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SEQUENTIALLY CONTINUOUS MAPPINGS OF PRODUCT SPACES

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§ 0. INTRODUCTION

It is trivial that any sequentially continous mapping between metric spaces is continuous. It is natural to pose the following question : for what classes of product of metric spaces are the sequentially continuous mappings (e.g. into metric spaces) continuous ? At first this problem in a proper way was posed in an extremely interesting paper of S. Mazur [1]. For about 20 years this paper was the most advanced in this direction. Mazur had shown that this problem can be reduced to the investigation of some special topological and set theoretical properties. After this in the classical review of Keisler and Tarski [2] the Mazur results were quoted and some concrete questions about sequentially continuous mappings of $2^{\Delta} \to 2$, $2^{\Delta} \to \mathbb{R}$ were given. Now we formulate these questions.

0.1 Is the existence of sequentially continuous but not continuous mapping $2^{\Delta} \to 2$ equivalent to the Ulam measurability of $|\Delta|$?

Cardinal $|\Delta|$ satisfying the conditions of 0.1 is called strongly sequential.

0.2 Is the existence of sequentially continuous but not continuous mapping of $2^{\Delta} \to \mathbb{R}$ equivalent to real mesurability of $|\Delta|$ (i.e. to the existence of real valued complete countably additive measure on Δ).

Cardinal $|\Delta|$ satisfying the condition of 0.2 is called sequential.

0.3 Let θ_{α} be the hierarchy of all weakly inacessible cardinals. Are all the cardinals $|\Delta| < \theta_{\theta_1}, \theta_{\theta_{\theta_1}}, \ldots$ non-sequential (i.e. any sequentially continuous mapping $2^{\Delta} \to \mathbb{R}$ is continuous).

We'll give below the review on positive results in this direction and some of their generalizations.

§ 1. EXPOSITION OF MAZUR RESULTS.

Now we explain why most of the information about the sequentially continuous mappings between metric spaces is given by sequential cardinals. The basic result on this reduction were established by Mazur [1]

Definition 1.1 : A mapping $f : X \to Y$ is called sequentially continuous, if for every sequence $\{x_n\}_{n\in\omega}$ of X, from $\lim_{n\to\infty} x_n = x_o$ it follows that $\lim_{n\to\infty} f(x_n) = f(x_o)$.

Mazur was the first, who has found (in [1]) a well-known and now largely used "representation theorem" generalized later by Gleason, Isbell and others.

Theorem 1.2 (Mazur [1]) : Let B be a metric space and $\{A_t\}_{t\in T}$ be a family of second countable Hausdorff spaces and $A = \prod_{t\in T} A_t$. Let $|T|$ be a non-sequential cardinal and f be a sequentially continuous mapping from A to B. Then f is continuous and depends on countably many coordinates. In other words, there is a countable set $P\subset T$ such that $f = f_o \cdot \pi_p$, where π_p is the projection of A onto $\prod_{t\in p} A_t$ and f_o is a continuous map from $\pi_p A$ to B.

For the case of a general (not only metric) space B instead of the property "$|T|$ is not sequential" Mazur introduced a general property of $\mathfrak{A}$-reducibility.

Definition 1.3 : Let $\mathfrak{A}$ be a property of classes of sets satisfying the following conditions :

a) if $M\subset P(\Delta)$ has the property $\mathfrak{A}$, then M is sequentially closed and is a G_δ set in the sequential topology of $P(\Delta)$;

b_1) if $M\subset P(\Delta)$ satisfies $\mathfrak{A}$, then for any $\Delta'\subset\Delta$ the set $M\cap P(\Delta')$ also has the property $\mathfrak{A}$;

b_2) let $\varphi : \Delta \to \Delta'$ and $M\subset P(\Delta)$ satisfies $\mathfrak{A}$. Then $M_\varphi = \{E\subset\Delta' : \varphi^{-1}(E)\in M\}\subseteq P(\Delta')$ satisfies $\mathfrak{A}$ too.

1.4 A class of sets satisfying the property $\mathfrak{A}$ is called $\mathfrak{A}$-class.

Definition 1.5 : A set Δ is $\mathfrak{A}$-reducible, if for any $\mathfrak{A}$-class $M \subseteq P(\Delta)$ that contains all finite subsets of Δ, we have $\Delta \in M$.

An important example of property $\mathfrak{A}$ is the following. Let C be a Hausdorff space and $H \subseteq C$ a closed G_δ-set in the sequential topology of C. A set $M \subset P(\Delta)$ has the property $[C,H]$ if there is a sequentially continuous mapping $\Psi : P(\Delta) \to C$ such that $M = \{E : \Psi(E) \in H\}$.

