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AN APPLICATION OF GROTHENDIECK'S INEQUALITY TO A PROBLEM IN HARMONIC ANALYSIS

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§ 1. TENSOR PRODUCTS OF C(X)-SPACES

Let X_i , $i = 1,2$, be compact Hausdorff spaces. We shall denote then by

$C(X_i)$ the Banach algebra of continuous complex-valued functions on X_i (point-wise multiplication, uniform norm),

$S(X_i)$ the group of unimodular functions in $C(X_i)$, i.e. $S(X_i) = \{f \mid f \in C(X_i)$ and for all $x_i \in X_i$, $|f(x_i)| = 1\}$,

$V(X_1 \times X_2) = C(X_1) \hat{\otimes} C(X_2)$ is the projective tensor product of the Banach spaces $C(X_1)$ and $C(X_2)$.

We recall the following well-known and/or easily established facts, concerning the above spaces :

(i) $V = V(X_1 \times X_2)$ is a semi-simple Banach algebra with Gelfand space $X_1 \times X_2$,

(ii) the convex hull of $S(X_i)$ is uniformly dense in the unit ball of $C(X_i)$ and therefore

(iii) every element $F \in V(X_1 \times X_2)$ has a representation $F = \Sigma\, a_k f_k \otimes g_k$, with $\Sigma |a_k| < \infty$, $f_k \in S(X_1)$, $g_k \in S(X_2)$.

We finally recall the following theorem, sometimes called "the fundamental theorem in the metric theory of tensor products",

<u>Theorem G</u> (Grothendieck [1]) : Let X_i , $i = 1,2$, be compact Hausdorff spaces and let H be a complex Hilbert space with inner product $<.|.>$. Let further $\varphi_i \in C(X_i, H)$ and let $\Phi \in C(X_1 \times X_2)$ be defined by

$$\Phi(x_1,x_2) = <\varphi_1(x_1)|\varphi_2(x_2)> \quad .$$

Then $\Phi \in V(X_1 \times X_2)$ and $\|\Phi\|_\Lambda = \|\Phi\|_{V(X_1 \times X_2)}$ satisfies the inequality

$$\|\Phi\|_\Lambda \leq K_C \cdot \|\varphi_1\|_{C(X_1,H)} \cdot \|\varphi_2\|_{C(X_2,H)}$$

where K_C is a universal constant (the complex Grothendieck constant) for which the bound $K_C < 1.607$ is known [2].

<u>Remark</u> : $C(X_i,H)$ is the space of H-valued continuous functions and is a Banach space if we define $\|f\|_{C(X_i,H)} = \max_{x \in X_i} \|f(x)\|_H$.

§ 2. A PROBLEM IN HARMONIC ANALYSIS

Let G be a compact Abelian group with dual group Γ, and let K be a closed subset of G. We shall say that the set K is a

(i) Kronecker set if $\Gamma|_K$ is uniformly dense in S(K) (S(K) being as above the group of unimodular continuous functions on K).

(ii) Helson(α) set if the convex hull of $\Gamma|_K$ is uniformly dense in the ball of radius α in C(K).

It follows from the Hahn-Banach theorem that K is a Helson(α) set if for every measure μ on G supported by K, we have

$$\sup_{\gamma\in\Gamma} \left|\int_G (g,\gamma)\,d\mu(g)\right| \geq \alpha \,.\, \|\mu\|_M \quad ,$$

where $\|\mu\|_M$ is the total variation of μ.

We recall that $\ell^1(\Gamma)$ is a commutative semi-simple Banach algebra with unit, having G as its Gelfand space, so that $\ell^1(\Gamma)$ may be identified with a Banach algebra A(G) of continuous functions on G. If K is a closed subset of G we shall write I(K) to denote the ideal in A(G) of all functions vanishing on K, and we shall write A(K) to denote the quotient algebra A(G)/I(K). It is clear from this definition that if $f\in C(K)$, then $f\in A(K)$ iff there exists $\tilde f\in A(G)$ such that $\tilde f|_K = f$.
It is clear from the above definitions that K is a Helson(α) set if for every $f\in C(K)$, $\|f\|<1$, there exists $\tilde f\in A(G)$, $\|\tilde f\|_{A(G)}$ = (by definition) $\|\hat{\tilde f}\|_{\ell^1(\Gamma)} < \alpha$.

