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THE SPACE OF ALL BOUNDED OPERATORS ON HILBERT SPACE DOES NOT HAVE THE APPROXIMATION PROPERTY

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A Banach space X is said to have the approximation property if the identity operator on X can be approximated uniformly on every compact subset of X by finite rank operators.

We prove the result stated in the title of this talk (or, rather, present the main ideas leading to the proof).

1. INTRODUCTION

Grothendieck discovered [2] that a Banach space X does not have the approximation property if and only if there exists $\beta \in X^* \hat{\otimes} X$ such that

(1) $$\operatorname{tr} \beta = 1 \quad \text{and} \quad \|\beta\|_{\vee} = 0$$

where, for $\beta = \Sigma\, \psi_\alpha \otimes x_\alpha$ with $\Sigma\, \|\psi_\alpha\|\, \|x_\alpha\| < \infty$, $\psi_\alpha \in X^*$, $x_\alpha \in X$, we set

$$\operatorname{tr} \beta = \beta(\mathrm{Id}_X) = \Sigma\, \psi_\alpha(x_\alpha) \quad ,$$

$$\|\beta\|_{\vee} = \sup\{\Sigma\, \psi_\alpha(x) x^*(x_\alpha) \ : \ x^* \in X^*,\ x \in X,\ \|x^*\| \le 1,\ \|x\| \le 1\} \quad .$$

(We regard, as usual, a $\beta \in X^* \hat{\otimes} X$ as a functional on $L(X,X)$ = the space of bounded linear operators from X into X where, for $T \in L(X,X)$,

$$\beta(T) = \Sigma\, \psi_\alpha(Tx_\alpha) \quad \text{if} \quad \beta = \Sigma\, \psi_\alpha \otimes x_\alpha \quad .)$$

Enflo solved the approximation problem [1], apparently, quite independently of the ideas of [2]. Enflo's idea, however, can be seen as a development of Grothendieck's :

The difficult part of (1) is, of course, the condition $\|\beta\|_{\vee} = 0$. This is, in a way, an extrinsic condition, i.e. it depends on the whole space X rather then on β alone. Enflo circumvented this difficulty in the following way : suppose that $\beta_n \in X^* \otimes X$, $n = 1,2,\ldots$ satisfy conditions :

Standard notation : $\mathbb{C}$ = complex numbers, $\mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$ for a set A, $|A|$ = the cardinality of A, 1_A = the indicator function of A. $[t]$ = entier of t.

(*) $$\operatorname{tr} \beta_n = 1 \quad \text{for} \quad n = 1,2,\ldots$$

(**) $$\lim_n \beta_n(T) = 0 \quad \text{if} \quad \operatorname{rk} T = 1$$

(***) $$\sum_{n=1}^{\infty} \|\beta_{n+1} - \beta_n\|_{\wedge} < \infty \quad .$$

Then $\beta = \beta_1 + \sum_{n=1}^{\infty} (\beta_{n+1} - \beta_n) = \lim \beta_n$ belongs to $X^* \hat{\otimes} X$ and satisfies (clearly) condition (1) and therefore X fails the approximation property. The crucial point of Enflo's method is that the condition (**) is quite easy to control. To illustrate this, let us look at the typical situation where

$$\beta_n = 2^{-n} \sum_{j=2^n}^{2^{n+1}-1} y_j^* \otimes y_j$$

with $\|y_j^*\| = \|y_j\| = y_j^*(y_j) = 1$ for all j.

Then (**) is obviously satisfied if either $y_j^* \xrightarrow{w^*} 0$ or $y_j \xrightarrow{w} 0$, which usually follows automatically from (***). In this way the whole problem is, practically speaking, reduced to the condition (***). This condition is already "intrinsic", i.e. it can be settled by looking at a single representation $\beta_n - \beta_{n+1} = \sum \varphi_a \otimes u_a$.

We shall proceed from these ideas.

2. A CRITERION FOR FAILING THE APPROXIMATION PROPERTY.

It will be convenient to work with the uniform version of condition (**). This amounts to

$$\|\beta_n\|_{\vee} \longrightarrow 0 \quad .$$

For a finite set J and $\Phi = (\varphi_a, z_a : a \in J)$ with $\varphi_a \in X^*$, $z_a \in X$, we denote $\beta(\Phi) = \sum_{a \in J} \varphi_a \otimes z_a \in X^* \otimes X$ and $\operatorname{tr} \Phi = \operatorname{tr} \beta(\Phi) = \sum \varphi_a(z_a)$. We shall use the following simple estimate of $\| \ \|_{\vee}$. For Φ like above let

$$\sigma(\Phi) = \max_{|\varepsilon(a)|=1} \Big\| \sum_{a \in J} \varepsilon(a)\varphi_a \Big\| \max_{a \in J} \|z_a\| \quad .$$

We have

(2.0) $$\|\beta(\Phi)\|_{\vee} \leq \sigma(\Phi) \quad .$$

To see it, let $x^* \in X^*$, $x \in X$. Put $\varepsilon(a) = \overline{\varphi_a(x)}|\varphi_a(x)|^{-1}$. We have

$$\Sigma\ |\varphi_a(x)x^*(z_a)| \le \|x^*\|\ \max \|z_a\|\ \Sigma\ |\varphi_a(x)| =$$

$$= \|x^*\|\ \max \|z_a\|\ \Sigma\ \varepsilon(a)\ \varphi_a(x) = \|x^*\|\ \max \|z_a\|\ (\Sigma\ \varepsilon(a)\ \varphi_a)(x) \le$$

$$\le \max \|z_a\|\ \|\Sigma\ \varepsilon(a)\varphi_a\|\ \|x\|\ \|x^*\| \le \sigma(\Phi)\ \|x\|\ \|x^*\| \quad .$$

In estimating the norms $\|\ \|_\wedge$ we shall use the following two standard lemmas. Let A be a finite set, let X and Y be Banach spaces and let $u_a \in X$, $\varphi_a \in Y$ for $a \in A$. The set $(\varphi_a, u_a : a \in A)$ will be called <u>sufficiently unconditional</u> if there exist functions (changes of signs) $\varepsilon_1, \dots, \varepsilon_\ell : A \to \mathbb{T}$ such that

$$(2.1) \qquad \|\sum_{a\in A} \overline{\varepsilon_j(a)}\ u_a\| = \|\sum_{a\in A} u_a\| \quad \text{for} \quad j = 1,\dots,\ell \quad ,$$

$$(2.2) \qquad \|\sum_{a\in A} \varepsilon_j(a)\ \varphi_a\| = \|\sum_{a\in A} \varphi_a\| \quad \text{for} \quad j = 1,\dots,\ell \quad ,$$

$$(2.3) \qquad \sum_{j=1}^{\ell} \varepsilon_j(a)\varepsilon_j(b) = 0 \qquad \text{for} \quad a \ne b \quad .$$

<u>Lemma 2.1</u> : If $(\varphi_a, u_a : a \in A)$ is sufficiently unconditional, then

$$\|\sum_{a\in A} \varphi_a \otimes u_a\|_\wedge \le \|\sum_{a\in A} \varphi_a\|\ \|\sum_{a\in A} u_a\| \quad .$$

<u>Proof</u> : It is an obvious application of the invariance of the trace. Let $\varepsilon_1, \dots, \varepsilon_\ell$ be like in the definition. We have, by (2.3),

$$\sum_j [(\sum_a \varepsilon_j(a)\varphi_a) \otimes (\sum_a \overline{\varepsilon_j(a)} u_a)] = \sum_j \sum_{a,b} \varepsilon_j(a)\ \overline{\varepsilon_j(b)}\varphi_a \otimes u_b$$

$$= \sum_{a,b} (\sum_j \varepsilon_j(a)\ \overline{\varepsilon_j(b)})\varphi_a \otimes u_b = \ell \sum_a \varphi_a \otimes u_a \quad .$$

Therefore, by (2.1) and (2.2),

$$\ell\ \|\Sigma\ \varphi_a \otimes u_a\|_\wedge \le \sum_j \|\Sigma\ \varepsilon_j(a)\varphi_a\|\ \|\Sigma\ \overline{\varepsilon_j(a)} u_a\| =$$

$$= \ell\ \|\Sigma\ \varphi_a\|\ \|\Sigma\ u_a\| \quad ,$$

which proves the lemma.

