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Séminaire d'analyse fonctionnelle (Polytechnique) (1977-1978), exp. nº 9, p. 1-14
<http://www.numdam.org/item?id=SAF_1977-1978____A8_0>

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1977-1978

SYMMETRIC STRUCTURES IN SOME BANACH LATTICES

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Exposé No IX 6 Janvier 1978

The purpose of these notes is to present certain results which will appear in a joint paper with W.B. Johnson, B. Maurey and G. Schechtman [4].

The starting point is a characterization of symmetric basic sequences of finite length in a Banach lattice of type 2. Instead of stating this theorem in the framework of Banach lattices of type 2 we prefer to use the equivalent context of Banach lattices which are 2-convex and q-convex for some $q<\infty$.

We recall that a Banach lattice Z is said to be r-convex for $1\le r\le\infty$ if there exists a constant $M<\infty$ such that, for every choice of $\{z_i\}_{i=1}^n$ in Z,

$$\Big\|\Big(\sum_{i=1}^n |z_i|^r\Big)^{1/r}\Big\| \le M\Big(\sum_{i=1}^n \|z_i\|^r\Big)^{1/r}, \quad \text{for } r<\infty$$

respectively

$$\Big\|\max_{1\le i\le n} |z_i|\Big\| \le M \max_{1\le i\le n} \|z_i\| \quad , \quad \text{for } r=\infty \quad .$$

The smallest possible value taken by M is denoted by $M^{(r)}(Z)$ and is called the r-convexity constant of Z. The dual notion of r-concavity is defined by replacing the sign $\le$ with $\ge$ and requiring that $M>0$. Again, the smallest possible value for M^{-1} is denoted by $M_{(r)}(Z)$ and is called the r-concavity constant of Z.

These notions were essentially introduced in [3] but most of their basic properties were described in [6] (see also [8]). The equivalence between the notion of type 2 for a lattice, on one hand, and that of 2-convexity together with some $q<\infty$-concavity, on the other, follows easily from a generalization, due to B. Maurey [8], of the classical inequality of Khintchine. The actual value of q in this equivalence is of no importance ; what really counts is to ensure that the lattice does not contain uniformly isomorphic copies of ℓ_∞^n for all n.

We also recall that a sequence $\{z_i\}_{i=1}^n$ is called K-symmetric provided that, for every permutation π of the integers and every choice of scalars $\{a_i\}_{i=1}^n$,

$$\Big\|\sum_{i=1}^n a_{\pi(i)} z_i\Big\| \le K \Big\|\sum_{i=1}^n a_i z_i\Big\| \quad .$$

Theorem 1 : Let X be a Banach lattice which is 2-convex and 2m-concave for some integer m. Then, for every $K \geq 1$, there exists a constant $D = D(K) < \infty$ so that, for every normalized K-symmetric basic sequence $\{x_i\}_{i=1}^n$ in X and every choice of scalars $\{a_i\}_{i=1}^n$, we have

$$\|\sum_{i=1}^n a_i x_i\| \overset{D}{\sim} \max\{(\sum_\pi \|\max_{1\le i\le n} |a_{\pi(i)} x_i|\|^{2m}/n!)^{1/2m}, w_n(\sum_{i=1}^n |a_i|^2)^{1/2}\} ,$$

where $w_n = \|\sum_{i=n}^n x_i\|/n^{1/2}$ and $\sum_\pi$ means summation over all n! permutations π of the first n integers. The notation $A \overset{D}{\sim} B$ is used instead of writing $D^{-1}A \le B \le DB$. The constant D depends only on K, m, $M^{(2)}(X)$ and $M_{(2m)}(X)$.

Proof : Let $|||.|||$ be a K-equivalent new norm on the linear span $U = [x_i]_{i=1}^n$ of $\{x_i\}_{i=1}^n$ such that, endowed with the new norm, $\{x_i\}_{i=1}^n$ is 1-symmetric and still normalized. Clearly, the 2-convexity constant $M^{(2)}(U)$ of U endowed with $|||.|||$ depends only on K and the type 2-constant of X. Moreover, if $\sum_\sigma$ means summation over all the cyclic permutations of $\{1,2,\ldots,n\}$ then, for every choice of $\{a_i\}_{i=1}^n$,

$$|||\sum_{i=1}^n a_i x_i||| =$$

$$= (\sum_\sigma |||\sum_{i=1}^n a_{\sigma(i)} x_i|||^2/n)^{1/2} \ge M^{(2)}(U)^{-1} n^{-1/2} |||\sum_{i=1}^n (\sum_\sigma |a_{\sigma(i)}|^2)^{1/2} x_i|||$$

$$= M^{(2)}(U)^{-1} n^{-1/2} (\sum_{i=1}^n |a_i|^2)^{1/2} |||\sum_{i=1}^n x_i||| .$$

This of course proves that, for some constant $D_1 < \infty$ and every choice of $\{a_i\}_{i=1}^n$,

$$D_1 \|\sum_{i=1}^n a_i x_i\| \ge w_n(\sum_{i=1}^n |a_i|^2)^{1/2} .$$

The above inequality is the only part in the proof of Theorem 1 which actually requires the assumption that X is 2-convex. For the remainder of the proof it would suffice to assume that X is B-convex (i.e. that it has some non-trivial type).

