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A RESULT OF HAYDON AND ITS APPLICATIONS.

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We begin by recalling the following theorem of Rosenthal and O'dell [5] :

Theorem : Let X be a separable Banach space. Then X does not contain a subspace isomorphic to ℓ_1 (written $\ell_1 \not\subseteq X$) if and only if every element of X^{**} is the weak* ($\sigma(X^{**},X^*)$) limit of a sequence in X.

If $X = c_0(\Gamma)$, Γ uncountable, then $X \not\supseteq \ell_1$ but not every element of X^{**} is a weak* sequential limit of elements of X.

Below we shall give a non-separable version of the above theorem, due to Haydon [3], which requires only the following lemma :

Lemma (Rosenthal [6]) : Let X be a Banach space. Then $X \supseteq \ell_1$ if and only if there exist a bounded non-empty subset S of X^*, $x^{**} \in X^{**}$, r real number, $\delta > 0$ such that for any weak* open subset U of X^*, $U \cap S \neq \emptyset$, we have

$$\sup_{x^* \in \bar{c}^*(U \cap S)} x^{**}(x^*) < r < r+\delta < \sup_{x^* \in \bar{c}^*(U \cap S)} x^{**}(x^*) \quad .$$

(If M is a subset of X^*, $\bar{c}^*(M)$ is the weak* closed convex hull of M.)

By K we denote the unit ball of X^* in the weak* topology. A measure on K is always a complete, regular, Borel measure.

Theorem (Haydon [3]) : Let X be a Banach space. Then the following are equivalent :

(i) $X \not\supseteq \ell_1$;

(ii) every $x^{**} \in X^{**}$ is universally measurable as a function on K ;

(iii) for every $x^{**} \in X^{**}$, every measure μ on K there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$, $\|x_n\| \leq \|x^{**}\|$ for all n, and $x_n \to x^{**}$ μ-a.e. on K ;

(iv) every $x^{**} \in X^{**}$ is universally measurable as a function on K and for every measure on K,

$$\int_K x^{**}(x^*) \, d\mu(x^*) = x^{**}(r\mu)$$

where $r\mu$ is the unique element of X^* such that

$$\int_K x^*(x)\ d\mu(x^*) = (r\mu)(x) \quad \text{for all } x \in X \quad ;$$

(v) for every $x^{**} \in X^{**}$, every measure μ on K, every $\varepsilon > 0$, there exists $K_o \subseteq K$, K_o weak* compact and convex such that x^{**} is continuous on K_o and $|\mu|(K_o) > (1-\varepsilon)\|\mu\|$.

<u>Proof</u> : (v) $\Rightarrow$ (iii). Let μ be a measure on K (which we assume to be a probability measure throughout the proof). Choose $K_n \subseteq K$, K_n compact, convex, $\mu(K_n) \to 1$, and x^{**} is continuous on each K_n. Let $R_n : X \to C(K_n)$ be the canonical operator ; that is, $R_n(x)(x^*) = x^*(x)$ for $x^* \in K_n$. Then we have $R_n^{**}(x^{**}) \in C(K_o)$, or, $R_n^{**}(x^{**})$ is in the weak closure of $\{R_n x : \|x\| \le \|x^{**}\|\}$ which is a convex set. Thus $R_n^{**}(x^{**})$ is in the norm closure of this set, so for any $\varepsilon_n > 0$ there exists $x_n \in X$, $\|x_n\| \le \|x^{**}\|$ and $|x^*(x_n) - x^{**}(x^*)| \le \varepsilon_n$ for all $x^* \in K_n$. If $\varepsilon_n \to 0$ then $x_n \to x^{**}$ μ-a.e. on K.

(v) $\Rightarrow$ (iv). There exist K_n, $\mu(K_n) \to 1$, such that x^{**} is continuous on K_n. Since x^{**} is continuous on K_n, $x^{**}(r\mu_n) = \int_K x^{**}(x^*)\ d\mu_n(x^*)$ where $\mu_n = \mu(K_n)^{-1}\ \chi_{K_n} . \mu$. (Note that $r\mu_n \in K_n$.) Since $r\mu_n \to r\mu$ in norm and clearly

$$\int_K x^{**}(x^*)\ d\mu_n(x^*) \to \int_K x^{**}(x^*)\ d\mu(x^*) \quad ,$$

we have that that $x^{**}(r\mu) = \int_K x^{**}(x^*)\ d\mu(x^*)$.