Lemma 1.6 ([1]) : The property $[C,H]$ satisfies the conditions a) and b_1), b_2) of the 1.3. A set Δ is $[C,H]$-reducible if every sequentially continuous mapping $P(\Delta) \to C$ transforming all finite subsets of Δ into H transforms the Δ into H.

The definition 1.5 is important in view of the general theorem of Mazur [1].

Let B be a Hausdorff space with the property

(D) the diagonal D of $B \times B$ is a G_δ-set in the sequential topology of the product $B \times B$.

Theorem 1.7 : Let $\{A_t\}_{t \in T}$ be a family of second countable Hausdorff spaces and $A = \prod_{t \in T} A_t$. Let f be a sequentially continuous map from A into B. If T is $[B \times B, D]$-reducible, then f is continuous.

Thus f depends only on countably many coordinates. In other words, there is a countable set $P \subset T$ such that $f = f_o \cdot \pi_p$ where π_p is the projection of A onto $\prod_{t \in p} A_t$ and f_o is a continuous map from $\pi_p A$ into B.

The main result of Mazur concerning $\mathfrak{A}$-reducibility is the following.

Theorem 1.8 ([1]) : The cardinal ω_o is $\mathfrak{A}$-reducible. If m is $\mathfrak{A}$-reducible and $n \leq m$, then n is $\mathfrak{A}$-reducible. If $m_\xi : \xi < n$ are $\mathfrak{A}$-reducible and n is $\mathfrak{A}$-reducible, then $m = (\sum_{\xi<n} m_\xi)^+$ is $\mathfrak{A}$-reducible. In particular all cardinals $< \theta_1$ are $\mathfrak{A}$-reducible.

The proof uses the so-called Ulam matrix on cardinal α^+. In fact, for $\alpha \geq \omega_0$, $\zeta < \alpha^+$ and $\eta < \alpha$ there are such $\{A_\eta^{(\zeta)} : \eta < \alpha\}$, that $\bigcup_{\eta<\alpha} A_\eta^{(\zeta)} \cup (\zeta + 1) = \alpha^+ : \zeta < \alpha^+$ and $A_\eta^{(\zeta)} \cap A_\eta^{(\xi)} = \emptyset : \eta < \alpha, \zeta \neq \xi$. These sets are constructed as follows : for $\xi < \alpha^+$ let f_ξ be a one-to-one function from ξ to α and let us $A_\eta^{(\zeta)} = \{\xi : \zeta < \xi < \alpha^+ : f_\xi(\zeta) = \eta\}$ for $\zeta < \alpha^+$ and $\eta < \alpha$. It is evident that this definition of $A_\eta^{(\zeta)}$ satisfies all the properties of Ulam's matrix.

Theorem 1.8 of Mazur was generalized in 1970 by the author using Hajnal's analogue of Ulam's matrix for inaccessible cardinals. Let $\overline{AC}$ be the class of cardinals that are not weakly inaccessible and SN the class of singular cardinals α [i.e. $\alpha = \sum_{\gamma<\beta} n_\gamma$, where $\beta < \alpha$ and $n_\gamma < \alpha : \gamma < \beta$]. For a cardinal α we denote by $C(\alpha)$ the family of closed and unbounded in α subsets of α in the order topology of α.

<u>Theorem 1.9</u> [4] : Let α be inaccessible cardinal and let there exists a set $A \subseteq \alpha \cap SN$ such that $A \in C(\alpha)$. If any cardinal $\beta < \alpha$ is $\mathfrak{A}$-reducible, then α is $\mathfrak{A}$-reducible.

In particular $\theta_1, \theta_2, \ldots, \theta_{\theta_1}, \ldots$ are all $\mathfrak{A}$-reducible. The proof of theorem 1.9 uses generalization of Ulam's matrix for this inaccessible cardinal α : there are such $\{A_\eta^{(\xi)} : \eta < \xi\}$, $\xi \in A$ that $\bigcup_{\eta<\xi} A_\eta^{(\xi)} \cup (\xi + 1) = A$ and $A_\eta^{(\xi)} \cap A_\eta^{(\zeta)} = \emptyset$ for $\xi \neq \zeta$, $\xi, \zeta \in A$. It was shown in [10] that for $A \subset \alpha \cap SN$, $A \in C(\alpha)$ such a matrix $\{A_\eta^{(\xi)} : \eta < \xi , \xi \in A\}$ exists.

