{\mathcal{U}} \frac{1}{2N+1} \sum_{a=-N}^{N} \rho_n(T^a x, T^a y)$$

are positive Hermitian forms and they satisfy

$$\|\Phi(x \otimes y)\|^2 \le \|S\|^2 \cdot \tilde{\rho}_1(x,x) \cdot \tilde{\rho}_2(y,y) \quad . \tag{**}$$

By definition, $\tilde{\rho}_n$ is invariant under the shift operator. As in Bochner's theorem on positive definite functions, this implies that $\tilde{\rho}_n$ must be of the form

$$\tilde{\rho}_n(x,y) = \int \hat{x} \cdot \bar{\hat{y}} \, d\mu_n$$

for some positive measure μ_n on the circle group $\mathbb{T}$ dual to $\mathbb{Z}$. In this case, inequality (**) becomes

$$\int |f_1 \cdot f_2| \, dm \leq \|S\|^2 \cdot \int |f_1| \, d\mu_1 \cdot \int |f_2| \, d\mu_2$$

for $f_1, f_2 \in C(\mathbb{T})$ and the Haar measure m. This certainly implies that m is absolutely continuous with respect to $\mu = \mu_1 + \mu_2$, say $m = g \cdot \mu$ for $g \in L_1(\mu)$. Thus

$$\int |f_1 \cdot f_2 \cdot g| \, d\mu \leq \|S\|^2 \cdot \int |f_1| \, d\mu \cdot \int |f_2| \, d\mu \quad .$$

It is readily established that this can only hold if μ is purely atomic and, since m is not purely atomic, this gives a contradiction.

REFERENCES

[1] T.K. Carne : Tensor products and the geometry of Banach spaces, Thesis, Cambridge (1979).

[2] T.K. Carne : Not all H'-algebras are operator algebras , Proc. Cambridge Philos. Soc. 86 (1979), 243-249.

[3] Ph. Charpentier : Q-algèbres et produits tensoriels topologiques, Thèse, Orsay (1973).

[4] A. Grothendieck : Résumé de la théorie métrique des produits tensoriels topologiques, Bol. Soc. Mat. São Paulo 8 (1956), 1-79.

[5] A.M. Tonge : Banach algebras and absolutely summing operators, Proc. Cambridge Philos. Soc. 80 (1976), 465-473.

[6] A.M. Tonge : Sur les algèbres de Banach et les opérateurs p-sommants, Exposé No 13, Séminaire Maurey-Schwartz 1975-76, Ecole Polytechnique, Palaiseau.

[7] N. Th. Varopoulos, A theorem on operator algebras, Math. Scand. 37 (1975), 173-182.