We have the following well known and obvious :

Lemme 2.2 : Let $A \subset C \times D$ and let $u_{c,d}$, $\varphi_{c,d} : c \in C,\ d \in D$ be such that for any $\theta : C \to \mathbb{T}$, $\eta : D \to \mathbb{T}$,

$$\Big\| \sum_{(c,d)\in A} \theta(c)\ \eta(d)\ u_{c,d} \Big\| = \Big\| \sum_{(c,d)\in A} u_{c,d} \Big\| ,$$

$$\Big\| \sum_{(c,d)\in A} \theta(c)\ \eta(d)\ \varphi_{c,d} \Big\| = \Big\| \sum_{(c,d)\in A} \varphi_{c,d} \Big\| .$$

Then $(\varphi_a, u_a : a \in A)$ is sufficiently unconditional.

Now we can formulate our main technical proposition. We shall use the Enflo's pattern from § 1 with $\beta_n = \beta(\Phi_n)$ where $\Phi_n = (\varphi_a, z_a : a \in J_n)$ with $\varphi_a \in X^*$, $z_a \in X$. In our proposition we combine two simple ideas :

1o) $(\varphi_a : a \in J_n)$ and $(\varphi_a : a \in J_{n-1})$ are related by a "martingale condition" : we assume that there exist $\varkappa_n : J_n \xrightarrow{\text{onto}} J_{n-1}$ such that

(2.4) $$\varphi_a = \sum_{\{b : \varkappa_n(b) = a\}} \varphi_b \quad \text{for every } a \in J_{n-1},\ n = 2,3,\ldots$$

Then we have, obviously,

$$\beta_{n-1} - \beta_n = \sum_{b\in J_n} \varphi_b \otimes \mathring{z}_b \qquad \text{where}$$

$$\mathring{z}_b = z_{\varkappa_n(b)} - z_b \quad \text{for} \quad b \in J_n \ ,\quad n = 2,3,\ldots$$

2o) To estimate $\| \sum_{b\in J_n} \varphi_b \otimes \mathring{z}_b \|_\wedge$, we partition J_n as, let us say, $J_n = A_1 \cup A_2 \cup \ldots \cup A_\nu$, A_j pairwise disjoint, and estimate the norms $\| \sum_{b\in A_j} \varphi_b \otimes \mathring{z}_b \|_\wedge$ separately using Lemma 2.1. The main idea behind "partitioning" is that, when the sizes of A_j are small enough, then there is, practically speaking, no dependence between $\sum_{b\in A_j} \varphi_b$ and $\sum_{b\in A_j} \mathring{z}_b$, and therefore, their norms can be made small simultaneously.

We summarize these remarks in the following :

Proposition 2.3 : Let J_n, $n = 1,2,\ldots$ be finite sets, let Φ_n, $\varkappa_n$, $\mathring{z}_b$ be as above (in particular, we assume that the "martingale condition" (2.4) is satisfied). Assume that

(2.5) $$\operatorname{tr}(\Phi_n) = 1 \quad \text{for} \quad n = 1,2,\ldots$$

(2.6) $$\sigma(\Phi_n) \longrightarrow 0 \quad \text{as } n \to \infty .$$

For $n = 1,2,\ldots$ let Δ_n be a partition of J_n such that

(2.7) the set $(\varphi_a, \overset{\circ}{z}_a : a \in A)$ is sufficiently unconditional for every $A \in \Delta_n$, $n = 1,2,\ldots$

(2.8) $$\sum_{n=1}^{\infty} |\Delta_n| \max_{A\in\Delta_n} \|\sum_{a\in A} \overset{\circ}{z}_a\| \, \|\sum_{a\in A} \varphi_a\| < \infty .$$

Then X does not have the approximation property.

<u>Proof</u> : We take $\beta_n = \beta(\Phi_n)$ and check (*), (**), (***). (*) is just (2.5) and (**) follows from (2.6), by (2.0). Therefore we should only check condition (***). We have

$$\beta_{n-1} - \beta_n = \sum_{b\in J_n} \varphi_b \otimes \overset{\circ}{z}_b = \sum_{A\in\Delta_n} \sum_{b\in A} \varphi_b \otimes \overset{\circ}{z}_b .$$

By (2.7) and Lemma 2.1

$$\|\sum_{b\in A} \varphi_b \otimes \overset{\circ}{z}_b\| \le \|\sum_{b\in A} \varphi_b\| \, \|\sum_{b\in A} \overset{\circ}{z}_b\| .$$

Therefore $$\|\beta_{n-1} - \beta_n\| \le |\Delta_n| \max_{A\in\Delta_n} \|\sum_{a\in A} \varphi_a\| \, \|\sum_{a\in A} \overset{\circ}{z}_a\|$$

and (***) follows by (2.8).

<u>Remark</u> : A simple form of the "martingale condition" (2.4) (used in [9] but not in the present paper is : $J_n = \{2^n + 1, \ldots, 2^{n+1}\}$ for $n = 1,2,\ldots$ and $\varphi_j = \varphi_{2j-1} + \varphi_{2j}$.

3. $\mathbb{B}(H)$, NOTATION AND SIMPLE FACTS.

The inner product in any Hilbert space will be denoted $\langle f|g\rangle$; $f \perp g$ means $\langle f|g\rangle = 0$ and, for subspaces H_1 and H_2, $H_1 \perp H_2$ means $f \perp g$ for every $f \in H_1$, $g \in H_2$.

Given Hilbert spaces H_1, H_2, we denote by $\mathbb{B}(H_1,H_2)$ the space of bounded linear operators from H_1 to H_2, equipped with the operator norm $\| \ \|_\infty$.

Let H_1, H_2 be Hilbert spaces, let $\mathbb{B} = \mathbb{B}(H_1,H_2)$. Let $x \in \mathbb{B}$. If $\operatorname{rk} x < \infty$, then we can define its "inner product" with any $y \in \mathbb{B}$ by the formula

(3.1) $$\langle y,x\rangle \overset{\text{def}}{=\!=} \operatorname{tr} x^* y .$$

By this formula x will be identified as an element of $\mathbb{B}^*$, denoted here by $\underline{x}$. It is well known that

$$\|\underline{x}\|_{\mathbb{B}^*} = \|x\|_1 \overset{\text{def}}{=\!=} \operatorname{tr}(xx^*)^{1/2} .$$

By $\mathcal{R}(x)$, $\mathcal{D}(x)$ we denote the range and the domain of x, respectively. We shall only use some most elementary facts about the norms $\| \ \|_p$:

(3.2) if rk x = 1, then $\|x\|_1 = \|x\|_\infty = \langle x,x\rangle^{1/2}$.

(3.3) if $x = \sum_{a\in A} x_a$ with $\mathcal{R}x_a \perp \mathcal{R}x_b$, $\mathcal{D}x_a \perp \mathcal{D}x_b$ for $a \neq b$, $a,b \in A$,

then $$\|x\|_\infty = \max_{a\in A} \|x_a\|_\infty .$$

(3.4) if y, z are isometries (onto) of H_1, H_2, respectively, then

$$\|zxy\|_p = \|x\|_p \quad \text{for } p = 1,\infty \text{ and } \langle zxy,zxy\rangle = \langle x,x\rangle .$$

As a corollary of (3.4) we note

(3.5) let $x = \sum_{(c,d)\in A} x_{c,d}$ with

$$\mathcal{R}x_{c,d} \perp \mathcal{R}x_{e,f} \quad \text{if } c \neq e \quad \text{and} \quad \mathcal{D}x_{c,d} \perp \mathcal{D}x_{e,f} \quad \text{if } d \neq f .$$

Then for every choice of signs $\theta(c) \in \mathbb{T}$, $\eta(d) \in \mathbb{T}$,

$$\|x\|_p = \Big\|\sum_{(c,d)\in A} \theta(c)\eta(d)\, x_{c,d}\Big\|_p \quad \text{for } p = 1,\infty .$$

Notice that (3.5) is indeed a consequence of (3.4) :

The assumptions of (3.5) say that there exist direct sum decompositions

$$H_1 = \sum_d \oplus H_1^d \ , \quad H_2 = \sum_c \oplus H_2^c$$

so that $\mathcal{R}x_{c,d} \subset H_2^c$, $\mathcal{D}x_{c,d} \subset H_1^d$ for every c,d.