We also notice, for every permutation π of the first integers,

$$K\|\sum_{i=1}^n a_i x_i\| \ge \int_0^1 \|\sum_{i=1}^n a_{\pi(i)} r_i(u) x_i\| du \ge$$

$$\ge \|\int_0^1 |\sum_{i=1}^n a_{\pi(i)} r_i(u) x_i| du\| \ge \|\max_{1\le i\le n} |a_{\pi(i)} x_i|\| .$$

Hence, by averaging in the sense of $\ell_{2m}^{n!}$ over all possible π, it follows that

$$K\|\sum_{i=1}^{n} a_i\, x_i\| \geq (\sum_{\pi} \|\max_{1\leq i\leq n} |a_{\pi(i)}\, x_i|\,\|^{2m}/n!)^{1/2m} \quad .$$

This proves one side of the inequality. The proof of the other inequality is still quite elementary but technically more difficult. Let $\{a_i\}_{i=1}^{n}$ be a fixed sequence of scalars. Then, for every permutation π of $\{1,2,\ldots,n\}$, it follows, by using Khintchine's inequality and the 2m-concavity of X, that

$$\begin{aligned}
\|\sum_{i=1}^{n} a_i\, x_i\| &\leq K\int_0^1 \|\sum_{i=1}^{n} a_{\pi(i)}\, r_i(u)\, x_i\|du \leq \\
&\leq K(\int_0^1 \|\sum_{i=1}^{n} a_{\pi(i)}\, r_i(u)\, x_i\|^{2m}du)^{1/2m} \leq \\
&\leq K\, M_{(2m)}(X)\, \|(\int_0^1 |\sum_{i=1}^{n} a_{\pi(i)}\, r_i(u)\, x_i|^{2m}du)^{1/2m}\| \leq \\
&\leq K\, M_{(2m)}(X)\, B_{2m}\, \|(\sum_{i=1}^{n} |a_{\pi(i)}\, x_i|^2)^{1/2}\| = \\
&= K\, M_{(2m)}(X)\, B_{2m}\, \|((\sum_{i=1}^{n} |a_{\pi(i)}\, x_i|^2)^m)^{1/2m}\| \quad .
\end{aligned}$$

Hence, by expanding the inner sum to the m-th power (here is where we use that m is an integer) and by averaging in the sense of $\ell_{2m}^{n!}$ over all possible π, we get that

$$\|\sum_{i=1}^{n} a_i\, x_i\| \leq K\, M_{(2m)}(X)\, B_{2m}(n!)^{-1/2m}\, (S_1 + S_2) \quad ,$$

where

$$S_1 = (\sum_{\pi} \|(\sum_{\substack{i_1,\ldots,i_m \\ \text{distinct}}} |a_{\pi(i_1)}\, x_{i_1}|^2 \cdots |a_{\pi(i_m)}\, x_{i_m}|^2)^{1/2m}\|^{2m})^{1/2m}$$

and

$$S_2 = (\sum_{\pi}\|(\sum_{\ell=1}^{m-1} \sum_{\substack{m_1+\cdots+m_\ell=m \\ m_1,\ldots,m_\ell\geq 1}} \sum_{\substack{i_1,\ldots,i_\ell \\ \text{distinct}}} |a_{\pi(i_1)}\, x_{i_1}|^{2m_1} \cdots$$

$$\cdots |a_{\pi(i_\ell)}\, x_{i_\ell}|^{2m_\ell})^{1/2m}\|^{2m})^{1/2m} \quad .$$

The expression S_1 can be estimated as follows :

$$S_1 \le M_{(2m)}(X)\|(\sum_{\pi} \sum_{\substack{i_1,\dots,i_m \\ \text{distinct}}} |a_{\pi(i_1)}x_{i_1}|^2 \dots |a_{\pi(i_m)}x_{i_m}|^2)^{1/2m}\| =$$

$$= M_{(2m)}(X)\ \|(\sum_{\substack{i_1,\dots,i_m \\ \text{distinct}}} (\sum_{\pi} |a_{\pi(i_1)}|^2 \dots |a_{\pi(i_m)}|^2)|x_{i_1}|^2 \dots |x_{i_m}|^2)^{1/2m}\| .$$

