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SUBSPACES OF L^p WHICH DO NOT CONTAIN L^p-ISOMORPHICALLY

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The work discussed here is due jointly to G. Schechtman and myself. To formulate our main result, let me first introduce some definitions and notation. Given Banach spaces X and Y, $X \hookrightarrow Y$ means : X is isomorphic (linearly homeomorphic) to a subspace of Y. Given a class $\mathcal{K}$ of Banach spaces and a Banach space B, we say that B is universal for $\mathcal{K}$ if $E \hookrightarrow B$ all $E \in \mathcal{K}$. As usual, L^p denotes $L^p([0,1])$ (or L^p of any atomless separable probability space). "Subspace" means "closed infinite dimensional linear subspace" unless otherwise stated.

Our principal discovery is as follows :

<u>Main Theorem</u> : <u>Let</u> $1 \leq p < \infty$, $p \neq 2$, <u>and let</u> $\mathcal{K}_p$ <u>denote the class of all subspaces</u> X <u>of</u> L^p <u>such that</u> $L^p \not\hookrightarrow X$. <u>Let</u> B <u>be a separable Banach space universal for</u> $\mathcal{K}_p$. <u>Then</u> $L^p \hookrightarrow B$.

We shall just outline the essential steps in the proof. Full details and additional information may be found in [6].

An immediate consequence of the Theorem is that $\mathcal{K}_p$ contains no element universal for $\mathcal{K}_p$ itself. It then follows easily by transfinite induction that there exists a family $(R_\alpha^p)_{\alpha<\omega_1}$ of subspaces of L^p so that for $\alpha < \beta < \omega_1$, $R_\alpha^p \subset R_\beta^p$ and $R_\beta^p \not\hookrightarrow R_\alpha^p$ (ω_1 denotes the first uncountable ordinal). In reality, we prove the theorem by first constructing such a family (or rather, just that for each α, there exists a $\beta > \alpha$ with $R_\beta^p \not\hookrightarrow R_\alpha^p$). The family that we obtain does have certain additional special properties ; for example, for $1 < p < \infty$, the spaces R_α^p all have unconditional bases ; for $p = 1$, the spaces all have the Radon-Nikodym property. However we do not know if, for $1 < p < \infty$, the spaces are all complemented in L^p ; nor do we know if the word "complemented" can be inserted before "subspaces" in the statement of the Main Theorem.

There are two essential desiderata to be satisfied in carrying out the construction of the R_α^p's. The first is to guarantee that $L^p \not\hookrightarrow R_\alpha^p$ all α. The second is to guarantee that if B separable is such that $R_\alpha^p \hookrightarrow B$ all $\alpha < \omega_1$, then $L^p \hookrightarrow B$.

To insure the first desideratum, we obtain the following result, of independent interest :

Theorem 1 : Let $1 < p < \infty$ and let X be a Banach space with an unconditional Schauder decomposition (X_j) (that is, for each $x \in X$, there exists a unique sequence (x_j) with $x_j \in X_j$ for all j, such that $\sum x_j = x$, the series unconditionally converge). Assume that L^p is isomorphic to a completemented subspace of X. Then one of the following holds :

(1) there exists an i such that L^p is isomorphic to a complemented subspace of X_i ;

(2) a block basic sequence of the X_i's is equivalent to the Haar basis of L^p and has closed linear span complemented in X.

(A sequence (b_i) in X is called a block basic sequence of the X_i's if there exist $x_j \in X_j$ and integers $n_1 < n_2 < \dots$ with

$$b_i = \sum_{j=n_i}^{n_{i+1}-1} x_j \quad \text{for all } i \quad .)$$

We do not know if Theorem 1 holds for $p = 1$. To employ Theorem 1 in our proof of the Main Theorem, we also make crucial use of the result established in [3] : If $1 < p < \infty$ and $Y \subset L^p$ is such that $L^p \hookrightarrow Y$, then there exists a $Z \subset Y$ with Z isomorphic to L^p and complemented in L^p.

To insure the second desideratum, we employ natural ideas concerning partially ordered sets. The ideas have their origin in the classical discussion of analytic sets, and were recently introduced in Banach space theory by J. Bourgain [1].

Before proceeding to these ideas, however, we wish to define the spaces R^p_α. We obtain them by alternating the construction of independent sums and disjoint sums of spaces of random variables.