(iii) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (ii) are trivial.

(ii) $\Rightarrow$ (i). If $X \supseteq \ell_1$ it is an old argument of Sierpinski that any weak* cluster point of a sequence equivalent to the usual basis of ℓ_1 is not measurable for an appropriately chosen measure μ on K. (See [3].)

(i) $\Rightarrow$ (v). For $x^{**} \in X^{**}$ and r a real number we shall denote by $\{x^{**} > r\}$ the set $\{x^* \in K : x^{**}(x^*) > r\}$. Let $x^{**} \in X^{**}$, μ a measure on K (again, assumed to be a probability measure). Let S be the support of μ, and r, δ_n real numbers with $\delta_n > 0$. Note that

$$\mathcal{K} = \{K' : K' \text{ compact, convex, } K' \subseteq K,$$
$$K' \subseteq \{x^{**} > r\}\ \}$$

is a directed set. (The convex hull of a finite union of elements of $\mathcal{K}$ is an element of K.) Choose a subset K_n' of $\{x^{**} > r\}$ which is the union of an increasing sequence of compact convex subsets of $\{x^{**} > r\}$ such that for any convex, compact subset L of $\{x^{**} > r\}$, $\mu(L \setminus K_n') = 0$. Similarly, choose $K_n'' \subseteq \{x^{**} < r+\delta_n\}$. We shall show that $\mu(K_n' \cup K_n'') = 1$. Suppose $\mu[S\setminus(K_n' \cup K_n'')] < 1$. Choose $S' \subseteq S\setminus K_n' \cup K_n''$, S' compact, and for every V weak* open, $V \cap S' \neq \emptyset$, $\mu(V \cap S') > 0$. By Rosenthal's Lemma there exists V open, $V \cap S' \neq \emptyset$, $\bar{c}(V \cap S')$ is a subset $\{x^{**} > r\}$ or $\{x^{**} < r+\delta\}$. Thus, $\mu[\bar{c}(V\cap S')\setminus(K_n' \cup K_n'')] = 0$. Contradiction. Choose δ_n decreasing to 0. Define $L = \bigcup_{n=1}^{\infty} K_n'$ and $M = \bigcap_{n=1}^{\infty} \bigcup_{i \geq n} K_i''$. Then $L \subseteq \{x^{**} > r\}$ and $M \subseteq \{x^{**} \leq r\}$ and $\mu(L \cup M) = 1$. This proves the following : for $\eta > 0$, there exists a compact, convex $C \subseteq \{x^{**} > r\}$ such that $\mu[\{x^{**} > r\}\setminus C] < \eta$. Repeating the argument above for $-x^{**}$ we obtain that for every $\eta > 0$, every r, s, $r < s$, every $x^{**} \in X^{**}$ there exists $C \subseteq \{r \leq x^{**} < s\}$, C compact, convex, and $\mu[\{r \leq x^{**} < s\}\setminus C] < \eta$. Let $\varepsilon > 0$. Choose $\sum_{n=1}^{\infty} \varepsilon_n < \varepsilon$ and positive integers $K(n)$ such that $K(n)\varepsilon_n > \|x^{**}\|$. By the above there exist compact, convex $C_{n,m}$, $n = 1,2,\ldots$, $-K(n) \leq m \leq K(n)$ such that

$$C_{n,m} \subseteq \{m\varepsilon_n \leq x^{**} < (m+1)\varepsilon_n\}$$

and

$$\mu(\bigcup_m C_{n,m}) > 1-\varepsilon_n \quad .$$

Let $C_n = c(\bigcup_m C_{n,m})$ = the convex hull of $\bigcup_m C_{n,m}$ which is compact. Let $C = \bigcap_{n=1}^{\infty} C_n$. Then $\mu(C) > 1-\varepsilon$ and it is a routine computation to show that x^{**} is continuous on C.

Corollary (Haydon [3]) : Suppose $X \not\supseteq \ell_1$ and M is a weak* compact subset of K. Then $\bar{c}^*(M) = \overline{c(M)}$ (closure of the convex hull of M in the norm topology).