We assume now (without loss of generality) that n satisfies $n/2 > m$. In this case $n-m+1 > n/2$ and therefore we get that

$$S_1 \le M_{(2m)}(X)((n-m)!)^{1/2m}(\sum_{i=1}^{n} |a_i|^2)^{1/2} \|(\sum_{i=1}^{n} |x_i|^2)^{1/2}\| \le$$

$$\le M_{(2m)}(X)(n!)^{1/2m} \cdot 2^{1/2}\ n^{-1/2}\ A_1^{-1} \int_0^1 \|\sum_{i=1}^{n} r_i(u)\ x_i\|du \le$$

$$\le 2\ M_{(2m)}(X)\ K(n!)^{1/2m}\ w_n\ (\sum_{i=1}^{n} |a_i|^2)^{1/2} ,$$

where $A_1(=1/\sqrt{2})$ is the Khintchine coefficient in $L_1(0,1)$.

To evaluate the expression S_2 we first fix $1 \le \ell < m$ and $m_1,\dots,m_\ell$ and put

$$S(\ell;m_1,\dots,m_\ell) =$$

$$= (\sum_{\pi} \|(\sum_{\substack{i_1,\dots,i_\ell \\ \text{distinct}}} |a_{\pi(i_1)}x_{i_1}|^{2m_1} \dots |a_{\pi(i_\ell)}x_{i_\ell}|^{2m_\ell})^{1/2m}\|^{2m})^{1/2m} .$$

Then, by applying Hölder's inequality in the lattice $\ell_{2m}^{n!}(X)$ (see e.g. [6]) for $\theta_j = m_j/m$ and the vectors $z_j = (z_j(\pi))_\pi \in \ell_{2m}^{n!}(X)$; $j = 1,2,\dots,\ell$, where

$$z_j(\pi) = (\sum_{i=1}^{n} |a_{\pi(i)}\ x_i|^{2m_j})^{1/2m_j} ,$$

we get that

$$S(\ell;m_1,\dots,m_\ell) \le \|z_1^{\theta_1} \dots z_\ell^{\theta_j}\|_{\ell_{2m}^{n!}(X)} \le \|z_1\|_{\ell_{2m}^{n!}(X)}^{\theta_1} \dots \|z_\ell\|_{\ell_{2m}^{n!}(X)}^{\theta_\ell} .$$

But $\ell < m$ and thus at least one of the integers m_j, $j = 1,2,\dots,\ell$, say m_1, is strictly larger than 1. Therefore, by estimating the norm in $\ell_{2m_j}^n$ by that in ℓ_4^n and the norm in $\ell_{2m_j}^n$, $1<j\le\ell$, by that in ℓ_2^n, it follows that

$$S(\ell;m_1,\dots,m_\ell) \le \Big(\sum_\pi \Big\|\Big(\sum_{i=1}^n |a_{\pi(i)}x_i|^4\Big)^{1/4}\Big\|^{2m}\Big)^{\theta_1/2m} \times$$
$$\times \Big(\sum_\pi \Big\|\Big(\sum_{i=1}^n |a_{\pi(i)}x_i|^2\Big)^{1/2}\Big\|^{2m}\Big)^{(1-\theta_1)/2m} \le$$
$$\le \big(A_1^{-1}K(n!)^{1/2m}\big)^{1-\theta_1} \Big\|\sum_{i=1}^n a_i x_i\Big\|^{1-\theta_1} \cdot \Big(\sum_\pi \Big\|\Big(\sum_{i=1}^n |a_{\pi(i)}x_i|^4\Big)^{1/4}\Big\|^{2m}\Big)^{\theta_1/2m} .$$

Using the estimate for S_1 we get that

$$\Big\|\sum_{i=1}^n a_i x_i\Big\| \le 4K^2 M_{(2m)}(X)^2 B_{2m} \max\Big\{w_n\Big(\sum_{i=1}^n |a_i|^2\Big)^{1/2}, (n!)^{-1/2m} S_2\Big\} .$$

Hence, if the maximum is not attained at the first term then, by using the estimate obtained for S_2, it follows that, for some constant $C = C(K,m,M^{(2)}(X),M_{(2m)}(X))$,

$$\Big\|\sum_{i=1}^n a_i x_i\Big\| \le C(n!)^{-\theta_1/2m} \Big\|\sum_{i=1}^n a_i x_i\Big\|^{1-\theta_1} \Big(\sum_\pi \Big\|\Big(\sum_{i=1}^n |a_{\pi(i)}x_i|^4\Big)^{1/4}\Big\|^{2m}\Big)^{\theta_1/2m}$$

