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AN ANALOGUE IN COMMUTATIVE HARMONIC ANALYSIS OF THE UNIFORM BOUNDED APPROXIMATION PROPERTY OF BANACH SPACE

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§ 0. PROLOGUE.

Recall [5] that a Banach space X has the ubap (= uniform bounded approximation property) if

(*) there are a k with $1 \leq k < \infty$ and a positive sequence $(q(m))$ such that given a finite dimensional subspace $E \subset X$ there exists an operator $u : X \to X$ satisfying the following conditions

(i) $u(x) = x$ for $x \in E$,

(ii) $\|u\| \leq k$,

(iii) $\dim u(X) = q(\dim E)$.

It is known ([5] [4]) that the L^p-spaces and $C(K)$-spaces and reflexive Orlicz spaces have the ubap. In contrast with the usual (bounded) approximation property, X has ubap iff the dual X^* has ubap iff every ultrafilter modeled on X has ubap ([3] [4]).

The study of translation invariant function spaces on compact Abelian groups led us to consider the translation invariant analogue of the ubap ; roughly speaking we modify (*) assuming that X, E, and u are translation invariant.

§ 1. PRELIMINARIES.

In the sequel G is a compact Abelian group, Γ its dual -the discrete group of characters of G (= the continuous homomorphisms of G into the circle group)- m, the normalized Haar measure of G. For $a \in G$ the translation operator τ_a is defined by $(\tau_a f)(x) = f(x - a)$ for f m-measurable function on G and for $x \in G$. A vector space X of m-equivalence classes of m-measurable functions on G is translation invariant if $\tau_a X \subset X$ for every $a \in G$. An operator $u : X \to Y$ acting between translation invariant vector spaces is translation invariant if $\tau_a u = u \tau_a$ for every $a \in G$.

By $L^1(G)$ we denote as usual the Banach space of the m-equivalence classes of m-measurable and m-absolutely integrable complex-valued functions on G ; with the norm $\|f\|_1 = \int_G |f(x)| m(dx)$. For f, $g \in L^1(G)$ we define the convolution $f * g \in L^1(G)$ by $(f * g)(x) = \int_G f(x-a)\, g(a)\, m(da)$.

We shall deal with translation invariant Banach spaces for which the operator of convolution by an L^1 function has the operator norm bounded by the norm of the function. To this end it suffices to impose the following conditions on the space ; they can be obviously weakened in various ways.

<u>Definition 1.1</u> : A translation invariant Banach space X is called regular if

(h.0) if $f \in X$ then $f \in L^1(G)$; moreover the inclusion $X \to L^1(G)$ is one to one and continuous ;

(h.1) the translation $\tau_a : X \to X$ is an isometry ;

(h.2) for every $f \in X$ the map $a \to \tau_a f$ (from G into X) is continuous.

Note that : 1^o) Every closed translation invariant subspace of a regular translation invariant Banach space is regular itself.
2^o) If E is a finite dimensional translation invariant subspace of a regular translation invariant Banach space, then $E = \{f = \sum_{\gamma \in M} c_\gamma \gamma,\ c_\gamma$ complex numbers, $M = E \cap \Gamma\}$. (Hint : Use (h.0) and check 2^o) for the space $L^1(G)$.)

Next we have :

<u>Proposition 1.1</u> : Let X be a regular translation invariant Banach space. For every $g \in L^1(G)$ define the operator u_g of convolution with g by the X-valued integral

$$u_g(f) = \int \tau_a f \cdot g(a)\, m(da) \qquad \text{for } f \in X \quad .$$

Then $u_g : X \to X$ is a bounded linear operator, precisely $\|u_g\| \le \|g\|_1$; regarding $u_g(f)$ as a function in $L^1(G)$ we have $u_g(f) = f * g$ for every $f \in X$.