Let $\Gamma_1 = \sum_d \eta(d)\, \mathrm{Id}_{H_1^d}$, $\Gamma_2 = \sum_c \theta(c)\, \mathrm{Id}_{H_2^c}$. Then, clearly, Γ_1 and Γ_2 are isometries of H_1, H_2, respectively, and we have

$$\Sigma\, \theta(c)\, \eta(d)\, x_{c,d} = \Gamma_2 \circ x \circ \Gamma_1 \quad .$$

An $x \in \mathbb{B}$ will be called an <u>α-homothety</u> if $\|x(f)\| = \alpha\|f\|$ for every $f \in H_1$. It will be called a <u>partial homothety</u> if it is a homothety on its domain (i.e. if it is the form yp where y is a homothety and p is an orthogonal projection). It is easy to see that

(3.6) if x is a partial homothety, then $\|x\|_1 \; \|x\|_\infty = \langle x,x\rangle^{1/2}$.

Otherwords, a partial homothety is <u>selfnormalizing</u>. By (3.2), rank one operators are also selfnormalizing. Let (K,μ) be a measure space. By i_K we denote the identity on $L_2(K,\mu)$. If $S \subset K$, then 1_S denotes the indicator function on S and p_S denotes the projection in $L_2(K,\mu)$ defined by $p_S f = f \cdot 1_S$.

Let K be a finite set, let the measure μ_K be defined by $\mu(\{a\}) = |K|^{-1}$ for all $a \in K$. We define $L_2(K) = L_2(K,\mu_K)$.

Let A, B be finite sets. By M(A,B) we denote the set of all $A \times B$ matrices, i.e. of functions from $A \times B$ into $\mathbb{C}$. Given an $x \in \mathbb{B}(L_2(B),L_2(A))$ we shall identify it in the usual way as an element $x \in M(A,B)$. For $a \in A$, $b \in B$ we define $\varepsilon_{a,b} \in M(A,B)$ by

$$\varepsilon_{a,b}(c,d) = \begin{cases} 1 & \text{if } a = c,\ b = d\ , \\ 0 & \text{otherwise} \quad . \end{cases}$$

Let $x \in M(A,B)$, $y \in M(C,D)$. We define $x \otimes y \in M(A \times C, B \times D)$ as usual :

$$(x \otimes y)(a,c\, ;\, b,d) = x(a,b)\; y(c,d) \quad .$$

We shall need the following simple facts

(3.7) $$\langle x \otimes y, u \otimes v\rangle = \langle x,u\rangle\langle y,v\rangle \quad ,$$

(3.8) $$\|x \otimes y\|_p = \|x\|_p \; \|y\|_p \qquad \text{for } p = 1,\infty \quad .$$

(3.9) if x and y are homotheties, then so is $x \otimes y$.

(3.10) if $\mathcal{R}x \perp \mathcal{R}u$, then $R(y \otimes x) \perp R(z \otimes u)$ for every y, z .

For use in formula (7.9) we introduce the following ad hoc notation : let $G = F \times F$ where we write $\theta \in G$ as $\theta = (\theta^0, \theta^1)$ with θ^0, $\theta^1 \in F$. For $x, y \in M(F,F)$ we define $x \overset{!}{\otimes} y \in M(G,G)$ by

$$(3.11) \qquad (x \overset{!}{\otimes} y)(\theta^0, \theta^1 ; \zeta^0, \zeta^1) = x(\theta^1, \zeta^0) y(\theta^0, \zeta^1) \quad .$$

Clearly, $\overset{!}{\otimes}$ has the properties (3.7)-(3.10).

If $x \in M(A,B)$, then $x^t \in M(B,A)$ denotes the transpose of x. Clearly

$$(3.12) \qquad \|x^t\|_p = \|x\|_p \quad , \quad \langle x^t, y^t \rangle = \langle x, y \rangle \quad \text{for all } x, y, p \quad .$$

4. THE FORMAL PATTERN OF THE CONSTRUCTION

The construction is done in two steps :

1^o defining for an arbitrary $\ell < \infty$ $\Phi_1, \dots, \Phi_{2\ell}$ so that the conditions (2.4)-(2.8) of Proposition 2.4 are satisfied (with estimates in (2.6), (2.8) independent of ℓ).

2^o passing with ℓ to ∞.

Step 1^o is the bulk of the construction ; step 2^o involves some further technical complications and we skip it in this note.

Let $r_1, r_2, \dots$ be some natural numbers. Let $G_n = \{1, \dots, r_n\}$ and let $\mu_n = \mu_{G_n}$ i.e. $\mu_n(\{j\}) = r_n^{-1}$ for $j = 1, \dots, r_n$.

Let us put $K_n = G_1 \times \dots \times G_n$. We shall work with the space of matrices $M(K_\ell, K_\ell)$ which is identified with $\mathbb{B}(L_2(K_\ell), L_2(K_\ell))$ as indicated in § 3. For $\xi = (\xi_1, \dots, \xi_m) \in K_m$ we define $I_\xi \subset K_\ell$ and $p_\xi \in \mathbb{B}(L_2(K_\ell), L_2(K_\ell))$ by

$$(4.1) \qquad I_\xi = \{\eta = (\eta_1, \eta_2, \dots) \in K_\ell : \eta_1 = \xi_1, \dots, \eta_m = \xi_m\} \quad \text{and} \quad p_\xi = p_{I_\xi} \quad .$$

We define also $K^n = G_n \times G_{n+1} \times \dots \times G_\ell$. We set

$$(4.2) \qquad J_1 = K_1 \ , \quad J_{2n} = K_n \times K_n \ , \quad J_{2n+1} = K_n \times K_{n+1} \quad \text{for } n = 1, 2, \dots$$

Let us make the following notational convention :

When we write $a = (\xi,\eta) \in J_m$, we always mean

$\xi = (\xi_1,\dots,\xi_{[\frac{1}{2}(m)]})$, $\eta = (\eta_1,\dots,\eta_{[\frac{1}{2}(m+1)]})$ with $\xi_j,\eta_j \in G_j$.

We define $\varkappa_m : J_m \to J_{m-1}$ in the following way : for $\xi \in K_m$ let $\dot{\xi} = (\xi_1,\dots,\xi_{m-1})$. For $b = (\xi,\eta) \in J_m$ we define

$$\varkappa_m b = \begin{cases} (\dot{\xi},\eta) & \text{if } m \text{ is even} , \\ (\xi,\dot{\eta}) & \text{if } m \text{ is odd} . \end{cases}$$

In § 5 we shall define a matrix $z \in M(K_\ell,K_\ell)$ which is the main ingredient of the whole construction. We set then for $\xi \in K_m$, $\eta \in K_n$

(4.3) $$z_{\xi,\eta} = p_\xi \, z \, p_\eta \quad , \quad \varphi_{\xi,\eta} = \underline{z}_{\xi,\eta} \quad .$$

We see that (2.4) is evidently satisfied. Condition (2.5) is equivalent to

(4.4) $$\langle z,z\rangle = 1 \quad .$$

We shall construct z so that

(4.5) all entries of z have absolute value K_ℓ^{-1} which obviously implies (4.4).

To see what becomes of condition (2.6), let $\varepsilon(a)$ be any numbers of absolute value 1, for $a \in J_n$. Then, by (4.5), all entries of the matrix $\sum_{a\in J_n} \varepsilon(a)\, z_a$ have absolute value $|K_\ell|^{-1}$. Hence

(4.6) $$\sigma(\Phi_n) \le |K_\ell|^{1/2} \max_{a\in J_n} \|z_a\|_\infty \quad .$$

To see that this leads to a desired estimate, let us anticipate the following fact, proved in § 6

(4.7) For $\xi \in K_m$, $\eta \in K_n$, $n \ge m$, $z_{\xi,\eta}$ is a homothety of $L_2(I_\eta)$ onto $L_2(I_\xi)$.

In this case, the ∞-norm of $z_{\xi,\eta}$ is very easy to compute :

$\|z_{\xi,\eta}\|_\infty = K_\ell^{1/2} \|z_{\xi,\eta} 1_{\{\zeta\}}\|$ for any $\zeta \in I_\eta$ and the last norm is evidently equal to

$$|K_\ell|^{-1}(|G_{m+1}| \ldots |G_\ell|)^{1/2} = |K_\ell|^{-1/2} . |K_m|^{-1/2} \quad .$$

We can thus conclude

$$\|z_a\|_\infty = |K_\ell|^{-1/2} . |K_n|^{-1/2} \quad \text{for } a \in J_{2n} \;, \; a \in J_{2n+1} \;,$$

hence, by (4.6)

$$(4.8) \qquad \sigma(\Phi_{2n}) \le |K_n|^{-1/2} \;, \quad \sigma(\Phi_{2n+1}) \le |K_n|^{-1/2}$$

which evidently must go to 0.