Consequently,

$$\Big\|\sum_{i=1}^n a_i x_i\Big\| \le C\Big(\sum_\pi \Big\|\Big(\sum_{i=1}^n |a_{\pi(i)}x_i|^4\Big)^{1/4}\Big\|^{2m}/n!\Big)^{1/2m} ,$$

from which, by applying another variant of Hölder's inequality (see e.g. [6]) for the vectors $v_i = (v_i(\pi))_\pi \in \ell_{2m}^{n!}(X)$, $i = 1,2,\dots,n$ defined by $v_i(\pi) = a_{\pi(i)}x_i/(n!)^{1/2m}$, we get that

$$\Big\|\sum_{i=1}^n a_i x_i\Big\| \le C \Big\|\Big(\sum_{i=1}^n |v_i|^4\Big)^{1/4}\Big\|_{\ell_{2m}^{n!}(X)} \le$$
$$\le C \Big\|\Big(\sum_{i=1}^n |v_i|^2\Big)^{1/2}\Big\|^{1/2}_{\ell_{2m}^{n!}(X)} \cdot \Big\|\max_{1\le i\le n} |v_i|\Big\|^{1/2}_{\ell_{2m}^{n!}(X)} .$$

This completes the proof since, as is easily checked,

$$\Big\|\Big(\sum_{i=1}^n |v_i|^2\Big)^{1/2}\Big\|_{\ell_{2m}^{n!}(X)} \le KA_1^{-1} \Big\|\sum_{i=1}^n a_i x_i\Big\| . \qquad \square$$

The importance of Theorem 1 can be illustrated by presenting some of its applications. For example, in the particular case when $X = L_p(0,1)$ for $p > 2$ Theorem 1 can be restated in the following simplified form.

Theorem 2 : For every $p > 2$ and $K \ge 1$ there exists a constant $D = D(p,K)$ such that, for every normalized K-symmetric basic sequence $\{x_i\}_{i=1}^n$ in

$L_p(0,1)$ and every choice of scalars $\{a_i\}_{i=1}^n$,

$$\|\sum_{i=1}^n a_i x_i\| \overset{D}{\sim} \max\{(\sum_{i=1}^n |a_i|^p)^{1/p}, w_n(\sum_{i=1}^n |a_i|^2)^{1/2}\} ,$$

where $w_n = \|\sum_{i=1}^n x_i\|/n^{1/2}$.

Proof : This is an immediate consequence of the inequality

$$\|\max_{1\le i\le n} |a_{\pi(i)}x_i|\| \le \|(\sum_{i=1}^n |a_{\pi(i)} x_i|^p)^{1/p}\| = (\sum_{i=1}^n |a_i|^p)^{1/p}$$

together with the fact that, for $p>2$, $L_p(0,1)$ is of cotype p which implies that

$$K\|\sum_{i=1}^n a_i x_i\| \ge (\sum_{i=1}^n |a_i|^p)^{1/p} . \quad \square$$

Theorem 2 already shows that symmetric basic sequences in $L_p(0,1)$, $p>2$ have a very special form. While in the infinite dimensional case they must be equivalent to the unit vector basis of either ℓ_p or ℓ_2 (cf. [5]) it follows from Theorem 2 that, in the finite dimensional case, they generate what, in the terminology of H.P. Rosenthal [10], is called an X_p-space. The results of H.P. Rosenthal from the above mentioned paper show e.g. that each such space is isomorphic to a complemented subspace of $L_p(0,1)$ (with the isomorphism and projection constants independent of the dimension of the subspace).

Theorem 2 can be also used to establish a series of results on the uniqueness of some symmetric structures. One such application is related to the well-known fact that each of the spaces ℓ_p, $1\le p<\infty$ has, up to equivalence, a unique symmetric basis. D.R. Lewis asked, in connection with this fact, whether a similar result holds also for the ℓ_p^n-spaces, $n=1,2,\dots$ To give a precise meaning to the notion of uniqueness for symmetric bases in finite dimensional spaces we introduce the following definition.

Definition 3 : Let $\mathfrak{F}$ be a family of finite-dimensional Banach spaces each of which has a normalized 1-symmetric basis. We say that each element of $\mathfrak{F}$ has a unique symmetric basis if there exists a function $D(K)$, $K\ge 1$, so that whenever a space $F\in\mathfrak{F}$ has a normalized K-symmetric basis $\{f_i\}_{i=1}^n$ then $\{f_i\}_{i=1}^n$ is $D(K)$-equivalent to the 1-symmetric basis of F.

In order to state the result on the uniqueness of symmetric bases of finite length we also need the following very simple known fact.