In fact it is true more general

<u>Theorem 1.10</u> : For $X \subset Card$, $M'(X) = \{\beta\text{-regular cardinals} : X \cap \beta$ contains a set from $C(\beta)\}$, $M^1(X) = X \cup M'(X)$, $M^{\xi+1}(X) = M^1(M^\xi(X))$ and $M^\delta(X) = \bigcup_{\xi<\delta} M^\xi(X)$ for limit ordinals δ. Let

$$\alpha_0 = \min(M^{\omega_0+1}(\overline{AC}) \backslash M^{\omega_0}(\overline{AC})) \quad ,$$

then all cardinals $< \alpha_0$ are $\mathfrak{A}$-reducible.

Using Jensen's [5] we obtain assuming the axiom of constructivity $V = L$, that :

<u>Corollary 1.11</u> : $(V = L)$ All the cardinals $\alpha < C_0$ -the first compact (weakly compact) are $\mathfrak{A}$-reducible.

§ 2. STRONGLY SEQUENTIALLY CARDINALS.

We'll give a positive answer to the problem of Keisler-Tarski : whether properties "strongly sequential" and "Ulam measurable" are equivalent ?

First of all by Mazur's result 1.7 we have :

Proposition 2.1 : There is a sequentially continuous but not continuous mapping $P(\Delta) \to 2$ if and only if there is a sequentially continuous mapping $F : P(\Delta) \to 2$ such that

(S^2) $F(X) = 0$ for all finite $X \subset \Delta$ and $F(\Delta) = 1$.

Such sequentially continuous mappings $F : P(\Delta) \to 2$ that satisfy (S^2) are called Mazur's mappings.

In one direction we have a trivial answer for Keisler-Tarski questions, because a lot of examples of Mazur's mappings are given by measures on Δ :

Proposition 2.2 : (i) Let $|\Delta|$ be an Ulam measurable cardinal, i.e. there is a $\{0,1\}$-valued non-trivial countable additive measure $\mu : P(\Delta) \to 2$. Then μ is a Mazur map.

(ii) Let $|\Delta|$ be real-measurable cardinal ; then there is a measure $\mu : P(\Delta) \to \mathbb{R}$ which is a sequentially continuous mapping, satisfying (S^2).

The structure of an arbitrary Mazur's mapping is not so simple as in 2.2 -they may be different from measures. For example in 1970 there was an attempt by Noble [9] to solve the Keisler-Tarski problem, he supposed that for any Mazur's map (of type 2.1 - (S^2)) its restriction is a Ulam measure. This is wrong. We'll construct a Mazur's map no restriction of which is Ulam measure :

Proposition 2.3 (assuming the existence of MC-measurable cardinals) : There are Mazur's mappings such that no restrictions of them are Ulam measures.

Proof : Really, let $\alpha \notin U$ be an Ulam measurable cardinal and $|A_1| = |A_2| = ($

$A_1 \cap A_2 = \emptyset$ and we have $\mu_i : P(A_i) \to 2 : i = 1,2$ be the Ulam's measures. Let $A = A_1 \cup A_2$, then we define a sequentially continuous mapping $\sigma : P(A) \to 2$ by putting :

for $E \subset A = A_1 \cup A_2$,

$$\sigma(E) = \min\{\mu_1(E \cap A_1),\ \mu_2(E \cap A_2)\} \quad .$$

Then $\sigma(E)$ is sequentially continuous as μ_i are sequentially continuous and σ is Mazur map, because $\sigma(A) = 1$ and $\sigma(X) = 0$ for finite $X \subset A$.

But no restriction of σ is a Ulam measure. Let, on the contrary, suppose that $B \subset A$ and let $\sigma \wedge B : P(B) \to 2$ be Ulam measure. So $\sigma(B) = 1$. But

$$B = (B \cap A_1) \cup (B \cap A_2) \quad ,$$

and for any $E \subseteq A$, $\sigma(E) \le \mu_i(E \cap A_i)$. We put $E = B \cap A_1$ and $\sigma(B \cap A_1) \le \mu_2(B \cap A_1 \cap A_2) \le \mu_2(\emptyset) = 0$. Analogously $\sigma(B \cap A_2) = 0$. Thus $\sigma(B \cap A_1) = \sigma(B \cap A_2) = 0$ and $\sigma(B) = 1$. So no restriction of σ is a Ulam measure

This example shows that a Mazur's mapping of $P(\Delta)$ into 2 can be a "pasting" of measure. We use the reverse idea -of "unsticking"- to show that all Mazur's mapping can be obtained by "pasting". We have positive solution of Keisler-Tarski problem :

<u>Theorem 2.4</u> : A cardinal α is strongly sequential iff α is Ulam-measurable.