Concerning condition (2.7), we have the following trivial

<u>Lemma 4.1</u> : Condition (2.7) is satisfied provided (4.9)

(4.9) $\quad \varkappa_n$ is 1-1 on every $B \in \Delta_n$, i.e. for $a,b \in B$, $a \ne b$ implies $\varkappa_n a \ne \varkappa_n b$.

<u>Proof</u> : We shall use (3.5) and Lemma 2.2. Let us take $A = \varkappa_n B$, $C = K_{[\frac{1}{2}(n-1)]}$, $D = K_{[\frac{1}{2}n]}$. We have obviously

$$\mathcal{R} z_{c,d} = L_2(I_c) \quad , \quad \mathcal{D} z_{c,d} = L_2(I_d) \quad ,$$

therefore the assumptions of (3.5) are clearly satisfied for $x_{c,d} = z_{c,d}$. For $a \in B$, $\mathcal{R} z_a$, $\mathcal{R} \mathring{z}_a$ are contained in $\mathcal{R} z_{\varkappa_n a}$ and $\mathcal{D} z_a$, $\mathcal{D} \mathring{z}_a$ in $\mathcal{D} z_{\varkappa_n a}$, therefore the assumptions of (3.5) are, a fortiori, satisfied for $x_a := z_{\varkappa_n^{-1}(a)}$, $x_a := \mathring{z}_{\varkappa_n^{-1}(a)}$, $a \in A$. Now we can apply Lemma 2.2.

5. <u>THE DEFINITION OF</u> z <u>AND OF</u> Δ_n<u>'S</u>.

We shall need a further detail. We assume that $G_n = F_n \times F_n$, i.e. every $\theta \in G_n$ is written as $\theta = (\theta^o, \theta^1)$ with $\theta^o, \theta^1 \in F_n$. We require that the following "independence condition" is satisfied :

(5.0) for every $A \in \Delta_{2n+1}$ and for every $B \in \Delta_{2n+2}$:

ξ_n^1 and η_n^o are constant for $(\xi,\eta) \in A$,

ξ_n^o and η_n^1 are constant for $(\xi,\eta) \in B$.

We define z by the formula

$$z(\xi,\eta) = |K_\ell|^{-1} \prod_{n=1}^{\ell-1} v_n(\xi_{n+1},\xi_n^1 ; \eta_n^o) v_n(\eta_{n+1},\eta_n^1 ; \xi_n^o) \cdot V(\xi_\ell,\eta_\ell) \quad , \tag{5.1}$$

where $v_n \in M(G_{n+1} \times F, F)$ are certain unimodular matrices defined in § 7 and $V \in M(G_\ell, G_\ell)$ can be an arbitrary symmetric, unimodular, homothetic matrix.

Let us now indicate how the Δ_m's are constructed. Let $m = 2n+1$ or $2n+2$, let $c,d \in K_{n-1}$, i.e. $c = (c_1,\dots,c_{n-1})$, $d = (d_1,\dots,d_{n-1})$ with $c_j, d_j \in G_j$ and let $g \in G_{n+1}$, $C,D \subset G_n$. We define

$$B^{2n+1}(c,d,g,C,D) = \Big\{(\xi,\eta) \in J_{2n+1} \ : \ \eta_1 = d_1,\dots,\eta_{n-1} = d_{n-1} ,$$
$$\xi_1 = c_1,\dots,\xi_{n-1} = c_{n-1},\ \eta_{n+1} = g,\ \eta_n \in D,\ \xi_n \in C\Big\}$$

(5.2)

$$B^{2n+2}(c,d,g,C,D) = \Big\{(\xi,\eta) \in J_{2n+2} \ : \ \xi_1 = d_1,\dots,\xi_{n-1} = d_{n-1} ,$$
$$\eta_1 = c_1,\dots,\eta_{n-1} = c_{n-1},\ \xi_{n+1} = g,\ \xi_n \in D,\ \eta_n \in C\Big\}$$

(let us notice that there is a slight lack of symmetry between $B^{2n+1}(\dots)$ and $B^{2n+2}(\dots)$: the second one has r_{n+1} times as many elements as the first one because the variable η_{n+1} is free in $B^{2n+2}(\dots)$ whence η_{n+1} is "bound" in $B^{2n+1}(\dots)$).

All elements of Δ_m will be of the form $B^m(c,d,g,C,D)$ for some $c,d \in K_{n-1}$, $g \in G_{n+1}$, $C,D \subset G_n$. Let us notice that $B^m(\dots)$ satisfy (4.9) and therefore (2.7) is automatically satisfied.

We pass now to the discussion of the main condition (2.8). For $B \in \Delta_m$ let us denote

$$\omega_B = \sum_{a \in B} z_a \quad , \quad w_B = \sum_{a \in B} \mathring{z}_a \quad .$$

Condition (2.8) can be thus formulated as

$$|\Delta_m| \ \|\omega_B\|_1 \ \|w_B\|_\infty \ \text{ is small for every } B \in \Delta_m . \tag{5.3}$$

For the sake of convenience we assume that

1^o, $|C|$ and $|D|$ are constant for all $B^m(c,d,g,C,D)$ in Δ_m, and

2^o, $B^{2n+1}(c,d,g,C,D) \in \Delta_{2n+1}$ iff $B^{2n+2}(c,d,g,C,D) \in \Delta_{2n+2}$.

By 1^o, $\langle\omega_B,\omega_B\rangle$ is constant for $B \in \Delta_m$, therefore

(5.4) $$\langle\omega_B,\omega_B\rangle = |\Delta_m|^{-1} \text{ for every } B \in \Delta_m \quad .$$

Since z is a symmetric matrix, 2^o implies that ω_B, w_B with $B \in \Delta_{2n+1}$ are just transposes of ω_B, w_B with $B \in \Delta_{2n+2}$. Therefore

(5.5) $$\max_{B\in\Delta_{2n+1}} |\Delta_{2n+1}| \, \|\omega_B\|_1 \, \|w_B\|_\infty = \max_{B\in\Delta_{2n+2}} |\Delta_{2n+2}| \, \|\omega_B\|_1 \, \|w_B\|_\infty$$

which lets us to restrict attention to the case of, for example, odd m, let us say $m = 2n+1$.

5A. Let $B = B^m(c,d,g,C,D) \in \Delta_m$. For $h \in G_{n+1}$ let us denote

(5.6) $$\omega^h = \omega_B^h = \sum_{a\in B^m(c,d,h,C,D)} z_a$$

thus

$$\omega_B = \omega^g \quad \text{and} \quad w_B = \sum_{h\neq g} \omega^h \quad .$$

We have obviously

(5.7) $$\|w_B\|_\infty \geq \max_{h\neq g} \|\omega^h\|_\infty \quad .$$

By (5.4) and (5.7), the following condition is necessary for (5.3) :

(5.8) $$\langle\omega^g,\omega^g\rangle^{-1} \, \|\omega^g\|_1 \, \|\omega^h\|_\infty \text{ is small for every } h \neq g$$

(let us notice that this quantity has to be big if $h = g$, namely ≥ 1 ; here we actually have the crux of the construction : making the ratio $\|\omega^h\|_\infty / \|\omega^g\|_\infty$ small for all $h \neq g$). Of course, (5.8) is useful only in case when (5.7) is not far from equality. This is settled in the following section.

6. THE ORTHOGONALITY CONDITION.

We shall define matrices $y_m \in M(K^m, K^m)$ by

$$y_m(\xi,\eta) = \prod_{n=m}^{\ell-1} v_n(\xi_{n+1},\xi_n^1 ; \eta_n^o) v_n(\eta_{n+1},\eta_n^1 ; \xi_n^o) \cdot V(\xi_\ell,\eta_\ell) \quad .$$

Thus $z = |K_\ell|^{-1} y_1$.

In the following Lemma, we use the notation of 5A.

Lemma 6.1 : We have $\|w_B\|_\infty = \max_{h \neq g} \|\omega^h\|_\infty$ provided

(6.0) y_{n+1} is a homothetic matrix.