<u>Proposition 4</u> : Let $\{y_i\}_{i=1}^n$ be a K-symmetric normalized basis of a subspace Y of $L_p(0,1)$, $p > 2$. Then

$$K^{-2}\, n^{1/2}/\|\sum_{i=1}^n y_i\| \le d(Y,\ell_2^n) \le K^3\, B_p^2\, n^{1/2}/\|\sum_{i=1}^n y_i\| \ .$$

<u>Proof</u> : The right hand side inequality is easily proved by using the formal identity map from Y onto ℓ_2^n (use the 2-convexity of $\{y_i\}_{i=1}^n$ as in the proof of Theorem 1 and the fact that Y is of type 2 with constant equal to the Khintchine constant B_p). Conversely, if T is an invertible operator from Y into $L_2(0,1)$ so that $\|T\|\cdot\|T^{-1}\| = d(Y,\ell_2^n)$ then, by transforming the vectors $\{Ty_i\}_{i=1}^n$ into a K-equivalent sequence of symmetrically exchangeable random variables in $L_2(0,1)$ (i.e. a sequence of n random variables whose joint distribution in R^n is invariant under permutations or changes of signs), we conclude that $d(Y,\ell_2^n) \ge K^{-1}\, n^{1/2}/\|\sum_{i=1}^n y_i\|$. We leave the details to the reader. □

<u>Theorem 5</u> : i) Each member of the family $\mathcal{G}_p$ of all subspaces of $L_p(0,1)$, $p > 2$, which have a normalized 1-symmetric basis, has a unique symmetric basis.

ii) Each member of the family $\mathcal{F}_p = (\ell_p^n)_{n=1}^\infty$, $1 \le p \le \infty$ has a unique symmetric basis.

<u>Proof</u> : Part (i) is an immediate consequence of Theorem 2. If $\{x_i\}_{i=1}^n$ is an arbitrary normalized K-symmetric basis of an $X \in \mathcal{G}_p$, $p > 2$ then, a priori, the expression $w_n = \|\sum_{i=1}^n x_i\|/n^{1/2}$, appearing when we apply Theorem 2 for $\{x_i\}_{i=1}^n$, depends on the particular K-symmetric basis used. However, in view of Proposition 4, w_n is essentially equal to $1/d(X,\ell_2^n)$ i.e. we get the same value for w_n (up to a constant which depends only on K and p) no matter which normalized K-symmetric basis of X has been used.

Part (ii) for $1 < p < \infty$ is an obvious corollary of (i) while for $p = 1$ and $p = \infty$ this fact has been proved in [7]. □

Remark : By using a more complicated argument it is possible to show that, in fact, even each member of the family $\bigcup_{p\geq 1} \mathfrak{F}_p$ has a unique symmetric basis i.e. that the constants appearing in the proof of Part (ii) of Theorem 5 can be chosen as to be independent of p. Whether a similar assertion is true also for $\bigcup_{p\geq 2} \mathcal{G}_p$ is not known. Actually, it is easily seen that a positive answer to this question would be equivalent to the fact that each member of the family of all finite dimensional spaces with a 1-symmetric basis has a unique symmetric basis. Although, as mentioned above, this problem is still open, some results of a positive nature were proved in [4]. It was shown there that, for given $q<2$ and $M<\infty$, each member of the class $\mathcal{C}_{q,M}$, of all finite dimensional Banach spaces with a normalized 1-symmetric basis which has q-concavity constant $\leq M$, has a unique symmetric basis.

Theorem 2 has additional applications to some questions concerning rearrangement invariant (r.i.) function spaces. If we restrict ourselves to countably generated measure spaces without atoms then we have to consider only the cases of r.i. function spaces on $[0,1]$ or on $[0,\infty)$. An r.i. function space X on an interval I, where I is either $[0,1]$ or $[0,\infty)$, will be a Banach lattice of measurable functions on I (endowed with the pointwise order) which, for simplicity, is supposed to satisfy the following two conditions.

(i) The integrable simple functions on I belong to X and form a dense set there.

(ii) The function $\chi_{[0,1]}$ has norm one.

Besides the class of $L_p(0,1)$ and $L_p(0,\infty)$-spaces the best known class of r.i. function spaces is that of Orlicz function spaces. For instance, it is known (cf. [1]) that any $L_p(0,1)$, $1\leq p<2$ contains a large class of Orlicz function spaces as well as other r.i. function spaces on $[0,1]$ or on $[0,\infty)$. For $p>2$ the situation is however completely different, as is shown by the following result.

Theorem 6 : Assume that an r.i. function space X on an interval I, where I is $[0,1]$ or $[0,\infty)$, is isomorphic to a subspace of $L_p(0,1)$, $p>2$. Then, up to an equivalent norm, X is equal to $L_p(I)$, $L_2(I)$ or to $L_p(I)\cap L_2(I)$, the last possibility being distinct only in the case when $I=[0,\infty)$.