By a "space of random variables" we mean a linear subspace of $L^0(P)$ for some probability space $(\Omega, \mathcal{S}, P)$; "$L^0(P)$" denotes the space of all (equivalence classes of) real-valued measurable functions defined on Ω. Given a random variable x defined on Ω, $\operatorname{dist} x$ denotes the probability measure defined on the Borel subsets of the reals by $(\operatorname{dist} x)(E) = P(\{\omega : x(\omega) \in E\})$. Given spaces of random variables X, Y on possibly different probability spaces, we say X and Y are distributionally isomorphic if there exists a linear bijection $T : X \to Y$ so that $\operatorname{dist} Tx = \operatorname{dist} x$ for all $x \in X$. It is important for the inductive definition of the R^p_α's that they are "distributionally presented" ; i.e. the isometric Banach space structure itself is not sufficient to define the family.

Now suppose $(\Omega,\mathcal{S},P)$ is an atomless probability space and B is a subspace of $L^p(\Omega)$. We denote by $(B\oplus B)_p$ a space of random variables Y on Ω such that there exist sets $E_i\in\mathcal{S}$ and spaces B_i of random variables with $E_1\cap E_2=\emptyset$, $P(E_i)=\frac{1}{2}$, $B_i\subset L^p(2P|E_i)$ (i.e. regarding $2P|E_i$ as a probability measure on $\mathcal{S}\cap E_i$ with $b(t)=0$, $t\notin E_i$ all $b\in B_i$) so that B_i is distributionally isomorphic to B for $i=1,2$; and $Y=B_1+B_2$. Phrased another way, consider the product probability space $\Omega\times[0,1]$; for b defined on Ω, f defined on $[0,1]$, let $b\otimes f$ denote the function :

$$(b\otimes f)(\omega,s) = b(\omega)f(s) \quad \text{all } \omega\in\Omega,\ s\in[0,1] \quad .$$

Then $(B\oplus B)_p$ is a space of random variables distributionnally isomorphic to the subspace Z of $L^p(\Omega\times[0,1])$ defined by

$$Z = \{b\otimes f : b\in B,\ f=\alpha\chi_{[0,\frac{1}{2}]}+\beta\chi_{[\frac{1}{2},1]} \quad \text{for } \alpha,\ \beta \text{ arbitrary reals}\} \quad .$$

We now define independent sums of sequences of spaces of random variables in L^p. Let $B_1,B_2,\dots$ subspaces of $L^p(\Omega)$. $(\sum_{i=1}^{\infty} B_i)_{\mathrm{Ind},p}$ denotes a space of random variables Y on Ω such that there exist independent σ-subalgebras $\mathcal{Q}_1,\mathcal{Q}_2,\dots$ of $\mathcal{S}$ and for each i a subspace $\overline{B}_i$ of $L^p(P|\mathcal{Q}_i)$ with B_i distributionally isomorphic to $\overline{B}_i$ and Y equal to the closed linear span in $L^p(\Omega)$ of $\overline{B}_1,\overline{B}_2,\dots$ Phrased another way, consider the infinite product probability space $\Omega^{\mathbb{N}}$ ($\mathbb{N}$ denotes the set of positive integers) endowed with the countable product of P with itself in $\mathcal{S}^{\mathbb{N}}$. Fix i, $b_i\in B_i$, and define $\overline{b}_i$ by $\overline{b}_i(\omega)=b_i(\omega_i)$ for all $\omega\in\Omega^{\mathbb{N}}$. Let $\overline{B}_i=\{\overline{b}_i : b_i\in B_i\}$. Then $(\sum B_i)_{\mathrm{Ind},p}$ is a space of random variables distributionnally isomorphic to the closed linear span of the $\overline{B}_i$'s in $L^p(\Omega^{\mathbb{N}})$.

It is worth pointing out that <u>if</u> $\int b\,dP=0$ for all i and $b\in B_i$, then $(\sum B_i)_{\mathrm{Ind},p}$ has a natural unconditional Schauder decomposition, $\overline{B}_1,\overline{B}_2,\dots$ in our above discussion. If however $1\in B_i$ for all i, the independent sum is not even a direct sum. In this case, we simply let $B_i^o=\{b\in B_i : \int b\,dP=0\}$. Then $(\sum B_i)_{\mathrm{Ind},p}=(\sum B_i^o)_{\mathrm{Ind},p}+[1]$ (here $[1]$ denotes the space of constant functions on Ω).