Let now K_i , i=1,2, be closed subsets of G, such that K_i is Helson (α_i) and let $K = K_1 \times K_2$ be the cartesian product which is a closed subset of $G\times G$. Since $A(G\times G) \approx A(G)\hat\otimes A(G)$ (isometrically) it follows from standard properties of the projective tensor norm that

$$A(G\times G)/I(K_1\times K_2) \approx C(K_1)\hat\otimes C(K_2)$$

in the sense that they are algebraically isomorphic, even though the isomorphism is not isometric.

We use now the fact that since G is abelian, the addition map $s : G\times G\to G$ (defined by $s(g_1,g_2) = g_1+g_2$) is a group homomorphism with adjoint $\hat{s} : \Gamma\to\Gamma\times\Gamma$, where $\hat{s}(\gamma) = \gamma\otimes\gamma$. (This is clear since $((g_1,g_2),\hat{s}(\gamma)) = (s(g_1,g_2),\gamma) = (g_1+g_2,\gamma) = (g_1,\gamma)(g_2,\gamma) = ((g_1,g_2),\gamma\otimes\gamma)$.)
The map $\hat{s}$ extends by linearity to an algebra homomorphism of $\ell^1(\Gamma)$ into $\ell^1(\Gamma\times\Gamma)$, i.e. of $A(G)$ into $A(G\times G)$, and if K_1 and K_2 are closed subsets of G, we obtain by restricting $\hat{s}$ to $K_1\times K_2$, an algebra homomorphism $\hat{s}_{(K_1,K_2)}$ of $A(K_1+K_2)$ into $A(K_1\times K_2)$.
Suppose now that K_1 and K_2 are disjoint closed subsets of G, such that the union $K_1\cup K_2$ is a Kronecker set. It was observed then by Varopoulos that not only is $\Gamma\times\Gamma|_{K_1\times K_2}$ uniformly dense in $S(K_1)\times S(K_2)$, but in fact this is already true for $\hat{s}(\Gamma)|_{K_1\times K_2}$. This implies that the map $\hat{s} : A(K_1+K_2)\to A(K_1\times K_2)$ is an isometric algebra homomorphism of $A(K_1+K_2)$ onto $A(K_1\times K_2)$ and since as we saw above $A(K_1\times K_2)\approx A(K_1)\hat{\otimes} A(K_2)\approx C(K_1)\hat{\otimes} C(K_2)$ it follows that $A(K_1+K_2)$ is isometrically and algebraically isomorphic to $C(K_1)\hat{\otimes} C(K_2)$ [4].

In the rest of this note we shall now study the following

<u>Problem</u> : Does there exist a number α_o, $0<\alpha_o<1$, such that whenever K_1 and K_2 are disjoint closed subsets of G such that $K_1\cup K_2$ is a Helson(α) , $\alpha>\alpha_o$, set, then $A(K_1+K_2)$ is algebraically isomorphic to $C(K_1)\hat{\otimes} C(K_2)$?

<u>Remark</u> : The purpose of posing the problem as a search for a number α_o , means that when proving that $A(K_1+K_2)$ is isomorphic to $C(K_1)\hat{\otimes} C(K_2)$ we are not allowed to use any additional assumptions on the sets K_1 and K_2 besides the assumption that $K_1\cup K_2$ is Helson (α) with $\alpha>\alpha_o$.

§ 3.