Proof : Let M be a weak* compact subset of K. Then $\bar{c}^*(M) = \{r\mu : \mu$ is a probability measure on K such that $\mu(K\setminus M) = 0\}$. Suppose $x^{**} \in X^{**}$ and $M \subseteq \{x^{**} \leq r\}$. By part (iv) of the theorem

$$x^{**}(r\mu) = \int_K x^{**}(x^*)\, d\mu(x^*) \leq r$$

if μ is a probability measure and $\mu(K\setminus M) = 0$. Therefore, $\bar{c}^*(M) \subseteq \{x^{**} \leq r\}$. Invoking the Hahn-Banach theorem proves $\bar{c}^*(M) = \overline{c(M)}$.

APPLICATIONS

The following appears to be a well known folk theorem. D.R. Lewis showed the author a proof several years ago. The following proof may be the same.

Lemma : Let M be a compact Hausdorff space and μ a probability measure on M. Let Y be a weakly compactly generated (wcg) Banach space and $\tau : M \to Y$ a scalarly measurable, bounded function. Then there is a $M_o \subseteq M$, M_o measurable, $\mu(M_o) = 1$, and $\tau(M_o)$ is norm separable.

Proof : Since Y is wcg (cf. [1]) for each separable subspace Y_o of Y and each separable subspace Z_o of Y^* there exists a projection $P : Y \to Y$ with P(Y) separable, $Y_o \subseteq P(Y)$, $Z_o \subseteq P^*(Y^*)$. If $\tau : M \to Y$ is scalarly measurable and $\|\tau(m)\| \leq 1$ a.e. then for any projection $P : Y \to Y$ with P(Y) separable, $P \circ \tau : M \to Y$ is separably valued and scalarly measurable ; hence $P \circ \tau$ is strongly measurable [2].

By Lusin's theorem there exists a $M_o \subseteq M$ such that $P \circ \tau$ is continuous on M_o ; that is $P(\tau(M_o))$ is compact. Then the set of functions $\{y^* : P^*y^* = y^* \ \|y^*\| \leq 1\}$ is equicontinuous on $\tau(M_o)$ because $y^*(P\tau(k)) = (P^*y^*)(\tau(k)) = y^*(\tau(k))$ for $k \in M_o$.

Claim 1 : $\{y^* : \|y^*\| \leq 1\}$ is a relatively compact set in the $L_1(M,\mu)$ norm. Suppose there exists $\{y_i^*\}_{i=1}^{\infty}$, $\|y_i^*\| \leq 1$ and $\int_M |y_i^*(\tau(m)) - y_j^*(\tau(m))| \, d\mu(m) > \eta > 0$ for all i,j, $i \neq j$. Let P be a projection in X with P(X) separable and $P^*y_i^* = y_i^*$. There exists $M_o \subseteq M$, $\mu(M_o) > 1 - \frac{\eta}{4(1+\eta)}$, and $\{y_i^*\}_{i=1}^{\infty}$ is an equicontinuous family on $\tau(M_o)$. That is, there exists $M_o \subseteq M$ such that $|y_j^*(\tau(m)) - y_i^*(\tau(m))| \leq \frac{\eta}{4}$ for $m \in M_o$ and all i,j, $i \neq j$; therefore

$$\int_M |y_j^*(\tau(m)) - y_i^*(\tau(m))| \, d\mu(m)$$

$$= \int_{M_o} |y_j^*(\tau(m)) - y_i^*(\tau(m))| \, d\mu(m) + \int_{M \setminus M_o} |(y_j^* - y_i^*)(\tau(m))| \, d\mu(m)$$

$$< \frac{\eta}{4} \mu(M_o) + 2 \, \mu(M \setminus M_o) < \eta \quad .$$

Contradiction.