<u>Proof</u> : It follows from (h.2) that the X-valued integral $\int \tau_a f\, g(a)\, m(da)$ exists. Thus, by (h.0), it equals $f * g$. Finally, by (h.1),

$$\|u_g(f)\| \le \int \|\tau_a f\|\ |g(a)|\ m(da) = \|g\|_1\ \|f\| \quad . \qquad \text{Q.E.D.}$$

§ 2. THE MAIN RESULT.

We begin by introducing the translation invariant analogue of ubap.

Definition 2.1 : A translation invariant Banach space X is said to have the invariant uniform approximation property, abreviated "inv. ubap" if

(**) there are a k with $1 \le k < \infty$ and a positive sequence $(q(m))$ such that given a finite dimensional translation invariant subspace E of X there exists a translation invariant operator $u : X \to X$ satisfying the following conditions

(i) $u(x) = x$ for $x \in E$,

(ii) $\|u\| \le k$,

(iii) $\dim u(X) \le q(\dim E)$.

Now we are ready to state the main result of this paper.

Theorem 2.1 : Every regular translation invariant Banach space has the inv. ubap.

To prove Theorem 2.1 it is enough in fact to establish it for the space $L^1(G)$ which is in fact equivalent to a result in Harmonic Analysis (cf. Theorem 2.2 below). Recall that the Fourier transform of a $g \in L^1(G)$ is the complex valued function $\hat{g}$ on Γ defined by $\hat{g}(\gamma) = \int_G g(x)\, \overline{\gamma}(x)\, m(dx)$. Let $S(g) = \{\gamma \in \Gamma : \hat{g}(\gamma) \neq 0\}$. If $M \subset \Gamma$, then $|M|$ denotes the cardinality of M.

Theorem 2.2 : For every k with $1 < k < \infty$, there exists a positive sequence $(q_k(n))$ such that for every finite set $M \subset \Gamma$ there exists a $g \in L^1(G)$ such that

(j) $\hat{g}(\gamma) = 1$ for $\gamma \in M$,

(jj) $\|g\|_1 \le k$,

(jjj) $|S(g)| \le q_k(|M|)$.

To derive Theorem 2.1 from Theorem 2.2 fix k and a translation invariant finite dimensional subspace E of X. By remark 2^o) after

Definition 1.1, $E = \{f \in L^1(G) : S(f) \subset M\}$ where $M = E \cap \Gamma$. Clearly $\dim E = |M|$. Pick $g \in L^1(G)$ satisfying (j) - (jjj) for this M and $u = u_g$. Then (j) implies (i), (jj) implies (ii) (via Proposition 1.1), and (iii) and (h.0) implies (jjj).

For the proof of Theorem 2.2 it is convenient to introduce more notation. For $M \subset \Gamma$ we denote by χ_M the characteristic function of M. By $\ell^1(\Gamma)$ we denote the Banach space of all complex valued functions φ on Γ with $\|\varphi\|_1 = \sup_M \sum_{\gamma \in M} |\varphi(\gamma)|$ where the supremum is taken over all finite subsets M of Γ. For $\varphi, \psi \in \ell^1(\Gamma)$ we define $\varphi * \psi \in \ell^1(\Gamma)$ by $(\varphi * \psi)(\gamma) = \sum_{\sigma \in \Gamma} \varphi(\gamma - \sigma)\psi(\sigma)$. For $\varphi \in \ell^1(\Gamma)$ the Fourier transform of φ is the function $\hat{\varphi}$ defined by $\hat{\varphi}(x) = \sum_{\gamma \in \Gamma} \varphi(\gamma)\gamma(x)$ for $x \in G$. Finally if M and N are subsets of Γ and $\gamma \in \Gamma$ then $M + N$, $\gamma + M$ and $-M$ have the usual meaning : $M + N = \{\sigma \in \Gamma : \sigma = \gamma_1 + \gamma_2$ with $\gamma_1 \in M$ and $\gamma_2 \in N\}$, $\gamma + M = \{\gamma\} + M$, and $-M = \{\sigma \in \Gamma : -\sigma \in M\}$.

