Séminaire d'analyse fonctionnelle
École Polytechnique

P. Enflo

On infinite-dimensional topological groups

Séminaire d'analyse fonctionnelle (Polytechnique) (1977-1978), exp. n° 10 et 11, p. 1-11
<http://www.numdam.org/item?id=SAF_1977-1978____A9_0>

ÉCOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES

PLATEAU DE PALAISEAU - 91128 PALAISEAU CEDEX

Téléphone : 941.82.00 - Poste N°

Télex : ECOLEX 691 596 F

S E M I N A I R E S U R L A G E O M E T R I E

D E S E S P A C E S D E B A N A C H

1977-1978

ON INFINITE-DIMENSIONAL TOPOLOGICAL GROUPS

P. ENFLO

(Institut Mittag-Leffler)

Exposés No X-XI 13-20 Janvier 1978

I - INTRODUCTION

In this seminar we will discuss infinite-dimensional topological groups. We will do this mainly in the spirit of Hilbert's fifth problem. In 1900 Hilbert asked among other questions the following : Is every topological group that is locally homeomorphic to R_n, a Lie group ? (For definitions we refer to [7]). Around 1950 this question was given an affirmative answer as the result of the joint efforts of several researchers (see [6] for a presentation of the solution). Before this result was proved it had been proved that even weak differentiability assumptions on the group operations imply that a group is a Lie group. In 1938 G. Birkhoff [1] proved that if in a locally Euclidean group $(x,y) \to xy$ is continuously differentiable then the group is a Lie group. Birkhoff's proof works also for groups which are locally Banach -the definitions of Lie group and local Lie group extend naturally to groups which are locally Banach. Much of the elementary theory also carries over without much extra work. As an example of the contrary we mention here in passing a fundamental problem to which we do not know the answer in the infinite-dimensional case.

It is a classical theorem that a local Lie group is locally isomorphic to a Lie group. In general, though, it is not true that a local group is locally isomorphic to a topological group.

It was proved by Malcev [5] that this is true if and only if the general associative law holds in some neighbourhood of the unit element, that is if $a_1(a_2(\ldots)a_n) = (a_1\ a_2)\ldots a_n$ for all combinations of brackets which make both sides well defined. We do not know, however, whether an infinite-dimensional local Lie group is locally isomorphic to an infinite-dimensional Lie group.

Between 1938 and 1950 Birkhoff's result was improved in several steps for locally Euclidean groups. We mention the result of I. Segal [8] from 1946 which says that a group for which $x \to xy$ is continuously differentiable for every fixed y is a Lie group. In contrast to Birkhoff's proof Segal's proof works only for finite-dimensional groups since it strongly uses local compactness, for instance via tools like Haar measure. We show below that Segal's result generalizes to the infinite-dimensional case if we assume that $(x,y) \to xy$ is uniformly continuous in some neighbourhood of the unit element. Without this assumption, though, it does not generalize even if we assume that $x \to xy$ is linear in x for every y.

Before showing this we make some definitions. Birkhoff's paper [1] motivates the first definition.

Definition 1 : A local group is an analytical local group if

A) the local group is a neighbourhood of 0 in a Banach space and 0 is unit element and

B) $(x,y) \to xy$ is continuously Fréchet differentiable is some neighbourhood of 0.

Definition 2 : A local group is a left differentiable local group if

A) the local group is a neighbourhood of 0 in a Banach space and 0 is unit element and

B) $x \to xy$ is continuously Fréchet differentiable for every y in a neighbourhood of 0.

Definition 3 : A topological group is an L-group (= left linear group) if

A) a neighbourhood of the unit element is a neighbourhood of 0 in a Banach space and 0 is unit element,

B) $xy = y + T_y x$ for all x and y in some neighbourhood of 0 where T_y is a linear transformation.

Obviously an L-group is left differentiable.