With these somewhat pedantic preliminaries out of the way, we can quickly construct our family of spaces (R^p_α).

Definition 1 : Let $1 \le p < \infty$. Let $R^p_0 = [1]$. Let α be an ordinal with $0 < \alpha < \omega_1$, and suppose R^p_γ has been defined for all $\gamma < \alpha$. If α is a successor ordinal, i.e. $\alpha = \gamma + 1$ for some γ, let $R^p_\alpha = (R^p_\gamma \oplus R^p_\gamma)_p$. If α is a limit ordinal, let $R^p_\alpha = (\sum_{\gamma<\alpha} R_\gamma)_{\mathrm{Ind},p}$.

Now it follows easily from Theorem 1 and the known result following it, that $L^p \not\hookrightarrow R^p_\alpha$ (for $1 < p < \infty$, $p \neq 2$). Indeed, suppose proved true for all $\gamma < \alpha$, α fixed. Then α cannot be a successor ordinal, since e.g. $\alpha = \gamma + 1$, $L^p \hookrightarrow R^p_\alpha \Rightarrow L^p \hookrightarrow R^p_\gamma$ by Theorem 1, possibility 1. Hence α must be a limit ordinal. It easily follows that then if $L^p \hookrightarrow R^p_\alpha$, $L^p \hookrightarrow (\sum_{\gamma<\alpha} (R^p_\gamma)^o)_{\mathrm{Ind},p}$; so again by Theorem 1, possibility 2, some block basic sequence (b_i) of the $\overline{(R^p_\gamma)^o}$'s is equivalent to the Haar basis of L^p. But (b_i) is a sequence of independent mean-zero random variables ; it follows easily from the techniques of [4] and [7] that $L^p \not\hookrightarrow [b_i]$ (in fact $(\ell^2 \oplus \ell^2 \oplus \dots)_p \not\hookrightarrow [b_i]$). For the case $p = 1$, it is easily proved by induction that R^1_α has the RNP for all α ; hence since L^1 fails the RNP, $L^1 \not\hookrightarrow R^1_\alpha$ all α.

As mentioned above, the spaces R^p_α have unconditional bases for all $1 < p < \infty$. In fact, we prove in [6] that for each α, there exists a martingale difference sequence (d^α_j) with R^p_α equal to the closed linear span of (d^α_j) in L^p. As we shall see later, this yields the following improvement of the Main Theorem : Let $1 < p < \infty$, $p \neq 2$ and $K^u_p = \{Y \subset L^p : Y$ has an unconditional basis and $L^p \not\hookrightarrow Y\}$. If B is separable and universal for K^u_p, then $L^p \hookrightarrow B$.

The result is definitely false for $p = 1$; in fact, it is known (see [8]) that K^u_1 has a universal element. Thus, for α sufficiently large, R^1_α has no unconditional basis. Since the R^1_α's all have the RNP, we obtain the following replacement :

Let $\mathcal{C}$ denote the class of all subspaces of L^1 with the RNP, and let B be a separable Banach space universal for $\mathcal{C}$. Then $L^1 \hookrightarrow B$.

In previous unpublished work, M. Talagrand has obtained that the class of all separable spaces with the RNP has no universal element .

As mentioned in the introduction, we do not know (for $1<p<\infty$, $p\neq 2$) if the spaces R^p_α are all complemented in L^p, or if they are all isomorphic to complemented subspaces of L^p ; that is, are they all $\mathcal{L}_p$-spaces ? (An affirmative answer would of course give a positive solution to the open problem : are there uncountably many isomorphically distinct $\mathcal{L}_p$ spaces, for $1<p<\infty$, $p\neq 2$? For the fact that there are infinitely many such see [7].)