The proof of Theorem 2.2 is based upon the next lemmas :

<u>Lemma 2.1</u> : Let $\varepsilon > 0$. Assume that for a finite set $M \subset \Gamma$ there exists a finite set W such that

$$|M + W| \le (1 + \varepsilon)|W| \quad . \tag{1}$$

Let

$$g = |W|^{-1} \widehat{\chi_{W+M} * \chi_{-W}} = |W|^{-1} \hat{\chi}_{W+M} \cdot \hat{\chi}_{-W} \quad .$$

Then

$$\hat{g}(\gamma) = 1 \text{ for } \gamma \in M \ , \quad \|g\|_1 \le (1 + \varepsilon)^{1/2} \ , \quad |S(g)| \le (1 + \varepsilon)|W|^2 \quad .$$

<u>Proof</u> : Clearly $\hat{g} = |W|^{-1} \chi_{W+M} * \chi_{-W}$. If $\gamma \in M$, then

$$\begin{aligned} |W|\hat{g}(\gamma) &= \sum_{\sigma \in \Gamma} \chi_{W+M}(\gamma - \sigma)\, \chi_{-W}(\sigma) \\ &= |\{\sigma : \gamma - \sigma \in W + M\} \cap \{\sigma : -\sigma \in W\} \\ &= |\{\sigma : -\sigma \in W\}| = |W| \quad . \end{aligned}$$

Thus $\hat{g}(\gamma) = 1$.

Using the Schwarz inequality, Parseval identity, and (1) we get

$$\begin{aligned}|W|\ \|g\|_1 &= \int_G |\hat{\chi}_{W+M}(x) \cdot \hat{\chi}_{-W}(x)|\,m(dx)\\ &\le \left(\int_G |\chi_{W+M}(x)|^2\, m(dx)\right)^{1/2} \left(\int_G |\chi_{-W}(x)|^2 m(dx)\right)^{1/2}\\ &\le |W+M|^{1/2}\ |W|^{1/2}\\ &\le (1+\varepsilon)^{1/2}\ |W| \quad .\end{aligned}$$

Hence $\|g\|_1 \le (1+\varepsilon)^{1/2}$.

Finally if $\hat{g}(\gamma) \neq 0$, then $\sum_{\sigma\in\Gamma} \chi_{W+M}(\gamma - \sigma)\chi_{-W}(\sigma) \neq 0$. Thus $\gamma \in \sigma + (W+M)$ for some $\sigma \in -W$. Hence $\gamma \in -W + (W+M)$. Thus, by (1), $|S(g)| \le |-W + (W+M)| \le |W|\,|W+M| \le (1+\varepsilon)|W|^2$. Q.E.D.

To complete the proof of Theorem 2.2, in view of Lemma 2.1 we have to construct for a given set $M \subset \Gamma$ a set $W \subset \Gamma$ so that (1) is satisfied and the cardinality of W depends on the cardinality of M only. Without loss of generality one may assume that M contains the neutral element 0 of Γ. The next Lemma goes back to Følner [2].

<u>Lemma 2.2</u> : Let m and n be positive integers. Let $M = \{\sigma_1, \sigma_2, \dots, \sigma_m\} \subset \Gamma$ with $\sigma_1 = 0$. Let

$$W_n = \{0,\sigma_1,2\sigma_1,\dots,n\sigma_1\} + \{0,\sigma_2,2\sigma_2,\dots,n\sigma_2\} + \dots$$
$$\dots + \{0,\sigma_m,2\sigma_m, \dots, n\sigma_m\} \quad .$$

Then

$$|M + W_n| \le \left(1 + \frac{m}{n+1}\right)|W_n| \quad .$$

Lemma 2.2 is an easy consequence of the next one :

<u>Lemma 2.3</u> : Let $0 \in F \subset \Gamma$ with $|F| < \infty$ and let $\sigma \in \Gamma$. Let $F_o = F$, $F_n = \{0,\sigma,2\sigma,\dots,n\sigma\} + F$ for $n = 1,2,\dots$ Then