2 - L-GROUPS

Example 1 : Let G be the group of continuously differentiable homeomorphisms of $[0,1]$ with $f(0) = 0$, $f(1) = 1$, $f'(x) > 0$ for all x. The group operation is $(f,g) \to f \circ g$ and the metric $d(f,g) = \sup_x |f'(x) - g'(x)|$. It is easy to check that G is a topological group. By the map $f \to f - x$ G is mapped onto a neighbourhood of 0 in the Banach space of continuously differentiable functions on $[0,1]$ with $h(0) = h(1) = 0$ and the norm $\|h\| = \sup|h'(x)|$. After the mapping G is an L-group since

$$h_1 h_2 = (h_1 + x) \circ (h_2 + x) - x = h_1 \circ (h_2 + x) + h_2 + x - x =$$

$$= h_2 + T_{h_2} h_1 \quad .$$

But G is as a topological group not isomorphic to an analytical group, which can be seen in many ways. For instance, in every neighbourhood of the unit element e there are sequences f_n, g_n s.t. $f_n g_n^{-1} \to e$ but $f_n^{-1} g_n$ does not converge to e. Moreover, arbitrarily close to e there are elements of G without square roots. Such elements are for instance given by f's which are linear with different derivatives a_1 resp. a_2 in the intervals $[0,\frac{1}{2} - \varepsilon]$ resp. $[\frac{1}{2} + \varepsilon, 1]$ where ε depends on a_1 and a_2. If $f = g \circ g$, $g \in G$ then G would have to be linear in the neighbourhoods of 0 and 1, which leads to a contradiction. In an analytical local group every element close to 0 lies on a unique local one-parameter subgroup and so, in particular, it has a square root.

Example 2 : Let G be the group of homeomorphisms of R with $f(-\infty) = -\infty$ and $f(+\infty) = +\infty$ and with

1) uniformly continuous derivatives
2) $\inf_x f'(x) > 0$
3) $\sup_x |f(x) - x| < \infty$.

The same pathologies as in Example 1 occur in this group. In this group. the element $f = x - C$, C constant has a continuum number of different square roots, in contrast to Theorem 3 below.

In [2] another type of L-group which is not an analytical group is given. Groups of m times continuously differentiable homeomorphisms of R_n can be equipped with many topologies which make them into L-groups. It might be of interest to investigate what Lie group properties are preserved by this class of groups. We give some examples on this. We say that a topological group does not have small subgroups if there is a neighbourhood U of e s.t. the only subgroup of G which is contained in U is $\{e\}$. An analytical group does not have small subgroups. The L-groups in Ex. 1 and Ex. 2 do not have small subgroups. We now prove that an L-group where the underlying Banach space is reflexive does not have small subgroups. We do not know the answer for L-groups on general Banach spaces.

<u>Theorem 1</u> : An L-group, for which the underlying Banach space is reflexive, does not have small subgroups.

<u>Proof</u> : Assume that $\{y^n\}$ is a subgroup in $U = \{x \mid \|x\| \leq \varepsilon\}$. Put $z_n = \frac{x + x^2 + \ldots + x^{n-1}}{n}$. With this we get $z_n\, x - z_n = \frac{x^n}{n}$ and $\|z_n\| \leq \varepsilon$. Let $\{z_{n_v}\}$ be a subsequence of z_n s.t. $z_{n_v} \to z$ weakly. Then

$$\|z \cdot x - z\| \leq \lim \|z_n \cdot x - z_n\| = 0$$

or

$$z \cdot x = z \quad .$$

Since $x \neq 0$ this is a contradiction.

An analytical local group has a one-parameter subgroup in every direction, that is, for every $z \in B$ $\exists$ a one-parameter subgroup $x(t)$ of G s.t. $x'(0) = z$. We do not know whether this is true for L-groups. Since in an L-group

$$x(t + \Delta t) = x(\Delta t) \cdot x(t) = x(t) + T_{x(t)} x(\Delta t) \quad ,$$

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} = T_{x(t)} \left(\frac{x(\Delta t)}{\Delta t} \right)$$

we see that with $x'(0) = z$ such a subgroup would satisfy the differential equation $x'(t) = T_x z$. In an L-group $x \to T_x z$ need not be 1st order Lipschitz in x and we do not know whether this equation has a solution.

After this discussion of L-groups we now turn to the generalisation of Segal's theorem.

3. LEFT DIFFERENTIABLE GROUPS

<u>Theorem 2</u> : A left differentiable local group is an analytical local group if and only if $(x,y) \to xy$ is uniformly continuous in some neighbourhood of 0.