<u>Proof</u> : In one side it is trivial : if $\alpha \notin U$, then by 2.2 α is strongly sequential.

Let α be the least strongly sequential cardinal. We'll show below that $\alpha \notin U$. By 2.1, there is a Mazur's map $\sigma : P(\alpha) \to 2$-sequentially continuous mapping satisfying (S^2). With the aid of σ we will construct a countable-additive measure on α. We show some lemmas :

<u>Lemma 1</u> : Let $\sigma : P(A) \to 2$ be Mazur's mapping. Then there exists such $A_o \subseteq A$, that $\sigma(A_o) = 1$ and

1) σ is monotonic on $P(A_o)$: if $B \subseteq C \subseteq A_o$, then $\sigma(B) \le \sigma(C)$ and
2) from $\sigma(B) = \sigma(C) = 1$ it follows that $B \cap C$-infinite.

Proof : Let us first show part 1). We must show that there is $A'_o \subseteq A$ such that $\sigma(A'_o) = 1$ and σ is monotonic on $P(A'_o)$, that is from $B \subseteq C \subseteq A'_o$ and $\sigma(B) = 1$ if follows that $\sigma(C) = 1$.

Assume that there is no such $A'_o \subseteq A$. Then we have

(*) for any $X \subseteq A$, $\sigma(X) = 1$ there are such $Y \subseteq Z \subseteq X$ that $\sigma(Y) = 1$ and $\sigma(Z) = 0$.

We use (*). We put $D_o = A$ and obtain from (*), $D_1 \subseteq D_1 \subseteq D_o$ with $\sigma(D_2) = 1$, $\sigma(D_1) = 0$. Then we apply (*) to D_2 etc. We get a descending sequence $\{D_n\}_{n=0}^{\infty}$ of subsets of α for which $\sigma(D_{2n}) = 1 : n < \infty$, $\sigma(D_{2n+1}) = 0 : n < \infty$. This is impossible because $\lim_{n\to\infty} D_{2n} = \lim_{n\to\infty} D_{2n+1}$ as $\lim_{n\to\infty} D_n$ exists. Then by the sequential continuity of σ, $1 = \lim_{n\to\infty} \sigma(D_{2n}) = \lim_{n\to\infty} \sigma(D_{2n+1}) = 0$, so $0 = 1$. Thus (*) is not true and there is $A'_o \subseteq A$, $\sigma(A'_o) = 1$, with monotonicity of σ on $P(A'_o)$. The proof of 2) is similar -if 2) is not true, we take $A'_o \subseteq A$, $\sigma(A'_o) = 1$, with $A_1, A_2 \subseteq A'_o$, $\sigma(A_1) = \sigma(A_2) = 1$, $A_1 \cap A_2$-finite, then $A_3, A_4 \subseteq A_1$, $\sigma(A_3) = \sigma(A_4) = 1$, $A_3 \cap A_4$-finite,... $A_{2n+1}, A_{2n} \subseteq A_{2n-1}$, $\sigma(A_{2n+1}) = \sigma(A_{2n}) = 1$, $A_{2n+1} \cap A_{2n} = E_n$-finite. Then $\{A_{2n} \setminus E_n\}$ are disjoint, so $\lim_{n\to\infty} \{A_{2n} \setminus E_n\} = \emptyset$; we take convergent subsequence $\{E_{n_k}\} \to E_\infty$;then $\lim A_{2n_k} = \lim_{k\to\infty} \{A_{2n_k} \setminus E_{n_k}\} \cup E_{n_k} = \emptyset \cup \lim E_{n_k}$. So $1 = \lim \sigma(A_{2n_k}) = \lim \sigma(E_{n_k}) = 0$ $\to$ impossible.

Now we describe "unsticking" : how to construct from a given Mazur's map and a pair of disjoint sets two new maps.

Let σ and A_o satisfy all the requirements of lemma 1.