Proof : Consider first the case $I=[0,1]$ and let T be an isomorphism from X into $L_p(0,1)$, $p>2$. For every integer n the sequence $\{T\chi_{[(i-1)2^{-n},i2^{-n})}\}_{i=1}^{2^n}$ is K-symmetric in $L_p(0,1)$ with $K \le \|T\| \cdot \|T^{-1}\|$. Therefore, once this sequence is normalized we are in a position to apply Theorem 2. It follows that there exists a constant $C<\infty$, independent of n, so that, for every step function Ψ of the form

$\Psi = \sum_{i=1}^{2^n} a_i \chi_{[(i-1)2^{-n},i2^{-n})}$, $n=1,2,\ldots$, we have

$$\|\Psi\|_X \overset{C}{\sim} \max\{\alpha_n \|\Psi\|_p , \|\Psi\|_2\} ,$$

where $\alpha_n = \|\chi_{[0,2^{-n})}\|_X \, 2^{n/p}$. For a fixed Ψ this formula is clearly valid even when n is replaced by any other integer $m>n$. Hence, we may as well put $\alpha = \liminf_n \alpha_n$ instead of α_n (observe that, by setting $\Psi \equiv 1$ in the above formula, we get $\alpha_n \le C$ for all n) and conclude that

$$\|\Psi\|_X \overset{C}{\sim} \max\{\alpha \|\Psi\|_p , \|\Psi\|_2\} ,$$

for any step function Ψ as above. Since the set of these step functions is dense in X it follows that

$$\|f\|_X \overset{C}{\sim} \max\{\alpha \|f\|_p , \|f\|_2\} ,$$

for all $f \in X$. Consequently, if $\alpha=0$ then $X=L_2(0,1)$ while for $\alpha>0$ we have that $X=L_p(0,1)$, up to an equivalent norm.

The case $I=[0,\infty)$ can be deduced from that of $I=[0,1]$. Indeed, observe first that, for every integer n, the restriction of X to the interval $[0,n]$ can be contracted to an r.i. function space on $[0,1]$. It follows that there exist β_n and γ_n such that, for every $f \in X$ which is supported by the interval $[0,n]$.

$$\|f\|_X \overset{C}{\sim} \max\{\beta_n \|f\|_p , \gamma_n \|f\|_2\}$$

and, again, for a fixed f, the numbers β_n and γ_n can be replaced by $\beta = \liminf_n \beta_n$, respectively $\gamma = \liminf_n \gamma_n$. Hence, for any $f \in X$,

$$\|f\|_X \overset{C}{\sim} \max\{\beta \|f\|_p , \gamma \|f\|_2\} .$$

The cases $\beta = 0$ and $\gamma = 0$ lead to $X = L_2(0,\infty)$, respectively $X = L_p(0,\infty)$, while when both β and γ are strictly positive we get that, up to an equivalent renorming, $X = L_p(0,\infty) \cap L_2(0,\infty)$. That $L_p(0,\infty) \cap L_2(0,\infty)$ is actually isomorphic to a subspace of $L_p(0,1)$ for every $p > 2$ can be seen by considering the subspace of $L_p(0,\infty) \oplus_\infty L_2(0,\infty)$ (which is clearly isomorphic to $L_p(0,1)$) consisting of those pairs (f,g) for which $f = g$.

<u>Corollary 7</u> : (i) The space $L_p(0,1)$, $1 \leq p \leq \infty$ has a unique representation as an r.i. function space on $[0,1]$.

(ii) The space $L_p(0,\infty)$, $1 < p \neq 2 < \infty$ has two distinct representations as an r.i. function space on $[0,\infty)$, namely $L_p(0,\infty)$ itself and $L_p(0,\infty) \cap L_2(0,\infty)$ if $p > 2$ or $L_p(0,\infty) + L_2(0,\infty)$ when $1 < p < 2$.

Assertion (i) is a direct consequence of Theorem 6 for $1 < p \neq 2 < \infty$. For $p = 1$ and $p = \infty$ the uniqueness was essentially proved in [9] ; the proof being based on [7]. The same argument also shows that $L_1(0,\infty)$ and $L_\infty(0,\infty)$ have each a unique representation as an r.i. function space on $[0,\infty)$.