A complete proof of this can be found in [2]. The steps are the following. With $xy = y + f_y(x)$ we first prove that $(x,y) \to f_y'(x)$ is continuous in both variables simultaneously in the norm topology for $f_y'(x)$. This uses the assumption that $(x,y) \to xy$ is uniformly continuous

and so it is not true in general for L-groups. Once this is done we know that $f'_y(x)$ is close to the identity operator if x and y are sufficiently close to 0, say $\|f'_y(x) - I\| \le \varepsilon$.
With this we get

$$\|(y.z^k - yz^{k-1}) - (y.z - y)\| =$$

$$= \|(z^{k-1} + f_{z^{n-1}}(yz) - (z^{k-1} + f_{z^{k-1}}(y)) - (yz - y)\| \le \varepsilon\|yz - y\| \quad .$$

This gives

$$\|y \cdot z^n - y - n(yz - y)\| \le n \ \varepsilon\|yz - y\|$$

and from this inequality we obtain that $x \to yx$ is first order Lipschitz uniformly in y. By repeating the argument a couple of times we get that in fact $x \to yx$ is continuously Fréchet differentiable and so that $(x,y) \to xy$ is continuously Fréchet differentiable.

4. GROUPS WITH LOCALLY UNIFORMLY CONTINUOUS GROUP MULTIPLICATION

In this section we remark that several results for locally compact groups are true also for groups with locally uniformly continuous group multiplication and several results for compact groups are true also for groups with uniformly continuous group multiplication. For a discussion of this and for uniform structures we refer to [3] and [4]. In view of example 2 where we do not have small subgroups but still $x^2 = y^2$, $x \neq y$ arbitrarily close to e we mention the following result,(see [2]).

<u>Theorem 3</u> : Let G be a topological group without small subgroups s.t. $(x,y) \to xy$ is locally uniformly continuous in some uniform structure on G. Then there is a neighbourhood U of e s.t. if $x \in U$, $y \in U$ and $x^2 = y^2$, then $x = y$.

5. CONSTRUCTION OF SQUARE ROOTS IN COMMUTATIVE GROUPS

In this section we will remove the differentiability assumption on the group operation and work instead with some Lipschitz condition or uniform continuity. There seems to be no result at all for non-commutative groups in this more general situation. So even the following problem seems to be open : is there a locally Hilbert local group where $(x,y) \to xy$ satisfies a first order Lipschitz condition in some neighbourhood of 0 and for which $x^3 = 0$ for all x ?

In such a group there would, of course, be no one-parameter subgroup.

For commutative groups, though, there is a method for constructing square roots by using n-dimensional cubes as defined below. The construction of square roots is a first step towards the construction of one-parameter subgroup . With some control over the position of the square root as in Theorem 6 below we get that every element of the group lies on a one-parameter subgroup. However the question of uniqueness of square roots under the assumptions of Theorem 6 remains open. The simple proof of Theorem 4 below gives the main argument in this section.

Definitions : We say that a set of 2^n-points in a metric linear space is an n-dimensional cube if the points are indexed by all n-vectors with the integers 0 and 1. There are 2^n such vectors. We say that a pair of points for which the indexes differ in exactly one coordinate is an edge. We say that a pair of points for which the indexes differ in exactly m coordinates is an m-diagonal.

Definition : We say that a Banach space has roundness p if for all quadruples of points a_{00} a_{01} a_{10} a_{11}, we have

$$\|a_{00} - a_{01}\|^p + \|a_{01} - a_{11}\|^p + \|a_{11} - a_{10}\|^p + \|a_{10} - a_{00}\|^p \geq$$

$$\geq \|a_{00} - a_{11}\|^p + \|a_{10} - a_{01}\|^p \quad .$$

Definition : We say that a Banach space is p-smooth if $\exists$ $C > 0$ such that

$$\|x + y\|^p + \|x - y\|^p \leq 2(\|x\|^p + C\|y\|^p)$$

for all x and y.

We have the following propositions

Proposition 1 : If in a Banach space B

$$\|x + y\|^p + \|x - y\|^p \leq 2(\|x\|^p + \|y\|^p)$$

for all x and y, then B has roundness p.

The proof is given in [3].