Lemma 2 : Suppose that E_1, $E_2 \subseteq A_o$, $E_1 \cap E_2 = \emptyset$ and $\sigma(E_1 \cup E_2) = 1$. We define two new s.c. mappings

$$\sigma_{E_2}^{(E_1)} : P(E_2) \longrightarrow 2 \ ; \ \sigma_{E_1}^{(E_2)} : P(E_1) \longrightarrow 2$$

by the following definitions :

$$\text{for } V_1 \subset E_1 \ , \qquad \sigma_{E_1}^{(E_2)}(V_1) = \sigma(V_1 \cup E_2) \ ;$$

$$\text{for } V_2 \subset E_2 \ , \qquad \sigma_{E_2}^{(E_1)}(V_2) = \sigma(V_2 \cup E_1) \ .$$

Then, either σ_{E_1} or σ_{E_2} is a Mazur's map.

Proof : It is clear that both σ_{E_1}, σ_{E_2} are seq. continuous since σ is s.c. Secondly, since $\sigma(E_1 \cup E_2) = 1$,

$$\sigma_{E_1}(E_1) = \sigma_{E_2}(E_2) = \sigma(E_1 \cup E_2) = 1 \quad .$$

Let us assume that neither σ_{E_1} nor σ_{E_2} satisfies (S^2). Then there exist finite sets

$$V_1 \subseteq E_1 \quad , \quad V_2 \subseteq E_2$$

such that $$\sigma_{E_1}(V_1) = \sigma_{E_2}(V_2) = 1$$

or the same

$$(**) \qquad \sigma(V_1 \cup E_2) = 1 \ ; \ \sigma(V_2 \cup E_1) = 1 \quad .$$

From lemma 1 it follows that $(V_1 \cup E_2) \cap (V_2 \cup E_1)$ is infinite. On the other hand, $(V_1 \cap E_2) \cap (V_2 \cup E_1) = V_1 \cup V_2$-finite [as $V_i \subseteq E_i$] -impossible-. Thus either σ_{E_1} or σ_{E_2} is a Mazur's map.

Using the idea od "unsticking" we prove the basic :

<u>Lemma 3</u> : There is a Mazur's mapping $\mu : P(\Delta) \to 2$, $|\Delta| = \alpha$, such that

(3) if $E, D \subseteq \Delta$ and $E \cap D = \emptyset$ and $\mu(E) = \mu(D) = 0$, then $\mu(E \cup D) = 0$.

<u>Proof</u> : We assume, on the contrary, that there are no Mazur's maps, satifying (3). Now let φ be an arbitrary Mazur's map $\varphi : P(A) \to 2$, then we find $A^o_\varphi = A_o \subseteq A$ such that for φ and A_o, lemma 1 is true and because (3) is not true, we find $A_1, A_2 \subseteq A_o$, such that

$$(4) \qquad \begin{aligned} &\varphi(A_1) = \varphi(A_2) = 0 \quad , \quad A_1 \cap A_2 = \emptyset \\ \text{but} \quad &\varphi(A_1 \cup A_2) = 1 \quad , \quad A_1, A_2 \subseteq A^o_\varphi \end{aligned}$$

(here (4) is the negation of (3)).

We set $\varphi_o = \sigma$. Then we have disjoint $A^o_o, A^o_1 \subseteq A_o$ such that $\sigma(A^o_o) = \sigma(A^o_1) = 0$, $\sigma(A^o_o \cup A^o_1) = 1$, $A^o_o \cap A^o_1 = \emptyset$. Now we apply to the pair (A^o_o, A^o_1) the lemma 2. So either $\sigma^{(A^o_1)}_{A^o_o}$ or $\sigma^{(A^o_o)}_{A^o_1}$ is a Mazur's map. Let this

Mazur's map be $\sigma_{A_1^o} : P(A_1^o) \to 2$. We denote $\sigma_{A_1^o}$ by φ_1.

Next we apply (4) to φ_1. Then we obtain $A_2^o = A^o{}_{\varphi_1} \subseteq A_1^o$ satisfying lemma 1 and A_o^1, $A_1^1 \subseteq A_2^o$ such that $A_o^1 \cap A_1^1 = \emptyset$;

$$\varphi_1(A_o^1) = \varphi_1(A_1^1) = 0 \quad , \quad \varphi_1(A_o^1 \cup A_1^1) = 1 \quad .$$

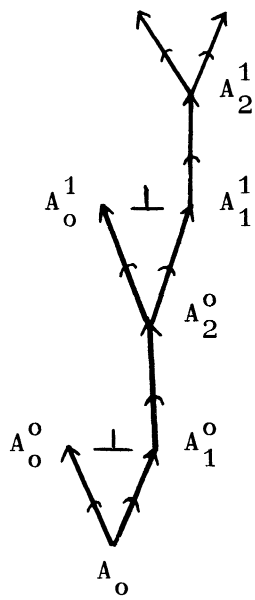
So, by induction we obtain for every $n \geq 0$ two sets A_o^n, A_1^n and a Mazur's mapping $\varphi_{n+1} : P(A_1^n) \to 2$, $\varphi_{n+1} = (\varphi_n)_{A_1^n}^{(A_o^n)}$ and $A_2^n \subseteq A_1^n$ satisfying together with φ_{n+1} lemma 1, and for which we have sets A_o^{n+1}, A_1^{n+1} with