(2) $$|(\sigma + F_n) \setminus F_n| \le (n+1)^{-1}\ |F_n| \quad .$$

<u>Proof</u> : Put $S_o = F$, $S_n = F_n \setminus F_{n-1}$ for $n = 1,2,\ldots$ Then, for $n \geq 1$,

(3) $$S_n \subset \sigma + S_{n-1} \quad .$$

The case $n = 1$ is trivial. Let $n \geq 2$ and let $\tau \in S_n$. Then $\tau \in F_n$ and $\tau \notin F_{n-1}$. Hence, for some $\varphi \in F$, $\tau = n\sigma + \varphi = \sigma + (n-1)\sigma + \varphi$. Claim : $(n-1)\sigma + \varphi \in S_{n-1}$. Otherwise, for some $\varphi_1 \in F$, $(n-2)\sigma + \varphi_1 = (n-1)\sigma + \varphi \in F_{n-2}$; thus $\tau = (n-1)\sigma + \varphi_1 \in F_{n-1}$, a contradiction. This proves (3).

Thus

(4) $$|F| = |S_o| \geq |S_1| \geq \ldots \geq |S_n| \geq |S_{n+1}| \quad .$$

Clearly

$$F_n = (F_n \setminus F_{n-1}) \cup (F_{n-1} \setminus F_{n-2}) \cup \ldots \cup (F_1 \setminus F_o) \cup F_o \quad .$$

Thus

$$\begin{aligned} |F_n| &= \sum_{k=0}^{n-1} |F_{n-k} \setminus F_{n-k-1}| + |F| \\ &= \sum_{k=0}^{n} |S_{n-k}| \quad . \end{aligned}$$

Hence, in view of (4),

(5) $$|F_n| \geq (n+1)\, |S_n| \quad .$$

Next observe that

(6) $$(\sigma + F_n) \setminus F_n = S_{n+1} \quad .$$

Indeed $(\sigma + F_n) \setminus F_n \subset F_{n+1} \setminus F_n = S_{n+1}$. Conversely if $\tau \in S_{n+1}$ then $\tau \notin F_n$ and $\tau = (n+1)\sigma + \varphi$ for some $\varphi \in F$. Thus $\tau = \sigma + n\sigma + \varphi \in \sigma + F_n$. Hence $S_{n+1} \subset (\sigma + F_n) \setminus F_n$. This proves (6).

Combining (4), (5) and (6) we get

$$|(\sigma + F_n) \setminus F_n| = |S_{n+1}| \leq |S_n| \leq (n+1)^{-1} |F_n| \quad . \qquad \text{Q.E.D.}$$

Proof of Lemma 2.2 : Fix n and for $i = 1,2,\dots,m$ put

$$F^i = \{0,\sigma_1,\dots,n\sigma_1\} + \{0,\sigma_2,\dots,n\sigma_2\} + \dots + \{0,\sigma_{i-1},\dots,n\sigma_{i-1}\} +$$

$$+ \{0,\sigma_{i+1},\dots,n\sigma_{i+1}\} + \dots + \{0,\sigma_m,\dots,n\sigma_m\} \quad .$$

Then $W_n = \{0,\sigma_i,\dots,n\sigma_i\} + F^i$ for $i = 1,2,\dots,m$. Thus applying Lemma 2.3 for $\sigma = \sigma_i$ and $F = F^i$ we get

$$|(\sigma_i + W_n)\setminus W_n| \le (n+1)^{-1}|W_n| \qquad (i = 1,2,\dots,m) \quad .$$