Proof : We shall use (3.1). Let $h, \chi \in G_{n+1}$, $h \neq \chi$. Obviously $\mathcal{D}\omega^h \perp \mathcal{D}\omega^\chi$. The fact that also $\mathcal{R}\omega^h \perp \mathcal{R}\omega^\chi$ follows easily from (6.0) and from (5.0) ; here is a formal argument :
For $h \in G_{n+1}$ let us denote

$$y^h = y_{n+1} \; p_{\{h\}\times G_{n+2}\times\ldots\times G_\ell} \quad .$$

Let us notice that, by (6.0)

(6.1) $$\mathcal{R} y^h \perp \mathcal{R} y^\chi \quad .$$

By (5.0), there exist $e, f \in F_n$ such that

$$\xi_n^o = e \quad , \quad \eta_n^1 = f \quad \text{for every } (\xi,\eta) \in B \quad .$$

We see that

(6.2) $$\omega^h = s^h \otimes (\Gamma \circ y^h)$$

where $s^h \in M(K_n, K_n)$ and $\Gamma \in M(G_{n+1}\times\ldots\times G_1 \, , \, G_{n+1}\times\ldots\times G_\ell)$ are defined by (at this point it really does not matter how s^h looks like)

$$s^h(\xi,\eta) = \prod_{j=1}^{n-2} v_j(d_{j+1},d_j^1 ; c_j^o)v_j(c_{j+1},c_j^1 ; d_j^o) \cdot v_{n-1}(\xi_n,c_{n-1}^1 ; d_{n-1}^o) \cdot$$

$$(6.3) \qquad \cdot v_{n-1}(\eta_n,d_{n-1}^1 ; c_{n-1}^o) \cdot v_n(h,\eta_n^1 ; \xi_n^o)$$

$$\text{if } \eta_n \in D,\ \xi_n \in C \text{ and } (\xi_1,\dots,\xi_{n-1}) = c,\ (\eta_1,\dots,\eta_{n-1}) = d$$

$$s^h(\xi,\eta) = 0 \text{ otherwise} \quad ;$$

$$\Gamma(\xi,\eta) = \begin{cases} v_n(\xi_{n+1},e ; f) & \text{if } \xi_{n+1} = \eta_{n+1},\dots,\xi_\ell = \eta_\ell \quad , \\ 0 \text{ otherwise} & . \end{cases}$$

Since Γ is an orthogonal transformation (it is just a diagonal matrix with all terms of absolute value 1), (6.1) implies that $\mathcal{R}(\Gamma \circ y^h) \perp \mathcal{R}(\Gamma \circ y^\chi)$ which, by (3.10), implies the desired conclusion $\mathcal{R}\,\omega^h \perp \mathcal{R}\,\omega^\chi$.

The "orthogonality condition" (6.0) seems to play an essential role in our construction. To clarify this condition we shall use the following description of y_m : let us define $\Gamma_n \in M(G_n \times G_{n+1}, G_n \times G_{n+1})$ and $T \in M(K_\ell,K_\ell)$ by

$$(6.4) \qquad \Gamma_n(\xi,\eta) = \begin{cases} v_n(\xi_{n+1},\xi_n^1 ; \eta_n^o) & \text{if } \xi_{n+1} = \eta_{n+1},\ \xi_n^o = \eta_n^1 \\ 0 \text{ otherwise} & \end{cases}$$

$$T(\xi,\eta) = \begin{cases} V(\xi_\ell,\eta_\ell) & \text{if } \xi_j^o = \eta_j^1,\ \xi_j^1 = \eta_j^o \text{ for } j<\ell \\ 0 \text{ otherwise} & \end{cases}$$

and let

$$(6.5) \qquad V_n = i_{K_{n-1}} \otimes \Gamma_n \otimes i_{K^{n+2}} \quad .$$

We have

$$(6.6) \qquad i_{K_{m-1}} \otimes y_m = V_m \circ V_{m+1} \circ \dots \circ V_{\ell-1} \circ T \circ V_{\ell-1}^t \circ \dots \circ V_{m+1}^t \circ V_m^t \quad .$$

For $g \in G_{n+1}$ let us define $v_n^g \in M(F_n,F_n)$ by

$$v_n^g(e,f) = v_n(g,e ; f) \quad .$$

Lemma 6.2 : The matrix y_n is homothetic provided

(6.7) v_m^g is a homothetic matrix for every $g \in G_{m+1}$ for all $m \geq n$.

Proof : Since Γ_m is equivalent to a direct sum of v_m^g, it is a homothety, by (6.7), for all $m \geq n$. Consequently, V_m are homotheties for $m \geq n$. Since T is also a homothety, so is $i_{K_{n-1}} \otimes y_n$, by the formula (6.6) and, consequently, y_n is homothetic.

The following lemma has been already announced in (4.7) ; as we proved there, (6.8) implies condition (2.6).

Lemma 6.3 : If (6.7) holds for every n, then the condition (4.7) is satisfied, i.e.

(6.8) for every $\xi, \eta \in K_n$, $z_{\xi,\eta}$ is a homothety of $L_2(I_\eta)$ onto $L_2(I_\xi)$.

Proof : We have

$$z_{\xi,\eta} = Q \cdot \varepsilon_{\xi,\eta} \otimes (\Gamma_2 \circ y_{n+1} \circ \Gamma_1)$$

where Q is a constant and $\Gamma_1, \Gamma_2 \in M(K^{n+1}, K^{n+1})$ are diagonal matrices defined by

$$\Gamma_1(\zeta,\upsilon) = \begin{cases} v_n(\upsilon_{n+1}, \eta_n^1 ; \xi_n^o) & \text{if } \zeta = \upsilon \\ 0 \text{ otherwise} & , \end{cases}$$

$$\Gamma_2(\zeta,\upsilon) = \begin{cases} v_n(\zeta_{n+1}, \xi_n^1 ; \eta_n^o) & \text{if } \zeta = \upsilon \\ 0 \text{ otherwise} & . \end{cases}$$

Since the matrix v_n is unimodular, Γ_1 and Γ_2 are isometries.

By lemma 6.2, y_{n+1} is a homothety, therefore $\Gamma_2 \circ y_{n+1} \circ \Gamma_1$ is a homothety and this clearly implies (6.8).

7. THE END OF THE CONSTRUCTION AND OF THE PROOF

So far we have been mainly concerned with the formal aspects of the construction. To recapitulate :

the matrix z is given by (5.1) where

(7.0) $$v_n^g \in M(F_n,F_n) \text{ defined by } v_n^g(e,f) = v_n(g,e;f) \quad ,$$

is an Hadamard matrix for every $g \in G_{n+1}$, every n (by an Hadamard matrix we mean a unimodular square matrix whose rows (columns) are mutually orthogonal) ;

the partitions Δ_m should satisfy the condition (5.0) plus the requirements (5.4), (5.5).

Then everything boils down to condition (5.8).

The rest of the construction is combinatorial. Let F be a finite set with $|F| = q^2$. A partition ∇ of F will be called regular if $|\nabla| = q$ and each element of ∇ has q elements. Let $ be a standard regular partition of F, let us say we write $F = H \times H$ and \$ = $\{\{h\} \times H : h \in H\}$.

Lemma 7.1 : Let q be a number of the form 2^{8p}, p an integer. Let F, \$ be like above and let G be a set with q^8 elements. There exist regular partitions ∇_g, $g \in G$, of F and matrices $v^g \in M(F,F)$, $g \in G$ so that

(7.1) $$v^g \text{ is an Hadamard matrix for every } g \in G \quad ,$$

(7.2) $$\|p_S\, v^g\, p_A\|_1 = q \quad \text{for every } A \in \nabla_g \text{, every } g \in G\text{, every } S \in \$ \quad .$$

(7.3) $$\|p_S\, v^h\, p_A\|_\infty \le q^{\frac{15}{16}} \quad \text{for every } A \in \nabla_g\text{, every } g \in G\text{, every } h \neq g\text{, every } S \in \$ \quad .$$

We postpone a (rather-simple) proof of this lemma to § 8.

Let us notice at this point that, by (7.2) and (7.3),

(7.4) $$\langle p_S\, v^g\, p_A, p_S\, v^g\, p_A\rangle^{-1} \|p_S\, v^g\, p_A\|_1\, \|p_S\, v^h\, p_A\|_\infty \le q^{-\frac{1}{16}} \quad ,$$

for every $A \in \nabla_g$, every $g \in G$, every $h \neq g$, every $S \in \$$, which seems to indicate that we are on a right track.