(5)
$$A_o^{n+1} \cup A_1^{n+1} \subseteq A_2^n \subseteq A_1^n \quad ; \quad A_n^{n+1} \cap A_1^{n+1} = \emptyset$$
$$\varphi_{n+1}(A_o^{n+1}) = \varphi_{n+1}(A_1^{n+1}) = 0 \quad ; \quad \varphi_{n+1}(A_o^{n+1} \cup A_1^{n+1}) = 1 \quad .$$

By the construction $\varphi_{n+1} = (\varphi_n)_{A_1^n}^{(A_o^n)}$ ... so we have

(6)
$$\varphi_{n+1}(E) = \sigma(E \cup A_o^o \cup \ldots \cup A_o^n)$$

for any $E \subseteq A_2^n$.



Now we put $E_n'' = A_1^{n+1} \cup \bigcup_{i=0}^{n} A_o^i$ and $E_n' = A_o^{n+1} \cup E_n'' = (A_o^{n+1} \cup A_1^{n+1}) \cup \bigcup_{i=0}^{n} A_o^i$.

From (5) and (6) it follows that

(7) $$\sigma(E''_n) = 0 \;;\; \sigma(E'_n) = 1 : n = 1,2,\ldots \quad .$$

The sequence $\{A_o^n\}_{n=1}^{\infty}$ is disjoint since $A_o^n \subseteq A_1^m$ for $n > m$ and $A_1^m \cap A_o^m = \emptyset$. So $\lim_{n\to\infty} A_o^n = \emptyset$. Next $\{E'_n\}_{n=1}^{\infty}$ is a descending sequence of sets : $E'_n \supseteq E'_{n+1}$: $n = 1,2,\ldots$. In fact

(8) $$E'_{n+1} = A_o^{n+2} \cup A_1^{n+2} \cup \bigcup_{i=0}^{n+1} A_o^i =$$

$$= \underbrace{A_o^{n+2} \cup A_1^{n+2}} \cup A_o^{n+1} \cup \bigcup_{i=0}^{n} A_o^i \subseteq \underbrace{A_1^{n+1}} \cup A_o^{n+1} \cup \bigcup_{i=0}^{n} A_o^i = E'_n$$

since $$A_o^{n+2} \cup A_1^{n+2} \subseteq A_1^{n+1} \quad .$$

So $\{E'_n\}_{n=1}^{\infty}$ is monotone and $\lim_{n\to\infty} E'_n$ exists. Now

(9) $$E'_{n-1} \setminus A_o^n = E''_{n-1} \quad ,$$

really, $E'_{n-1} \setminus A_o^n = (\underbrace{A_o^n} \cup A_1^n) \cup \bigcup_{i=0}^{n-1} A_o^i \setminus \underbrace{A_o^n} = A_1^n \cup \bigcup_{i=0}^{n-1} A_o^i = E''_{n-1}$ since different A_o^j are disjoint.

Because $\{A_o^n\}_{n=1}^{\infty}$ are disjoint :

(10) $$\lim_{n\to\infty} (E'_{n-1} \setminus A_o^n) = \lim_{n\to\infty} E'_{n-1} \setminus \lim_{n\to\infty} A_o^n = \lim_{n\to\infty} E'_{n-1} \quad ,$$

or (11) $$\lim_{n\to\infty} E''_n = \lim_{n\to\infty} E'_n \quad .$$

By the sequential continuity of σ,

(12) $$0 = \lim_{n\to\infty} \sigma(E''_n) = \lim_{n\to\infty} \sigma(E'_n) = 1 \quad .$$

Thus we come to a contradiction with (4). Lemma 3 is proved.

Applying to $\mu : P(\Delta) \to 2$ of lemma 3, also lemma 1 we obtain a set such that

(i) μ is a Mazur's map : $\mu : P(D) \to 2$;

(ii) μ is monotonous on $P(D)$;

(iii) $\mu(A) = \mu(B) = 0$ for $A \cap B = \emptyset$ and $A, B \subseteq D$ implies $\mu(A \cup B) = 0$;

(iv) $\mu(A) = \mu(B) = 1$ and $A, B \subseteq D$ implies $A \cap B \neq \emptyset$.