This question is equivalent to one involving a family of finite dimensional spaces, without any appearance of transfinite constructions. Let us recall that a subspace X of a Banach space Y is said to be K-complemented if there exists a surjective projection $P: Y\to X$ with $\|P\|\leq K$. For $B\subset L^p$, n a positive integer, we let $(\bigoplus_{i=1}^{n} B)$ denote the distributionally-defined sum of n disjoint copies of B (thus $(\bigoplus_{i=1}^{2} B)_p=(B\oplus B)_p$ as already defined), and similarly $(\sum_{i=1}^{n} B)_{\mathrm{Ind}}$ denotes the distributionally-defined sum of n independent copies of B. Now fixing n, we define a sequence of finite-dimensional spaces of random variables by

$$B_1^n=\left(\left[\,\overset{n}{\underset{i=1}{1}}\,\right]\right)_p$$

(i.e. B_1^n is the "natural" representation of ℓ_n^p in L^p) and if j is a positive integer, B_{2j-1}^n already defined, then

$$B_{2j}^n=\left(\sum_{i=1}^{n} B_{2j-1}^n\right)_{\mathrm{Ind}}\quad,$$

$$B_{2j+1}^n=\left(\bigoplus_{i=1}^{n} B_{2j}^n\right)_p\quad.$$

The question : Are the R^p_α's all complemented in L^p ? , is then equivalent to : <u>Is there a</u> $K_p<\infty$ <u>so that</u> B_m^n <u>is</u> K_p<u>-complemented in</u> L^p <u>for all</u> n, m ? It can be shown that B_m^n is $K_{p,m}$ complemented ; i.e. with constant independent of n.

We now proceed to the second desideratum. For B a separable Banach space and $1\leq p<\infty$, we define an ordinal number, $h_p(B)$, with $0<h_p(B)\leq\omega_1$, called the <u>local</u> L^p<u>-index of</u> B. This index is similar to one defined by Bourgain in [1].
For a discussion of the local L^∞-index and its connection with the classical theory of analytic sets, see our expository paper [5].
The main features of the index are summarized as follows :

__Theorem 2__ : __Let__ X, Y, __and__ B __be separable Banach spaces__, $1 \le p < \infty$,

(a) $h_p(B) = \omega_1$ __if and only if__ $L^p \hookrightarrow B$.

(b) __If__ $X \hookrightarrow Y$, $h_p(X) \le h_p(Y)$.

We then demonstrate that $h_p(R_\alpha^p) \ge \alpha + 1$ for all $\alpha < \omega_1$, all p. The Main Theorem follows immediately from our previous discussion, the above, and Theorem 2.

We now briefly sketch the definition of the local L^p-index. Let B be a separable Banach space, $D_n = \{(\varepsilon_1, \dots, \varepsilon_n) : \varepsilon_i = 0 \text{ or } 1 \text{ all } i\}$, $1 \le p < \infty$. We define a partial order in $\bigcup_{n=0}^{\infty} B^{D_n}$ as follows : if $u \in B^{D_n}$, let $|u| = n$; if $|u| = n$, $|v| = n+1$, we set $u < v$ if $u(\varepsilon) = \dfrac{v(\varepsilon,0) + v(\varepsilon,1)}{2^{1/p}}$ for all $\varepsilon \in D_n$. More generally, if $|u| = m$, $|v| = n$, we set $u < v$ provided $n < n$ and $u(\varepsilon) = 2^{-\frac{n-m}{p}} \sum_{\tau \in D_{n-m}} v(\varepsilon,\tau)$ all $\varepsilon \in D_m$.

Now let $0 < \delta \le 1$ and let $\overline{B}^\delta$ denote the set of all $u \in \bigcup_{n=0}^{\infty} B^{D_n}$ so that

$$\delta\Big(\sum_{x \in D_n} |c(x)|^p\Big)^{1/p} \le \Big\| \sum_{x \in D_n} c(x)\, u(x) \Big\| \le \Big(\sum_{x \in D_n} |c(x)|^p\Big)^{1/p}$$

for all $c \in \mathbb{R}^{D_n}$ provided $|u| = n$. For ease in notation, we set $\overline{B}^1 = \overline{B}$; evidently the elements u of $\overline{B}$ of rank 1, i.e. $|v| = n$, simply correspond to the 2^n-tuples of B isometrically equivalent to the usual $\ell^p_{2^n}$-basis. We identify the elements of $\overline{B}$ of rank 0 with the elements of B of norm one. The following is now easily established :

__Proposition 3__ : $L^p \hookrightarrow B$ __if__ (__and only if__) __there exist__ $0 < \delta \le 1$ __and elements__ $u_1, u_2, \dots$ __in__ $\overline{B}^\delta$ __with__ $u_n < u_{n+1}$ __for all__ n.