Using theorem G above we shall presently show that the answer to the problem raised above is yes, by proving the following

<u>Theorem</u> : Let K_1 and K_2 be disjoint compact subsets of the compact abelian group G, such that $K_1\cup K_2$ is a Helson($1-\beta$) set in G, where $\beta.(2+2K_C)<1$. Then $A(K_1+K_2)$ is algebraically isomorphic to $C(K_1)\hat{\otimes} C(K_2)$.

Before attempting to prove the theorem we shall see that there is an a priori lower bound for the possible values of α_0 for which the answer to the problem could be positive, by proving the following

<u>Proposition</u> : There exist closed subsets K_1 and K_2 of the circle group T, such that $K_1 \cup K_2$ is Helson ($2^{-1/2}$) while $A(K_1 + K_2)$ is not algebraically isomorphic to $C(K_1) \hat{\otimes} C(K_2)$.

<u>Proof</u> : A simple necessary condition for two Banach algebras to be algebraically isomorphic is that they have the same Gelfand space. In the present case the Gelfand space of $C(K_1) \hat{\otimes} C(K_2)$ is $K_1 \times K_2$ while the Gelfand space of $A(K_1 + K_2)$ is $K_1 + K_2$. These spaces are connected by the function s mapping $K_1 \times K_2$ onto $K_1 + K_2$. Since s is a continuous map of a compact space onto a Hausdorff space, s is bicontinuous whenever it is injective. We shall now first show that if the map $s : K_1 \times K_2 \to K_1 + K_2$ is not injective then $K_1 \cup K_2$ cannot be Helson (α) for any $\alpha > 2^{-1/2}$. We assume thus that $k_1, k_1' \in K_1$, $k_2, k_2' \in K_2$ and that

$$k_1 + k_2 = k_1' + k_2' \quad .$$

We define then the measure $\mu \in M(K_1 \cup K_2)$ as

$$\delta_{k_1} + \delta_{k_2} + \delta_{k_1'} - \delta_{k_2'} \quad ,$$

and we see that for any $\gamma \in \Gamma$, we have

$$\begin{aligned} |\hat{\mu}(\gamma)| &= |(k_1,\gamma) + (k_2,\gamma) + (k_1',\gamma) - (k_2',\gamma)| \\ &= |z + w + u - zw\bar{u}| = |u(z\bar{u} + w\bar{u} + 1 - (z\bar{u})(w\bar{u}))| \\ &= |1 + z' + w' - z'w'| \leq 8^{1/2} \quad , \end{aligned}$$

as is easily verified by direct calculation of $|\hat{\mu}(\gamma)|^2$.
The subset of T that satisfies the condition of the proposition is obtained by choosing 4 points in the circle, satisfying the above algebraic relation over the integers, but no other relation. It is a matter of elementary calculus (though not simple calculus) to prove that such a set is then Helson($2^{-1/2}$).

To prove the theorem we shall need the following

Lemma : Let X and Y be compact spaces, let $\{a_i\}_{i=1}^{\infty}$ be positive numbers, let $f_i \in C(X)$, $g_i \in C(Y)$, $\|f_i\|_\infty \leq 1$, $\|g_i\|_\infty \leq 1$, and let t be a positive number such that

$$\Sigma\, a_i = 1$$

$$\|1 - \Sigma\, a_i f_i\|_\infty \leq t, \qquad \|1 - \Sigma\, a_i g_i\|_\infty \leq t \quad .$$

Then

$$\|1 - \Sigma\, a_i f_i g_i\|_\wedge \leq (2 + 2K_C)t \quad .$$

Proof : Using the identity

$$(1 - f_i g_i) = (1 - f_i) + (1 - g_i) - (1 - f_i)(1 - g_i)$$

we have

$$(1 - \Sigma\, a_i f_i g_i) = \Sigma\, a_i (1 - f_i g_i) = (1 - \Sigma\, a_i f_i) + (1 - \Sigma\, a_i g_i)$$

$$- \Sigma\, a_i (1 - f_i)(1 - g_i) \quad .$$

We have therefore

$$\|1 - \Sigma\, a_i f_i g_i\|_\wedge \leq t + t + \|\Sigma\, a_i (1 - f_i)(1 - g_i)\|_\wedge \quad .$$