<u>Lemma 4</u> : A Mazur's map μ with (i)-(iv) is, in fact, a Ulam measure.

Proof : μ is finitely additive. Really, let $A,B \subseteq D$, $A \cap B = \emptyset$. If $\mu(A) = \mu(B) = 0$, then

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

by (iii), since $\mu(A \cup B) = 0$. If $\mu(A) = 1$, $\mu(B) = 0$, then $\mu(A \cup B) = 1$ by (ii). The case $\mu(A) = \mu(B) = 1$ is impossible by (iv). By the sequential continuity of μ it is also countable additive.

Finally μ is a non-trivial measure as μ is a Mazur's map.

Thus $|D| = \alpha$ and α is Ulam measurable, i.e. $\alpha \geq k_o$ - the first measurable cardinal. But k_o is strongly sequential, i.e. $k_o \geq \alpha$. So $\alpha = k_o$ and theorem 2.4 is completely proved.

By Mazur's theorem 1.7 and theorem 2.4 we have :

Corollary 2.5 : There is a non-continuous, sequentially continuous mapping from product $\prod_{i \in I} X_i$ of second countable Hausdorff spaces to 2 (or any discrete metric space) if and only if $|I|$ is Ulam measurable.

§ 3. SEQUENTIAL CARDINALS AND ARBITRARY SEQUENTIALLY CONTINUOUS MAPPINGS OF METRIC SPACES.

The methods of the previous theorem can be applied to arbitrary sequential cardinals. Recall that α is sequential iff there is a sequentially continuous but not continuous mapping of $P(\alpha)$ into $\mathbb{R}$.

By Mazur's theorem 1.7 we have

Proposition 3.1 : The cardinal α is sequential iff there exists

(S) a sequentially continuous mapping $F : P(A) \to \mathbb{R}$, $|A| = \alpha$, such that $F(X) = 0$ for any finite $X \subseteq A$, but $F(A) \neq 0$.

Unfortunately we are unable to prove that sequential cardinals are in fact real measurable, but we prove that they possess a set-theoretical property similar to this :

Definition 3.2 : As Keisler-Tarski [2] we denote $[\aleph_1, \alpha] \not\subseteq C_1^{[\omega_1]}$ the fact that there is a countably-complete-$\aleph_1$-saturated ideal over α.

The ideal I is $\aleph_1$-saturated iff any system $\{X_i : i \in J\}$ of disjoint elements not belonging to I : $\{X_i : i \in J\} \subseteq P(\alpha)\setminus I$ - is at most countable, $|J| \leq \aleph_0$.

<u>Example</u> : For a real-valued σ-additive measure μ on $P(\alpha)$, the ideal I of sets of zero-measure : $I = \{X \subset \alpha : \mu(X) = 0\}$ is countably complete and $\aleph_1$-saturated.

By the methods of the previous theorem we have the following result of the author [7] :

<u>Theorem 3.3</u> : If α is a sequential cardinal, then $[\aleph_1, \alpha] \not\subseteq C_1^{[\omega_1]}$.

Solovay [8] has shown that under the V = L-axiom of constructibility, there is no α such that $[\aleph_1, \alpha] \not\subseteq C_1^{[\omega_1]}$. So by theorem 3.3 all cardinals are non-sequential under V = L, so all $|T|$ are non-sequential and by Mazur's theorem 1.2 we have :

<u>Corollary 3.4</u> : Under V = L, any sequentially continuous mapping of the product of any number of Hausdorff second countable spaces into metric space is continuous.

Keisler-Tarski [2] showed that any cardinal α, satisfying $[\aleph_1, \alpha] \not\subseteq C_1^{[\omega_1]}$ is larger than small inaccessible cardinals. So by 3.3 any sequential cardinal is larger then small inaccessible. We obtain thus :

<u>Corollary 3.5</u> : Let $M^{\infty}(X) = \bigcup_{\xi \in ORD} M^{\xi}(X)\setminus\xi$, and let $\rho_0 = \min\{\alpha : \alpha \notin M^{\infty}(\overline{AC})\}$, $\rho_1 = \min\{\alpha : \alpha \notin (M^{\infty})^{\infty}(\overline{AC})\}$. Then for $\alpha < \rho_0$ or $\alpha < \rho_1$ the sequentially continuous mapping of product of α separable metric spaces to an arbitrary metric space is continuous. If $2^{\aleph_0} = {}_1$ or even $2^{\aleph_0} < \rho_0, \rho_1$, then Keisler-Tarski [2] have shown that all cardinals α, satisfying $[\aleph_1, \alpha] \not\subseteq C_1^{[\omega_1]}$ are Ulam measurable. Thus by 3.3 all sequential cardinals α have $[\aleph_1, \alpha] \not\subseteq C_1^{[\omega_1]}$ and so are Ulam measurable. In other words, if $2^{\aleph_0} < \rho_1$, sequentially is equivalent to the Ulam measurability.