An equivalent formulation : if $L^p \not\hookrightarrow B$, the every non-empty subset of $\overline{B}^\delta$ has a maximal element with respect to $<$. We now start "erasing" the maximal elements from $\overline{B}^\delta$.

__Definition 2__ : __Let__ $0 < \delta \le 1$ __and fix__ p, $1 \le p < \infty$. __Let__ $H_0^\delta(B) = \overline{B}^\delta$. __Suppose__ $\alpha > 0$ __and__ $H_\gamma^\delta(B)$ __defined for all__ $\gamma < \alpha$. __If__ $\alpha = \gamma + 1$ __for some__ γ, __let__ $H_\alpha^\delta(B) = \{u \in H_\gamma^\delta(B)$: __there exists a__ $v \in H_\gamma^\delta(B)$ __with__ $u < v\}$. __If__ α __is a limit ordinal__, __let__ $H_\alpha^\delta(B) = \bigcap_{\gamma < \alpha} H_\gamma^\delta(B)$.

Now assume $L^p \not\hookrightarrow B$. Since Proposition 3 implies that the $H_\alpha^\delta(B)$'s strictly decrease if non-empty, there must exist an α with $H_\alpha^\delta(B) = H_{\alpha+1}^\delta(B) = \emptyset$.

Definition 3 : *Let* $h_p(\delta,B)$ *equal the least ordinal* δ *with* $H_\alpha^\delta(B) = \emptyset$ *and set* $h_p(B) = \lim\limits_{\delta\to 0} h_p(\delta,B)$.

It is easily proved that $\delta < \overline{\delta}$ implies $h_p(\overline{\delta},B) \le h_p(\delta,B)$. It follows from the *boundedness principle* (see [2] and the discussion in [5]) that $h_p(\delta,B) < \omega_1$ and hence $h_p(B) < \omega_1$ (assuming $L^p \not\hookrightarrow B$ of course). We now simply define $h_p(B) = \omega_1$ if $L^p \hookrightarrow B$; Theorem 2 may now be readily established.

Rather than appealing to a general principle to establish $h_p(\delta,B) < \omega_1$, it is possible to give a direct proof based on simple though fundamental ideas concerning partial orderings. *A relation* $<$ *on a non-empty set* X *is said to be well-founded provided there do not exist* $x_1, x_2, \dots$ *in* X *with* $x_n < x_{n+1}$ *for all* n. We define the classes $H_\alpha(X)$ $(= H_\alpha(X,<))$ by $H_0(X) = X$; $H_{\alpha+1}(X) = \{x \in H_\alpha(X) :$ there exists $y \in H_\alpha(X)$ with $x < y\}$, $H_\alpha(X) = \bigcap\limits_{\beta<\alpha} H_\beta(X)$ for α a limit ordinal. Assuming $<$ is well-founded, there exists a least ordinal α, denoted $h(X)$, with $H_\alpha(X) = \emptyset$, we then have the following simple but crucial permanence property :

Proposition 4 : *Let* $<$ *and* $\overline{<}$ *be well-founded relations on* X *and* Y *respectively and let* $\tau : X \to Y$ *be an order-preserving map. That is, if* $u < v$, *then* $\tau u \,\overline{<}\, \tau v$. *Then* $h(X) \le h(Y)$. *In fact, for all ordinal* α, $\tau(H_\alpha(X)) \subset H_\alpha(Y)$.

To establish the boundedness of $h_p(\delta,B)$, it suffices to exhibit an order preserving map τ between $\overline{B}^\delta$ and a countable set M endowed with a well-founded partial $\overline{<}$. Let B_0 be a countable dense subset of B, and M_n the subset of $B_0^{D_n}$ so that $u \in M_n$ provided

$$\frac{\delta}{2}\Big(\sum_{x\in D_n} |c(x)|^p\Big)^{1/p} \le \Big\|\sum_{x\in D_n} c(x)\,u(x)\Big\| \le 2\Big(\sum_{x\in D_n} |c(x)|^p\Big)^{1/p}$$

for all $c \in \mathbb{R}^{D_n}$. Let $M = \bigcup\limits_{n=0}^{\infty} M_n$ and $\eta_k = \delta\, 2^{-(2k+3)}$ all k. We order M as follows : if $u, v \in M$, then $u \,\overline{<}\, v$ provided, if $|u| = k$, then $|v| = k + \ell$ for $\ell \ge 1$ and

$$\|u(\varepsilon) - 2^{-\frac{\ell}{p}} \sum_{\tau\in D_\ell} v(\varepsilon,\tau)\| \leq \sum_{i=0}^{\ell-1} 2^{i}\, \eta_{k+i}$$

for all $\varepsilon \in D_k$.