Using now theorem G we have

$$\|\Sigma\, a_i (1 - f_i)(1 - g_i)\|_\wedge \leq K_C \max_X \{(\Sigma\, a_i\ |1 - f_i|^2)^{1/2}\}$$

$$\times \max_Y \{(\Sigma\, a_i\ |1 - g_i|^2)^{1/2}\} \quad .$$

Now $\Sigma\, a_i |1 - f_i|^2 \leq |2(\Sigma\, a_i (1 - f_i)| \leq 2t$, and we get the same estimate for $\Sigma\, a_i |1 - g_i|^2$.

We have thus $\|1 - \Sigma\, a_i f_i g_i\|_\wedge \leq 2t + K_C(2t)^{1/2}(2t)^{1/2} = (2 + 2K_C)t$, so the lemma is proved.

To prove the theorem we now let K_1 and K_2 be compact subsets of G, such that $K_1 \cup K_2$ is a Helson$(1-\beta)$ set, with $\beta < (2+2K_C)^{-1}$. We choose now $\delta > \beta$, such that $\delta(2+2K_C) < 1$. To prove the theorem it suffices to find, for each $f \in C(K_1)$, $|f| = 1$, and $g \in C(K_2)$, $|g| = 1$, a function $F \in A(K_1 + K_2)$, such that $\|f \otimes g - F\|_\wedge \leq \delta(2+2K_C)$. Towards this we define $\varphi \in C(K_1 \cup K_2)$ by

$$\varphi|_{K_1} = f \ , \quad \varphi|_{K_2} = g \quad .$$

By the assumptions on $K_1 \cup K_2$, φ has a representation

$$\varphi = \Sigma\, b_i\, \gamma_i \ , \quad \gamma_i \in \Gamma \ , \quad \Sigma|b_i| = A \leq (1-\delta)^{-1} \quad .$$

We write $b_i = r_i . \exp(i\ \alpha_i)$, with $r_i > 0$. We then write

$$F = A^{-1} \Sigma\, r_i \exp(2i\ \alpha_i)\, \gamma_i \in A(K_1 + K_2) \quad .$$

Putting now $a_i = A^{-1} . r_i$, $f_i = \bar{f} . \exp(i\,\alpha_i)\,\gamma_i$, $g_i = \bar{g} . \exp(i\,\alpha_i)\,\gamma_i$. We see that the assumptions of the lemma are satisfied, so

$$\|1 - \Sigma\ a_i f_i g_i\|_\wedge \leq (2+2K_C)\delta \quad ,$$

and therefore also

$$\|f \otimes g - F\|_\wedge = \|(f \otimes g)(1 - \Sigma\, a_i f_i g_i)\|_\wedge \leq (2+2K_C)\delta \quad .$$

This proves the theorem, and using our estimate of K_C , we see that $A(K_1 + K_2)$ is algebraically isomorphic to $C(K_1) \hat{\otimes} C(K_2)$ if $K_1 \cup K_2$ is a Helson(α) set with $\alpha > 0.81$.

For a more extensive study of the problem considered in this note, including in particular a study of "the real case" which is somewhat different, and algebras of type $A(K_1 + K_2 + K_3)$ etc. we have to refer to [3].

REFERENCES

[1] A. Grothendieck, Résumé de la théorie métrique des produits tensoriels topologiques, Bol. Soc. Matem. Sao Paolo 8 (1956) 1-79.

[2] S. Kaijser, A note on the Grothendieck constant with an application to Harmonic Analysis, UUDM Report No 1973:10, (Uppsala University, mimeographed).

[3] S. Kaijser, Representations of tensor algebras as quotients of group algebras, Ark. f. Mat. 10 (1972) 107-141.

[4] N. Th. Varopoulos, Tensor algebras and Harmonic Analysis, Acta Math. 119, (1967) 51-112.