The space $L_p(0,\infty) + L_2(0,\infty)$ appearing in the statement (ii), is the dual of $L_q(0,\infty) \cap L_2(0,\infty)$, where $1/p + 1/q = 1$. The proof of (ii) follows also from Theorem 6 and the fact that, e.g. when $p > 2$, $L_p(0,\infty) \cap L_2(0,\infty)$ is isomorphic to $L_p(0,\infty)$. This latter fact can be proved by using the decomposition method provided we know that $L_p(0,\infty)$ contains a complemented copy of $L_p(0,\infty) \cap L_2(0,\infty)$. To check this fact one can use either Poisson processes as in [4] or the fact, due to H.P. Rosenthal [10], that X_p-spaces are isomorphic to complemented subspace of $L_p(0,\infty)$, for all $p > 2$. This suffices since $L_p(0,\infty) \cap L_2(0,\infty)$ is clearly a complemented subspace of any ultrapower of the subspaces generated by $\{\chi_{[(i-1)2^{-n}, i2^{-n})}\}_{i=1}^{n2^n}$; $n = 1,2,\ldots$ and each such subspace is an X_p-space (and their ultrapowers are complemented in some L_p-space).

Assertion (i) of Corollary 7 is actually valid for a larger class of r.i. function spaces on $[0,1]$. In [4] Section 5 it is shown, for example, that any $q < 2$ concave r.i. function space on $[0,1]$ has a unique representation as an r.i. space on $[0,1]$. Without the assumption of $q < 2$ concavity this result is not true in general. A 2-concave r.i. function space on $[0,1]$ with uncountably many mutually non-equivalent representations as an r.i. function space on $[0,1]$ was constructed in [4] Section 10. In fact, such an example can be built for every $1 < p < 2$ as to have, in addition, the property that, on one hand, it embeds isomorphically in $L_p(0,1)$ and, on the other, it contains a complemented copy of $L_p(0,1)$.

We conclude with two more applications of Theorem 1.

Theorem 8 : Suppose that a Banach lattice X is isomorphic to a subspace of a Banach lattice Y which is of type 2 and, in addition, satisfies an upper $r > 2$ estimate (i.e. $\|\sum_{i=1}^{m} y_i\| \le M(\sum_{i=1}^{m} \|y_i\|^r)^{1/r}$ for some constant $M < \infty$ and for every sequence of pairwise disjoint vectors $\{y_i\}_{i=1}^m$ in Y). Then either X itself satisfies an upper r-estimate or ℓ_2 is disjointly finitely representable in X.

In the case when $Y = L_r(0,1)$, $r > 2$ the theorem was originally proved in [2]. The statement is quite simple in this particular case.

Corollary 9 : Suppose that a Banach lattice X is linearly isomorphic to a subspace of $L_r(0,1)$, $r > 2$. Then either X is linearly isomorphic and order equivalent to an $L_r(\nu)$-space for a suitable ν or ℓ_2 is disjointly finitely representable in X.

A Banach lattice X as above clearly satisfies a lower r-estimate for disjoint elements (since $L_r(0,1)$ is of cotype r). On the other hand, if ℓ_2 is not disjointly finitely representable in X, then by Theorem 8, X satisfies also an upper r-estimate. Thus, by [9], it is linearly isomorphic and order equivalent to an $L_r(\nu)$-space.

The other application of Theorem 1 is a variant of Theorem 8 which is valid in the special case when X is an r.i. function space on [0,1].

Theorem 10 : Suppose that an r.i. function space X is isomorphic to a subspace of a Banach lattice Y which is of type 2 and $r > 2$ convex. Then either X itself is r-convex or $X = L_2(0,1)$, up to an equivalent renorming.

Since the proof of the Theorems 8 and 10 use the same basic ideas we shall present here only that of Theorem 10. Both proofs rely on the following observation.

Lemma 11 : Let $\{y_i\}_{i=1}^n$ be an arbitrary sequence of vectors in an r-convex Banach lattice Y and let $q \geq r$. Let $|||\cdot|||$ be a norm on R^n defined by

$$|||a||| = (\sum_{\pi} \| \max_{1\leq i\leq n} |a_{\pi(i)} y_i| \|^q)^{1/q} \quad ; \quad a = \{a_i\}_{i=1}^n \in R^n \quad .$$

Then, R^n endowed with the norm $|||\cdot|||$ forms an r-convex Banach lattice whose r-convexity constant is $\leq M^{(r)}(Y)$.