Thus

$$|M + W_n| \le |W_n| + |M + W_n \setminus W_n| \le |W_n| + \sum_{i=1}^{m} |(\sigma_i + W_n)\setminus W_n|$$

$$\le (1 + \frac{m}{n+1})|W_n| \quad . \qquad \text{Q.E.D.}$$

Proof of Theorem 2.2 : Put $\varepsilon = k^2 - 1$. Let $M \subset \Gamma$ with $0 \in M$ and $|M| = m < \infty$ be given. Pick n so that $\frac{m}{n+1} \le \varepsilon$, say $n = [\frac{m}{\varepsilon}] \le \frac{m}{k^2-1}$. If W_n is that of Lemma 2.2, then $|W_n| \le n^m$. For M and W_n construct g as in Lemma 2.1. Then

$$\|g\|_1 \le (1+\varepsilon)^{1/2} = k$$

and

$$|S(g)| \le (1+\varepsilon)|W_n|^2 \le k^2 n^{2m} \le k^2\left(\frac{m}{k^2-1}\right)^{2m} = q_k(m) \quad . \qquad \text{Q.E.D.}$$

§ 3. FINAL REMARKS.

1^o. A routine argument using duality between $L^1(G)$ and $L^\infty(G)$ gives that the assertion of Theorem 2.2 is equivalent to the following.

For every k with $1 < k < \infty$ there exists a sequence $(q_k(m))$ such that for every subset $M \subset \Gamma$ there exists a set W with $M \subset W \subset \Gamma$ and $|W| \le q_k(|M|)$ such that if $h \in L^\infty(G)$ and $\hat{h}(\gamma) = 0$ for $\gamma \in W \setminus M$ then $|\sum_{\gamma\in M} \hat{h}(\gamma)| \le k\,\|h\|_\infty$.

2^o. If for some set $M \subset \Gamma$ with $0 \in M$ there exists a $g \in L^1(G)$ such that $\hat{g}(\gamma) = 1$ for $\gamma \in M$, $|S(g)| < \infty$ and $\|g\|_1 = 1$ then M is contained in a finite subgroup Γ_o of Γ with $|\Gamma_o| \leqslant |S(g)|$.

Proof : Let $\varphi_n = \sum_{j=0}^{n} (n+1)^{-1}(\hat{g})^j$. Then φ_n tends (in $\ell^1(\Gamma)$) to the characteristic function of a subset, say Γ_o, of Γ. Clearly $M \subset \Gamma_o \subset S(g)$ and $\|\hat{\chi}_{\Gamma_o}\|_1 = \lim_n \|\hat{\varphi}_n\|_1 \le \|g\|_1 = 1$. Thus $\|\hat{\chi}_{\Gamma_o}\|_1 = 1$. Since $0 \in M \subset \Gamma_o$, the proof of Cohen's idempotent measure theorem (cf. [1]) yields that Γ_o is a subgroup of Γ. Q.E.D.

3^o. It follows from 2^o that for arbitrary compact Abelian group Theorem 2.2 can not be extended to the case $k = 1$. In fact we have :

Proposition 3.1 : A compact Abelian group G satisfies the assertion of Theorem 2.2 for $k = 1$ iff there exists a positive integer n_o such that G is a product of a family of cyclic groups $(Z_{n(\alpha)})_{\alpha \in A}$ with $n(\alpha) \le n_o$ for $\alpha \in A$.

Proof : The assumption on G yields that every finite subset $M \subset \Gamma$ generates a subgroup Γ_o of Γ with $|\Gamma_o| \le (n_o!)^{|M|}$. We define $g = \hat{\chi}_{\Gamma_o}$.

It follows from 2^o that the condition imposed on G is necessary.

4^o. In particular if G is the Cantor group $(\mathbb{Z}_2)^{\underline{n}}$ ($\underline{n}$ any cardinal number) then every M with $0 \in M$ and $|M| = m$ generates a subgroup Γ_o with $|\Gamma_o| \le 2^{m-1}$. Hence in this case one gets $q_1(m) = 2^{m-1}$ $(m = 1, 2, \dots)$. From this fact one gets that for $(\mathbb{Z}_2)^{\underline{n}}$ one has $q_k(m) \le (k+1)2^{m/k}$.

5^o. No satisfactory estimation from below for $(q_k(m))$ seems to be known even in the case of the Cantor group $(\mathbb{Z}_2)^{\underline{n}}$.

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