Proposition 2 : If a Banach space has roundness p, $p > 1$, then it is p-smooth. If a Banach space is p-smooth, $p > 1$, then it has roundness p_o for some $p_o > 1$.

Proof : The first part is trivial. For the second part, assume that there is no such p_o. Then by Prop. 1 we could find a sequence of points x_n, y_n, $\|y_n\| \leq \|x_n\| = 1$ and a sequence $p_n \searrow 1$ such that

$$\|x_n + y_n\|^{p_n} + \|x_n - y_n\|^{p_n} > 2(\|x_n\|^{p_n} + \|y_n\|^{p_n}) \quad .$$

It is easy to see that for any such sequence we would have $y_n \to 0$. But for $\|y_n\|$ sufficiently small and p_n sufficiently close to 1 this contradicts p-smoothness.

Proposition 3 : In an n-dimensional cube in a space with roundness p we have

$$\sum_{\alpha} s_{\alpha}^p \geq \sum_{\beta} d_{n,\beta}^p$$

where s_α runs through the lengths of the edges and $d_{n,\beta}$ runs through the lengths of the n-diagonals.

The proof is an easy induction by the dimension. We get

Corollary : In an n-dimensional cube in a space with roundness $p > 1$

$$n^{1/p} \cdot s_{max} \geq d_{n,min}$$

where s_{max} is the maximal length of an edge and $d_{n,min}$ is the minimal length of an n-diagonal.

We do not know whether Proposition 1 generalizes to higher dimensions, so we have the following question : if a Banach space B has type p is it then true that $\exists$ K such that for all n and all n-dimensional cubes in B we have $\sum_\alpha s_\alpha^p \geq K \sum_\beta d_{n,\beta}^p$?

We now turn to the construction of square roots.

<u>Theorem 4</u> : If a Banach space B with roundness p is given a commutative groups structure $(x,y) \to xy$ s.t. $\|xy - zy\| = O(\|x - z\|^{1/p})$ uniformly in y as $\|x - z\| \to 0$ then the set of elements of the form x^2 is dense in B.

<u>Proof</u> : We assume that 0 is the unit element of the group. Consider a $z \in B$ and for every n form the elements $y_k = \frac{kz}{n} \cdot \left(\frac{(k-1)z}{n}\right)^{-1}$, $k = 1,2,\ldots,n$. Then $y_1 y_2 \ldots y_n = z$. Form the n-dimensional cube consisting of the 2^n products

$$y_{k_1} y_{k_2} \ldots y_{k_r} \quad , \quad k_1 < k_2 < \ldots . < k_r \quad , \quad r \leq n \quad ,$$

where these 2^n points are indexed in an obvious way. In this n-dimensional cube we have by assumption $s_{max} = o\left(\frac{1}{n^{1/p}}\right)$ as $n \to \infty$. And so by the corollary of Proposition 3 we have $d_{n,min} = o(1)$ as $n \to \infty$. But if (d_1, d_2) is an n-diagonal then $d_1 \, d_2 = z$. Thus $\min\|d_1^2 - z\| \to 0$ and the theorem is proved.

<u>Remark</u> : In Theorem 4 we have given the whole Banach space a group structure. We do not know whether the local version of this theorem is true. The argument does not give a good control over the position of the square root.

In the next theorem we will only assume uniform continuity of the group multiplication. The conclusion is only that $\max d(x,M) < \infty$ where M is the set of elements of the form x^2 but on the other hand we can introduce a metric which gives us these "approximative square roots" at about half the distance to 0.

Theorem 5 : Let B be a Banach space with roundness p, $p > 1$, and let $(x,y) \to xy$ be a commutative group structure on B s.t. $(x,y) \to xy$ is uniformly continuous and 0 is unit. Then there is a group invariant metric d on B, $d \geq \| \ \|$, and constants K_1 and K_2 such that for each $z \in B$ there is an $x \in B$ such that

1) $d(x^2,z) < K_1$

2) $|d(z,0) - 2d(x,0)| < K_2(d(z,0))^{1/p}$.