Proof : Since $q \geq r$, it clearly suffices to show that R^n endowed with the norm

$$\|a\|_o = \| \max_{1\leq i\leq n} |a_i y_i| \|$$

is r-convex with r-convexity constant $\leq M^{(r)}(Y)$. If $a^k = \{a_i^k\}_{i=1}^n$, $k = 1,2,\ldots,m$ is a sequence of vectors in R^n then

$$\|(\sum_{k=1}^m |a^k|^r)^{1/r}\|_o = \| \max_{1\leq i\leq n} (\sum_{k=1}^m |a_i^k y_i|^r)^{1/r}\| \leq$$

$$\leq \|(\sum_{k=1}^m \max_{1\leq i\leq n} |a_i^k y_i|^r)^{1/r}\| \leq M^{(r)}(Y)(\sum_{k=1}^m \| \max_{1\leq i\leq n} |a_i^k y_i| \|^r)^{1/r} =$$

$$= M^{(r)}(Y)(\sum_{k=1}^m \|a^k\|_o^r)^{1/r} \quad . \quad \square$$

Proof of Theorem 10 : As already observed above, one can find an integer m so that Y is 2m-concave. Let T be an isomorphism from X into Y and fix n. By applying Theorem 1 to the $\|T\| \cdot \|T^{-1}\|$-symmetric sequence $\{T\chi_{[(i-1)2^{-n}, i2^{-n})}\}_{i=1}^{2^n}$ in Y we conclude the existence of a constant $C < \infty$ independent of n so that, for every step function $f \in X$ of the form $f = \sum_{i=1}^{2^n} a_i \chi_{[(i-1)2^{-n}, i2^{-n})}$, we have

$$(+) \quad \|f\|_X \overset{C}{\sim} \max\{(\sum_{\pi} \| \max_{1\leq i\leq 2^n} |a_{\pi(i)} T\chi_{[(i-1)2^{-n}, i2^{-n})}\|_Y^{2m}/(2^n!))^{1/2m}, \|f\|_2\} \quad .$$

If, for every integer n,

$$\|\chi_{[0,2^{-n})}\|_X \le 2CM \|\chi_{[0,2^{-n})}\|_2$$

where M is a constant exceeding both the r-convexity and 2m-concavity constants of Y then, for every step function f as above (with n arbitrary), we have

$$\|f\|_X \le M\Big(\sum_{i=1}^{2^n} \|a_i \chi_{[(i-1)2^{-n},i2^{-n})}\|_X^2\Big)^{1/2} \le 2CM^2 \|f\|_2 \quad .$$

On the other hand, (+) shows that also

$$C \|f\|_X \ge \|f\|_2 \quad .$$

This proves that, in this case, X is equal to $L_2(0,1)$, up to an equivalent renorming.

Suppose now that there exists an integer k so that

$$\|\chi_{[0,2^{-k})}\|_X > 2CM \|\chi_{[0,2^{-k})}\|_2 \quad .$$

Then, as easily verified, when we evaluate the norm of $\chi_{[0,2^{-k})} = \sum_{i=1}^{2^{n-k}} \chi_{[(i-1)2^{-n},i2^{-n})}$, $n \ge k$ by using the formula (+) then the maximum is attained by the first term i.e.

$$\|\chi_{[0,2^{-k})}\|_X \overset{C}{\sim} \Big(\sum_\pi \Big\| \max_{1\le i\le 2^{n-k}} |T\chi_{[(\pi(i)-1)2^{-n},\pi(i)2^{-n})}| \Big\|_Y^{2m}/(2^n!)\Big)^{1/2m} \quad .$$

By Lemma 11 the expression

$$|||a||| = \Big(\sum_\pi \Big\| \max_{1\le i\le 2^{n-k}} |a_i \chi_{[(\pi(i)-1)2^{-n},\pi(i)2^{-n})}| \Big\|_Y^{2m}/(2^n)!\Big)^{1/2m}$$

defines a lattice norm on $R^{2^{n-k}}$ whose r-convexity constant does not exceed M. Hence

$$M|||a||| \ge |||\underbrace{\{1,\dots,1\}}_{2^{n-k}\text{ times}}|||\Big(\sum_{i=1}^{2^{n-k}} |a_i|^r/2^{n-k}\Big)^{1/r} \quad ,$$

for every $a \in R^{2^{n-k}}$, which means that if $\Psi = \sum_{i=1}^{2^{n-k}} b_i \chi_{[(i-1)2^{-n},i2^{-n})}$ is an arbitrary step function entirely supported by the interval $[0,2^{-k})$ then, for $b=\{b_i\}_{i=1}^{2^{n-k}}$,

$$|||b||| \geq M^{-1}C^{-1}\,\|\chi_{[0,2^{-k})}\|_X\Big(\sum_{i=1}^{2^{n-k}}|b_i|^r/2^{n-k}\Big)^{1/r} \geq 2\,\|\Psi\|_2 \quad .$$

This means that if we evaluate the norm in X of functions entirely supported by the interval $[0,2^{-k})$ by using the formula (+) then the maximum is always attained in the first term. Thus, by Lemma 11, the restriction of X to $[0,2^{-k})$ is r-convex and so is all of X. □

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