We verify, assuming $L^p \not\hookrightarrow B$, that Z is a well founded relation on M. We now set $\varepsilon_k = \delta 8^{-(k+2)}$ for $k = 0,1,2,\ldots$ For each k, $x \in D_k$, $u \in \overline{B}^\delta$ with $|u| = k$, choose $v(x) \in B_0$ with $\|u(x) - v(x)\| < \varepsilon_k$. We then check that the $v \in B^{D_k}$ thus defined belongs to M_k's. Defining $\tau u = v$, we verify that $\tau : \overline{B}^\delta \to M$ is order preserving, thus obtaining $h_p(\delta,B) < \omega_1$ by Proposition 4.

We conclude by sketching the argument that $h_p(R^p_\alpha) \geq \alpha + 1$. We employ the simpler notation $H_\alpha(R^p_\alpha)$ for $H^1_\alpha(R^p_\alpha)$. It is easily seen, for any Banach space B, any α, that $H_\alpha(B) \neq \emptyset \Rightarrow H_\alpha(B) \cap B \neq \emptyset$ (recall that we identify the rank 0 elements of $\overline{B}$ with B itself).

<u>Theorem 5</u> : <u>Let</u> $1 \leq p < \infty$, $0 \leq \alpha < \omega_1$. <u>Then</u> $1 \in H_\alpha(R^p_\alpha)$.

Of course this shows that

$$h_p(R^p_\alpha) \geq h_p(1,R^p_\alpha) \geq \alpha + 1 \quad .$$

We prove the result by transfinite induction. The successor-ordinal case follows easily by the following general concatenation lemma :

<u>Lemma 6</u> : <u>Let</u> B <u>be a separable Banach space</u>, $1 \leq p < \infty$, $0 \leq \gamma < \omega_1$, $n \geq 0$, <u>and</u> $u \in H_\gamma(B)$ (<u>with respect to</u> p) <u>with</u> $|u| = n$. <u>Define</u> $\overline{u}$ <u>in</u> $(B \oplus B)^{D_{n+1}}$ <u>by</u> $\overline{u}(0,x) = u(x) \oplus 0$ <u>and</u> $\overline{u}(1,x) = 0 \oplus u(x)$ <u>all</u> $x \in D_x$. <u>Then</u> $\overline{u} \in H_\gamma((B\oplus B)_p)$.

The point of this is that if $|u| = 0$, then $|\overline{u}| = 1$; hence $\overline{u}$ has a (unique) predecessor, namely $v = \dfrac{\overline{u}(0) + \overline{u}(1)}{2^{1/p}}$; and thus $v \in H_{\gamma+1}((B\oplus B)_p)$; in our situation, if $u = 1$, also $v = 1$.

Now suppose α is a limit ordinal, $\alpha > 0$, and it has been shown that $1 \in H_\beta(R^p_\beta)$ all $\beta < \alpha$. To show that $1 \in H_\alpha(R^p_\alpha)$, it suffices to show that $1 \in H_\beta(R^p_\alpha)$ all $\beta < \alpha$. Now fixing $\beta < \alpha$, there exists an into-isometry $T : R^p_\beta \to R^p_\alpha$ with $T1 = 1$. The map $\tau : \overline{R^p_\beta} \to \overline{R^p_\alpha}$ defined by $(\tau u)(x) = T(u(x))$ for all k, $u \in \overline{R}^p_\beta$ with $|u| = k$, $x \in D_k$, is then order-preserving and $\tau 1 = 1$. So by Proposition 4 and the induction hypothesis : $1 \in \tau(H_\beta(R^p_\beta)) \subset H_\beta(R^p_\alpha)$, completing the proof.

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