Let now q_n be a sequence of numbers such that q_n is of the form 2^{8p}, p an integer, and

(7.5) $$q_n \longrightarrow \infty \text{ faster than any power of } n \quad ,$$

(7.6) $$q_{n+1} \le q_n^2 \quad .$$

We define $F_n = \{1,\dots,q_n^2\}$, $G_n = F_n \times F_n$. Let $\$_n$ be any regular partition of F_n. We apply lemma 7.1 to $q = q_n$, $F = F_n$, $G = G_{n+1}$, $\$ = \$_n$; we obtain thus the regular partitions ∇_g of F_n for $g \in G_{n+1}$ and matrices $v_n^g \in M(F_n,F_n)$ so that the respective conditions (7.1), (7.2), (7.3) are satisfied.

Now we can complete the definition of Δ_m's (see (5.2)). For $A \subset F_n$ and $e \in F_n$ let $C(A,e) = \{\theta \in G_n : \theta^o \in A,\ \theta^1 = e\}$, $D(A,e) = \{\theta \in G_n : \theta^o = e,\ \theta^1 \in A\}$. We define for $m = 2n+1$ or $2n+2$:

(7.7) $$\Delta_m = \{B^m(c,d,g,C(S,e),D(A,f)) : c,d \in K_{n-1},\ g \in G_{n+1},\ e,f \in F_n \text{ and } A \in \nabla_g,\ S \in \$\} \quad .$$

With this definition of Δ_m, (5.0), (5.4) and (5.5) are obviously satisfied.

Now we can prove (5.8). Let $m = 2n+1$, let $B \in \Delta_m$ be like in (7.7). We use the notation of 5A. We claim that for every $h \in G_{n+1}$,

(7.8) $$<\omega^g,\omega^g>^{-1} \|\omega^g\|_1 \ \|\omega^h\|_\infty = \\ = <p_S v_n^g p_A, p_S v_n^g p_A>^{-1} \|p_S v_n^g p_A\|_1 \ \|p_S v_n^h p_A\|_\infty \quad .$$

This quantity is, by (7.4), equal to $q_n^{-\frac{1}{16}}$. By (7.5), this implies (5.8).

To prove (7.8), we shall again use the formulas (6.2), (6.3) ; this time we pay more attention to s^h. We have

(7.9) $$s^h = Q \cdot \varepsilon_{c,d} \otimes [\varepsilon_{e,f} \overset{!}{\otimes} (\Gamma_1 \circ p_S v_n^h p_A \circ \Gamma_2)] \quad \text{where}$$

Q is a constant which does not depend on h,

$\overset{!}{\otimes}$ is defined by (3.11) (and behaves exactly like $\otimes$) ,

$\Gamma_1, \Gamma_2 \in M(F_n,F_n)$ are diagonal matrices defined by

$$\Gamma_1(\zeta,\nu) = \begin{cases} v_{n-1}((\zeta,e),c^1_{n-1}\,;\,d^o_{n-1}) & \text{if } \zeta = \nu \ , \\ 0 \quad \text{otherwise} \ , \end{cases}$$

$$\Gamma_2(\zeta,\nu) = \begin{cases} v_{n-1}((f,\nu),d^1_{n-1}\,;\,c^o_{n-1}) & \text{if } \zeta = \nu \ , \\ 0 \quad \text{otherwise} \ . \end{cases}$$

If we now put (6.2) and (7.9) together, then we get

$$\omega^h = Q \cdot \varepsilon_{c,d} \otimes [\varepsilon_{e,f} \overset{!}{\otimes} (\Gamma_1 \circ p_S\, v^h_n\, p_A \circ \Gamma_2)] \otimes (\Gamma \circ y^h)$$

or, writing it in a schematic way

$$\omega^h = Q \cdot X \otimes [Y \overset{!}{\otimes} Z^h] \otimes W^h \quad .$$

We have for every $h \in G_{n+1}$,

$$\|\omega^h\|_p = Q\ \|X\|_p\ \|Y\|_p\ \|Z^h\|_p\ \|W^h\|_p \quad \text{for } p = 1,\infty \quad ,$$

and

$$<\omega^g,\omega^g> = Q^2\ <X,X>\ <Y,Y>\ <Z^g,Z^g>\ <W^g,W^g> =$$

$$= Q^2\ \|X\|_1\ \|X\|_\infty\ \|Y\|_1\ \|Y\|_\infty\ <Z^g,Z^g>\ \|W^g\|_1\ \|W^g\|_\infty$$

(the last equality follows from the fact that X, Y, W^g are selfnormalizing, cf. (3.6) and the two lines following (3.6)).

Let us also notice that for every $h \in G_{n+1}$,

$$\|\Gamma \circ y^h\|_\infty = \|y^h\|_\infty = \|y\|_\infty$$

thus $\|W^h\|_\infty = \|W^g\|_\infty$. Now it is evident that

$$<\omega^g,\omega^g>^{-1}\ \|\omega^g\|_1\ \|\omega^h\|_\infty = <Z^g,Z^g>\ \|Z^g\|_1\ \|Z^h\|_\infty$$

and, since Γ_1, Γ_2 are isometries (onto), (7.8) follows by (3.4).

8. PROOF OF LEMMA 7.1.

The main ingredient here is the following combinatorial

Sublemma : There exist regular partitions ∇_g, $g \in G$, of F such that if $A \in \nabla_g$, $B \in \nabla_h$ with $g, h \in G$, $g \neq h$, then

(8.0) $$|A \cap B| \le q^{7/8} .$$

Proof : Let K be the Abelian field of order 2^p, i.e. $K = GF(2^p)$. Since $|F| = (2^p)^{16}$, we can identify F, as a set, with the vector space K^{16}. Let E and E' be two different 8-dimensional subspaces of $F = K^{16}$. Clearly $\dim_K(E \cap E') \le 7$ and therefore

(8.1) $$|E \cap E'| \le 2^{7p} = q^{7/8} .$$

It is a standard fact that, given a 2P-dimensional vector space V over a field of order β, there are at least β^{P^2} different P-dimensional subspaces of V. (To see this let us choose a basis for V, say $e_1, e_2, \ldots, e_{2P}$ and to a tuple $g = (g_{ij} : 1 \le i, j \le P)$ with $g_{ij} \in K$ let us assign

$$E_g \overset{\text{def}}{=\!=} \operatorname{span}\{ \sum_{j=1}^{P} g_{ij}\, e_j + e_{P+i} \;:\; i = 1, \ldots, P\} .$$

It should be clear that $E_g = E_h$ only if $g = h$ and we have obviously β^{P^2} different g's like above.)

In our case this means that there are at least $2^{64p} = q^8$ different 8-dimensional subspaces of $F = K^{16}$. Let us denote these by E_g, $g \in G$. Let ∇_g be the partition of F into 8-dimensional hyperplanes parallel to E_g. Then ∇_g are, obviously, regular partitions, and (8.0) follows from (8.1). □

Next let us notice that there exists an Hadamard matrix $w \in M(F,F)$ such that $\operatorname{rk} p_S\, w\, p_U = 1$ for every $S, U \in \$$ and, moreover,

(8.2) $\mathcal{R}(p_S\, w\, p_U) = \mathbb{C} \cdot \alpha_{S,U}$ where $\alpha_{S,U}$ with $S, U \in \$$ are pairwise orthogonal vectors.

Otherwords, all columns of the matrix $p_S\, w\, p_U$ are of the form $z \cdot \alpha_{S,U}$

where $z \in \mathbb{T}$ and $\alpha_{S,U} \perp \alpha_{S,T}$ if $U \neq T$.

To construct such w we take simply any $q \times q$ Hadamard matrix, say y and define for $e,f \in F$

$$w(e,f) = y(e_1,f_2)y(e_2,f_1)$$

(an $e \in F$ is written as $e = (e_1,e_2)$ with $e_1,e_2 \in H$).
We see that, if $S,U \in \$$ with $S = \{i\} \times H$, $U = \{j\} \times H$ then (8.2) is satisfied with

$$\alpha_{S,U}(e) = \begin{cases} y(e_2,j) & \text{if } e_1 = i \\ 0 & \text{otherwise} \end{cases}$$

(if we take $T \in \$$, $T \neq U$, say $T = \{k\} \times H$, then $\alpha_{S,U} \perp \alpha_{S,T}$ because the j-th and the k-th columns of y are orthogonal).

We shall also need the following, entirely trivial, remark :

(8.3) if $\mathcal{L}$ and ∇ are arbitrary regular partitions of F, then there exists a permutation ρ of F which carries ∇ onto $\mathcal{L}$, i.e. for every $B \in \nabla$, $\rho(B) \in \mathcal{L}$.