A complete proof of this can be found in [3]. We sketch it here. First introduce a metric d' in B by $d'(x,y) = \sup_z \|xz - yz\|$. So d' is group invariant and $d' \geq \| \ \|$. Then choose an $\varepsilon > 0$ and let $d(x,y)$ be infimum of lengths of ε-chains between x and y in the d'-metric. An ε-chain is a sequence of points $x = z_o, z_1, z_2 \dots z_k = y$ such that $d'(z_i, z_{i-1}) \leq \varepsilon$. The length of the chain is $\sum_i d'(z_i, z_{i-1})$. With this definition d is group invariant and $d \geq \| \ \|$. Moreover $\exists$ K such that if $d(x,y)$ then

$$d(x,y) \leq K\|x - y\| \quad .$$

To construct a square root of z, take an ε-chain $0 = z_o, z_1, \dots, z_n = z$ between 0 and z of almost minimal length. Let $y_j = z_j \, z_{j-1}^{-1}$ and form the n-dimensional cube with the points $y_{k_1} \, y_{k_2} \dots y_{k_r}$. Since the edges here have lengths $\leq \varepsilon$ in $\| \ \|$ the shortest n-diagonal (d_1, d_2) has length $\leq K\varepsilon . n^{1/p}$ in d which is of a smaller order of magnitude than $d(z,0)$ which is about εn. Since d_1 and d_2 are near each other their distance to 0 must be about $\frac{1}{2} d(z,0)$ since $z_o \, z_1 \dots z_n$ is an ε-chain of almost minimal length. Repetition of this argument with z replaced by $z \cdot d_1^{-2}$ will complete the proof.

With the stronger assumption that $(x,y) \to xy$ is first order Lipschitz we can get Theorem 5 for local groups.

Theorem 6 : Let U be a neighbourhood of 0 in a Banach space with roundness p, $p > 1$. Let $(x,y) \to xy$ be a local commutative group structure on U with 0 as unit, s.t. $(x,y) \to xy$ is first order Lipschitz.

Then there is a group invariant metric $d(x,y)$ on some neighbourhood U of 0 s.t. for every Z in V, and every $\varepsilon > 0$, $\exists$ x in V s.t. $z = x^2$ and $|d(z,0) - 2d(x,0)| \leq \varepsilon$.

The proof of this is similar to that of Theorem 5 but we replace the minimal length of ε-chain by minimal arc-length. It is clear that the assumption roundness > 1 can be weakened in Theorem 6; we do not know exactly how much. Perhaps the Radon-Nikodym property is enough.

6. AN APPLICATION TO UNIFORM HOMEOMORPHISMS BETWEEN TOPOLOGICAL LINEAR SPACES

The existence of approximate square roots at approximatively half the distance to 0 as is given by Theorem 5 gives us new information if we assume that the group is the additive group of a locally bounded topological linear space [as for instance the L_p-spaces, $0 < p < 1$]. In this situation Theorem 5 will enable us to construct a norm on the topological linear space and so we get the following

Theorem 7 : If a locally bounded topological linear space is uniformly homeomorphic to a Banach space with roundness > 1, then it is a normable space.

The complete proof of this is given in [3].

Corollary : $L_p(0,1)$ is not uniformly homeomorphic to $L_q(0,1)$, if $0 < p < 1$, $1 < q < \infty$.

We do not know whether roundness > 1 can be replaced by uniform convexity or even reflexivity in Theorem 7. And we do not know whether the Corollary is true for $q = 1$, or whether it is true that $L_p(0,1)$ is not uniformly homeomorphic to $L_q(0,1)$ if $p \neq q$, $0 < p < 1$, $0 < q < 1$.

REFERENCES

[1] G. Birkhoff, Analytical groups, Trans. A.M.S. 43 (1938) 61-101.

[2] P. Enflo, Topological groups in which multiplication on one side is differentiable or linear, Math. Scand. 24 (1969) 195-207.

[3] P. Enflo, Uniform structures and square roots in topological groups, Israel J. Math. 8 (1970) 230-272.

[4] J. Kelley, General topology, van Nostrand 1955.

[5] A. Malcev, Sur les groupes topologiques locaux et complets, Doklody Acad. Sc. URSS 32 (1941) 606-608.

[6] D. Montgomery and L. Zippin, Topological transformation groups, New York 1955.

[7] L. Pontrjagin, Topological groups.

[8] I. Segal, Topological groups in which multiplication on one side is differentiable, Bull. A.M.S. 52 (1946) 481-487.