§ 4. VARIOUS GENERALIZATIONS.

It is still unknown whether without any additional assumptions, sequentiability of cardinals is equivalent to real measurability. We have proved only that the sequentiability of α implies $[\aleph_1,\alpha]\not\subseteq C_1^{[\omega_1]}$ (i.e. the existence of countably complete $\aleph_1$-saturated ideal). On the other hand the real-measurability of α also implies $[\aleph_1,\alpha]\not\subseteq C_1^{[\omega_1]}$ (since real measurability $\Rightarrow$ sequentially of ideal or sets measure zero is $\aleph_1$-saturated).

But the converse is not true : from $[\aleph_1,\alpha]\not\subseteq C_1^{[\omega_1]}$ does not follow the real measurability of α. In fact Martin and Solvay have shown [9] that under Martin's axiom A there can be cardinals α, $[\aleph_1,\alpha]\not\subseteq C_1^{[\omega_1]}$, which are not real measurable.

Nevertheless, assuming Martin's axiom A, for sequential cardinals, we can give a complete answer to the Keisler-Tarski problem. Instead of Martin's axiom A we use it's consequence proved by Martin-Solovay [9] -so-called "strong Baire category theorem" :

SBCT : The intersection of $< 2^{\aleph_0}$ dense open substets of $\mathbf{R}$ is dense.

Theorem 4.1 (Assuming SBCT) : A cardinal α is sequential iff α is real-measurable iif α is Ulam measurable.

The coincidence of the real measurability and of the Ulam measurability assuming SBCT was proved by Martin-Solovay [9].

So theorem 3.3 is weak and in the particular case 4.1 is good.

Problem : It is completely unknown whether non $\mathfrak{A}$-reducibility for the arbitrary $\mathfrak{A}$ satisfying a), b_1), b_2) is equivalent to real measurability.

It is even unknown if an analogue of 3.3 holds for general non $\mathfrak{A}$-reducibility. However for a special $\mathfrak{A}$ we can obtain an analogue of 3.3.

Let us recall the property a) of $\mathfrak{A}$:

a) if $\mathbf{X}$ is a class of subsets of A satisfying $\mathfrak{A}$, then $\mathbf{X}$ is sequentially closed and a G_δ-set in the sequential topology of P(A), i.e. $P(A)\setminus\mathbf{X} = \bigcup_{n=1}^{\infty} X_n$, where the X_n are sequentially closed.

We replace a) by

a') if $\mathbf{X}$ is a class of subsets of A satisfying $\mathfrak{A}$, then X is sequentially closed and $P(A)\setminus\mathbf{X} = \bigcup_{n=1}^{\infty} X_n$, where the X_n are sequentially closed and G_δ-sets themselves in the sequential topology of P(A).

Theorem 4.2 : If $\mathfrak{A}$ satisfies a'), b_1), b_2), then the non $\mathfrak{A}$-reducibility of α implies $[\aleph_1, \alpha] \not\leq C_1^{[\omega_1]^1}$.

We have given the review of the results on the sequentially continuous mapping. These problems can have different applications. They are interesting in the analysis of sequential topology of various spaces and first of all to the analysis of the sequential topology of Tychonoff products. The presented results find already their application in the investigations of uniform spaces. Among the applications of the results are Huzek papers.

However there are many problems with the Tychonoff powers of $\mathbb{R}$, $\mathbb{N}$, ... We have such a problem :

Problem : Is the existence of a sequentially continuous, but not continuous mapping of $\mathbb{R}^\Delta$ in $\mathbb{R}$ equivalent to the sequentiability of $|\Delta|$? to the real measurability of $|\Delta|$?

We only know that by Mazur's theorem 1.2 from the non-sequentiability of $|\Delta|$ it follows that any sequentially continuous mapping $\mathbb{R}^\Delta \to \mathbb{R}$ is continuous.

But the converse is unknown : let any s.c. map $\mathbb{R}^\Delta \to \mathbb{R}$ be continuous. Must $|\Delta|$ be non-sequential or not ?

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