Now we can define v^g. Let ∇_g, $g \in G$, be the partitions of F from the Sublemma and, for $g \in G$, ρ_g be a permutation of F which carries ∇_g onto $\$$. We define v^g by

$$v^g(e,f) = w(e,\rho_g f) \quad ,$$

i.e. v^g is obtained by applying ρ_g^{-1} to the columns of w.

Let $S \in \$$, $h \in G$. Let us notice that

$$\mathcal{R}(p_S\, v^h\, p_B) = \mathcal{R}(p_S\, w\, p_{\rho_h B}) \quad \text{for } B \in \nabla_h \quad ,$$

therefore, by (8.2),

(8.4) $$\operatorname{rk} p_S\, v^h\, p_B = 1 \quad \text{if } B \in \nabla_h \quad ,$$

(8.5) $$\mathcal{R}(p_S\, v^h\, p_B) \perp \mathcal{R}(p_S\, v^h\, p_C) \quad \text{if } B \neq C;\ B,C \in \nabla_h \quad .$$

Now (7.2) follows by (8.4) and (3.2).

Let $g \in G_{n+1}$, $A \in \nabla_g$. For $B \in \nabla_h$, let us denote

$$u_B = p_S \, v^h \, p_{A \cap B} \quad ;$$

We have obviously

$$p_S \, v^h \, p_A = \sum_{B \in \nabla_h} u_B \quad .$$

By (8.5), $\mathfrak{R} u_B \perp \mathfrak{R} u_C$ if $B \neq C$. Since, obviously, also $\mathfrak{D} u_B \perp \mathfrak{D} u_C$ if $B \neq C$, by (3.3) we have

$$\| p_S \, v^h \, p_A \|_\infty = \max_{B \in \nabla_h} \| u_B \|_\infty \quad .$$

Clearly, u_B has $q \cdot |A \cap B|$ non zero entries, all of them of absolute value 1. Therefore, by (3.2) and (8.4),

$$\| u_B \|_\infty = (q \cdot |A \cap B|)^{1/2} \quad .$$

If now $h \neq g$, then, by (8.0), $|A \cap B| \leq q^{7/8}$ for every $B \in \nabla_h$ and this yields (7.3).

An expanded version of the present note will appear elsewhere.

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9. PASSING WITH l TO ∞.

There are, essentially, two technical problems to resolve:

1° to give meaning to the formula (5.1) for $l = \infty$.

2° to define a duality in $\mathbb{B}(H)$ so that we can define $\varphi_{\xi,\eta}$ in an analoguous way to (4.3).

A somewhat surprising fact is that, in order to settle 1°, it is more convenient to work with a space $\mathbb{B}(H_1,H_2)$ where H_1 and H_2 are two different Hilbert spaces.

Let G_n, μ_n and K_n have the same meaning as in §4. Let us denote

$$K = \prod_{j=1}^{\infty} G_j \ , \quad L_2(K) = L_2(K,\mu) \quad \text{where} \quad \mu = \prod_{j=1}^{\infty} \mu_j \ ;$$

$$K_\infty = \{\eta = (\eta_n) \in K : \eta_n = 1 \quad \text{from some } n \text{ on}\} .$$

Thus K_∞ is a countable set. For any countable set N we denote by $\ell_2(N)$ the Hilbert space of square summable functions on N. For $\eta \in N$ we define the unit vector $e_\eta \in \ell_2(N)$ by $e_\eta(\xi) = \delta_{\xi,\eta}$ (the Kronecker δ). Let us identify K_n with the subset $\{(\eta,1,1,\dots) : \eta \in K_n\}$ of K_∞ ; let H_n be the subspace of $\ell_2(K_\infty)$ spanned by $\{e_\eta : \eta \in K_n\}$.

To define our Z, we shall need that the matrices v^g from Lemma 7.1 satisfy, in addition, the following condition

(9.0) $\quad v^g(1,\eta) = 1$ and $v^g(\xi,1) = 1$ for every $g \in G$, $\xi,\eta \in F$.

(we prove at the end of this section that this can be done).

The resulting matrices $v_n \in M(G_{n+1}, F_n; F_n)$ satisfy then

$$v_n(g,\xi;\eta) = 1 \quad \text{if either} \quad \xi = 1 \quad \text{or} \quad \eta = 1.$$

Under this assumption, if $\eta \in K_\infty$, then the infinite product

$$z(\xi,\eta) = \prod_{n=1}^{\infty} v_n(\xi_{n+1}, \xi_n^1; \eta_n^0) v_n(\eta_{n+1}, \eta_n^1; \xi_n^0)$$

is well defined for every $\xi \in K$, because its terms are 1 from some n on.

It is thus natural to try to interprete z as an element of $\mathbb{B} \overset{\text{def}}{=\!=} \mathbb{B}(\ell_2(K_\infty), L_2(K))$ where we define

$$(ze_\eta)(\xi) = z(\xi,\eta).$$

It is clear that ze_η is a unimodular function in $L_2(K)$, thus

$$\|ze_\eta\| = 1 \quad \text{for every} \quad \eta \in K_\infty .$$

We shall soon prove that

(9.1) if $\eta \neq \nu$, then $ze_\eta \perp ze_\nu$.

This, obviously, implies that z is an isometry, thus, indeed $z \in \mathbb{B}$. Now we define $z_{\xi,\eta}$, $\xi \in K_m$, $\eta \in K_n$ as in §4:

For $\xi \in K_m$ let $I_\xi \subset K$, $I_\xi^\infty \subset K_\infty$ and the projections $P_\xi \in \mathbb{B}(L_2(K), L_2(K))$, $p_\xi \in \mathbb{B}(\ell_2(K_\infty), \ell_2(K_\infty))$ be defined by

$$I_\xi = \{\eta \in K : \eta_1 = \xi_1, \dots, \eta_m = \xi_m\}, \quad I_\xi^\infty = I_\xi \cap K_\infty ;$$

$$P_\xi f = f \cdot 1_{I_\xi} \quad , \quad p_\xi f = f \cdot 1_{I_\xi^\infty} \quad \text{for} \quad f \in L_2(K), \quad f \in \ell_2(K_\infty),$$

respectively.

We set for $\xi \in K_m$, $\eta \in K_n$

$$z_{\xi,\eta} = p_\xi \; z \; p_\eta \; .$$

To define $\varphi_{\xi,\eta}$, we introduce a duality in $\mathbb{B}$ in the following way: let Lim_n be a Banach limit, i.e. $\operatorname{Lim}_n \in l_\infty^*$ and, for $(t_n)_{n=1}^\infty \in l_\infty$

$$|\operatorname{Lim}_n t_n| \leq \limsup |t_n| .$$

In particular, $\operatorname{Lim}_n t_n = \lim_{n\to\infty} t_n$, if the ordinary limit exists.
We define for $x,y \in \mathbb{B}$

$$\underline{\underline{x}}(y) = \operatorname{Lim}_l |K_l|^{-1} \langle y, x_{|H_l} \rangle = \operatorname{Lim}_l |K_l|^{-1} \sum_{\eta \in K_l} \langle y e_\eta | x e_\eta \rangle \; .$$

Just for the sake of illustration let us make the following obvious remarks:

1 $(x,y) \to \underline{\underline{x}}(y)$ is a norm one sesqui-linear form on $\mathbb{B}\times\mathbb{B}$.

2 $\underline{\underline{x}}(y) = 0$ if either x or y is compact.

3 $\underline{\underline{x}}(x) = 1$ if x is an isometry (into).

For $x \in \mathbb{B}$ we denote $\|x\|_* = \|\underline{\underline{x}}\|_{\mathbb{B}^*}$.

We shall use the following simple estimates:

(9.2) $$\|x\|_* \leq \max_{\eta \in K_\infty} \|x e_\eta\|$$

(9.3) $$\|x\|_* \leq \lim_{l\to\infty} K_l^{-1} \|x_{|H_l}\|_1$$

(9.4) $$\|x\|_\infty = \lim_{l\to\infty} \|x_{|H_l}\|_\infty .$$

We define $\varphi_{\xi,\eta}$ for $\xi \in K_m$, $\eta \in K_n$ by

$$\varphi_{\xi,\eta} = \underline{\underline{z}}_{\xi,\eta} .$$

Let us now investigate the restrictions $z_{|H_l}$, $z_{\xi,\eta|H_l}$ etc. We shall show that most of the results of §6 apply to these operators as well. First let us notice that $z_{|H_l}$ is in a canonical way equivalent to the matrix $z^{(l)} \in M(K_{l+1}, K_l)$ defined by

$$z^{(l)}(\xi,\eta) = |K_l|^{-\frac{1}{2}} \prod_{j=1}^{l-1} v_j(\xi_{j+1},\xi_j^1;\eta_j^0)v_j(\eta_{j+1},\eta_j^1;\xi_j^0) \cdot$$
$$\cdot\ v_l(\xi_{l+1},\xi_l^1;\eta_l^0)v_l(1,\eta_l^1;\xi_l^0)$$

(the factor $|K_l|^{-\frac{1}{2}}$ arises from our normalization conventions: $M(K_{l+1},K_l)$ is identified with $\mathbb{B}(L_2(K_l),L_2(K_{l+1}))$ while $z_{|H_l} \in \mathbb{B}(\ell_2(K_l), L_2(K_{l+1}))$.

For $m = 1,2,\dots,l$ we define matrices $y_m^{(l)} \in M(G_m\times\dots\times G_{l+1} , G_m\times\dots\times G_l)$ by

$$y_m^{(l)}(\xi,\eta) = \prod_{j=m}^{l-1} v_j(\xi_{j+1},\xi_j^1;\eta_j^0)v_j(\eta_{j+1},\eta_j^1;\xi_j^0) \cdot$$
$$\cdot\ v_l(\xi_{l+1},\xi_l^1;\eta_l^0)v_l(1,\eta_l^1;\xi_l^0) ,$$

in particular

$$y_l^{(l)} = v_l \overset{!}{\otimes} v_l^1 ,$$

thus $y_l^{(l)}$ is a homothety. The formula (6.6) (with $T = i_{G_m\times\dots\times G_{l-1}} \otimes y_l^{(l)}$ and $y_m = y_m^{(l)}$ yields now:

(9.5) $y_m^{(l)}$ is a homothety for $m = 1,2,\dots,l$.

This, clearly, implies (9.1).

Now we shall derive the estimates needed in Proposition 2.3 from the corresponding estimates in §4-§7. In several places we repeat the former argument almost verbatim!

Ad(2.5). This is immediate because, for $l > n, m$,

$$\sum_{\xi\in K_m,\eta\in K_n} \sum_{\theta\in K_l} \langle z_{\xi,\eta}e_\theta | z_{\xi,\eta}e_\theta\rangle = \sum_{\theta\in K_l} \langle ze_\theta, ze_\theta\rangle = |K_l| .$$

Therefore $\sum_{(\xi,\eta)\in J_n} \varphi_{\xi,\eta}(z_{\xi,\eta}) = \operatorname{Lim}_l 1 = 1.$

Ad (2.6). An analogue of Lemma 6.3 is true, with an analogu-ous proof:

$$z_{\xi,\eta|H_l} \text{ is canonically equivalent to } z^{(l)}_{\xi,\eta} \overset{def}{=\!=} p_\xi z p_\eta$$

(this time, $p_\xi \in M(K_{l+1},K_{l+1})$, $p_\eta \in M(K_l,K_l)$ are defined by (4.1)). We have

$$z^{(l)}_{\xi,\eta} = Q\ \varepsilon_{\xi,\eta} \otimes (\Gamma_2\ Y^{(l)}_{n+1}\ \Gamma_1)$$

with $\Gamma_1 \in M(K_l,K_l)$ and $\Gamma_2 \in M(K_{l+1},K_{l+1})$ defined as in the proof of Lemma 6.3. We conclude, by the same argument, that $z^{(l)}_{\xi,\eta}$ is a homothety, equivalently, that $z_{\xi,\eta|H_l}$ is a homothety. This implies that $z_{\xi,\eta}$ is a homothety. Looking at $\|z_{\xi,\eta}e_\theta\|$ for a suitable θ we fined easily that

$$\text{(9.6)} \qquad \|z_{\xi,\eta}\|_\infty = |K_n|^{-\frac{1}{2}} \quad \text{if} \quad \xi \in K_m\ , \quad \eta \in K_n\ .$$

On the other hand, if $|\varepsilon(a)| = 1$ for $a \in J_n$, then the matrix $x = \sum_{a\in J_n} \varepsilon(a) z_a$ is unimodular, therefore

$$\|xe_\eta\| = 1 \quad \text{for every} \quad \eta \in K_\infty \quad \text{and, by (9.2),}$$

$$\Big\| \sum_{a \in J_n} \varepsilon(a) z a \Big\|_* \leq 1$$

This, together with (9.6), gives the desired estimate (4.8).

Ad (2.7). Although (3.4) is no longer true for $p = *$, it remains true if y and z are diagonal isometries. It is easy to see that this suffices for the argument of Lemma 4.1.

Ad (2.8). Let $m = 2n+1$ or $2n+2$, let B, ω_B, w_B, and ω^h be like in 5A. Let

$$(\omega^h)^{(1)} = \sum_{a \in B^m(c,d,h,C,D)} z_a^{(1)}$$

thus $(\omega^h)^{(1)}$ is canonically equivalent to $\omega^h_{|H_1}$. We have

$$(9.7)\quad (\omega^h)^{(1)} = \begin{cases} |K_1|^{-\frac{1}{2}}\, s^h \otimes [\Gamma\, y^{(1)}_{n+1} \quad (\varepsilon_{h,h} \otimes i_{G_{n+2} \times \ldots \times G_1})] & \text{if } m=2n+1 \\ |K_1|^{-\frac{1}{2}}\, s^h \otimes [(\varepsilon_{h,h} \otimes i_{G_{n+2} \times \ldots \times G_{1+1}})\, y^{(1)}_{n+1}\, \Gamma] & \text{if } m=2n+2\,. \end{cases}$$

Let us notice that the elements in the brackets are selfnormalizing (the first one is a partial homothety, the second one is the transpose of a partial homothety; we use (3.6)) and that their norms do not depend on $h \in G_{n+1}$. Therefore

$$(9.8)\quad \langle(\omega^g)^{(1)}, (\omega^g)^{(1)}\rangle^{-1} \|(\omega^g)^{(1)}\|_1\, \|(\omega^h)^{(1)}\|_\infty = \langle s^g, s^g\rangle^{-1} \|s^g\|_1\, \|s^h\|_\infty .$$

To obtain the desired estimate, it is now enough to make two remarks, both of which follow easily from (9.7):

$$(9.9)\qquad \|w_B^{(1)}\|_\infty = \max_{h\neq g} \|(\omega^h)^{(1)}\|_\infty ,$$

$$(9.10)\qquad <(\omega^g)^{(1)}, (\omega^g)^{(1)}> = |\Delta_m|^{-1}|K_1| .$$

To prove (9.9) we observe that the elements in the brackets in (9.7) satisfy the assumptions of (3.3), therefore, by (3.10),

$$((\omega^h)^{(1)}) \perp ((\omega^\chi)^{(1)}) \quad \text{and} \quad ((\omega^h)^{(1)}) \perp ((\omega^\chi)^{(1)}) \text{ if } h \neq \chi .$$

Now (9.9) follows by (3.3).

To prove (9.10) we notice that, by (9.7), $<(\omega^g)^{(1)},(\omega^g)^{(1)}>$ does not depend on B because neither $<s^g,s^g>$ nor $<[...],[...]>$ does. But $(\omega^g)^{(1)}$ is nothing but $\omega_B^{(1)}$ and

$$\sum_{B\in\Delta_m} <\omega_B^{(1)}, \omega_B^{(1)}> = <z^{(1)},z^{(1)}> = |K_1| .$$

Hence (9.10) follows.

In this way (9.8) becomes

$$|\Delta_m||K_1|^{-1}\|\omega_B^{(1)}\|_1 \|w_B^{(1)}\|_\infty = \max_{h\neq g} (<s^g,s^g>^{-1} \|s^g\|_1 \|s^h\|_\infty) .$$

Now we pass with 1 to ∞ and use formulas (9.3) and (9.4). We get

$$|\Delta_m| \|\omega_B\|_* \|w_B\|_\infty \leq \max_{h\neq g} (<s^g,s^g>^{-1}\|s^g\|_1 \|s^h\|_\infty) .$$

In §7 we have actually proved that

$$<s^g,s^g>^{-1}\|s^g\|_1 \|s^h\|_\infty \leq q_n^{-1/16} \quad \text{for every } h \neq g$$

and by (7.5), $\Sigma q_n^{-1/16} < \infty$. This proves (2.8).