Claim 2 : For each $\varepsilon > 0$, there exists M_o, $\mu(M_o) > 1-\varepsilon$ such that $\{y^* \circ \tau : \|y^*\| \leq 1\}$ is equicontinuous on M_o. Choose $\{y_i^*\}_{i=1}^{\infty}$ dense in the L_1 norm of $\{y^* \circ \tau : \|y^*\| \leq 1\}$. By the above, there exists M_o, $\mu(M_o) > 1-\varepsilon$ such that $y_i^* \circ \tau$ is equicontinuous on M_o. Let $y^* \in Y^*$, $\|y^*\| \leq 1$. There exists a subsequence $y_{i_j}^*$ such that

$$\int |y_{i_j}^*(\tau(m)) - y^*(\tau(m))| d\mu(m) \to 0.$$

But there exists $\{y_{i_{j_k}}^*\}$ that converges uniformly on M_o to a continuous function φ. Therefore $\int_{M_o} |y^*\tau(m) - \varphi(m)| d\mu(m) = 0$. Or $y^*\tau = \varphi$ a.e. on M_o. Therefore $\{y^* \circ \tau : \|y^*\| \leq 1\}$ is equicontinuous on M_o. In particular, $\tau(M_o)$ is relatively compact. So τ is essentially separably valued and, thus, is strongly measurable.

We state without proving, the following

Lemma : $T : X \to Y$ an operator, then the following are equivalent :

(i) for any bounded set $B \subseteq X$, $T(B)$ is dentable ;

(ii) for any probability space (S,Σ,ν), any operator $S : L_1(S,\Sigma,\nu) \to X$, TS is differentiable.

For convenience we shall call an operator satisfying (i) a denting operator.

Our principal application is the following :

Theorem : Let X, Y be Banach spaces, $X \not\supseteq \ell_1$, Y wcg. Then any operator $T : X^* \to Y$ is a dentable operator.

Proof : Let K the unit ball of X^* with the weak* topology. Let $R : X \to C(K)$ be the canonical operator, $R(x)(x^*) = x^*(x)$. Let (S,Σ,ν) be a probability space and $U : L_1(S,\Sigma,\nu) \to X^*$ an operator. As is well known there exists $\tilde{U} : L_1(S,\Sigma,\nu) \to C(K)^*$ such that $R^*\tilde{U} = U$ [4]. Let μ be a measure on K and consider $L_1(K,\mu)$ as a subspace of $C(K)^*$. The question as to whether TU is a denting operator is equivalent to whether $TR^* : L_1(K,\mu) \to Y$ is denting. Considering T as a function from K into Y we have that T is scalarly measurable $(X \not\supseteq \ell_1)$ and Y is wcg. So T is μ-strongly measurable. Hence there exists an operator $V : L_1(K,\mu) \to Y$, differentiable, $Vf = \int_{k\in K} f(k)\,T(k)\,d\mu(k)$.

We shall show $Vf = TR^*f$ for $f \in L_1(K,\mu)$. Choose $y^* \in Y^*$. Then $\langle Vf,y^*\rangle$ $= \int f(k)\,y^*(T(k))\,d\mu(k)$ and $\langle TR^*f,y^*\rangle = \langle R^*f,T^*y^*\rangle = \int T^*y^*(k)\,f(k)\,d\mu(k)$ $= \int y^*(T(k))\,f(k)\,d\mu(k)$. Thus $V = TR^*$ on $L_1(K,\mu)$ so TR^* is differentiable. Therefore, T is a denting operator.

<u>Corollary 1</u> : Suppose $X \not\supseteq \ell_1$ and $T : X \to Y$ is an absolutely summing operator. Then T is a 1-Radonifying.

<u>Proof</u> : If $T : X \to Y$ is absolutely summing then there exist a compact Hausdorff space M, a probability measure μ on K (K is the unit ball of X^*, as above), and operators I, R, V, J such that

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "R"'] & Y \arrow[r, "I"] & C(M) \\
C(K) \arrow[rr, "J"] & & L_1(K,\mu) \arrow[u, "V"']
\end{tikzcd}$$

is commutative, where R and J are canonical and I is an isometry.

Let $U : X^* \to L_1(S,\nu)$ be any operator where $L_1(S,\nu)$ is a finite measure space (hence wcg). Essentially, proving that T is 1-Radonifying is equivalent to proving $T^{**}U^*$ is nuclear. Regarding $L_1(K,\mu)$ as a subspace of $C(K)^*$ we have that UR^* is differentiable and bounded. That is, $U : K \to L_1(S,\nu)$ is μ-strongly measurable, $\int \|Uk\|\,d\mu(k) < +\infty$, and $UR^*f = \int_K f(k)\,U(k)\,d\mu(k)$ for all $f \in L_1(K,\mu)$. Since $U(k)$ is an element of $L_1(K,\mu)$ it is easy to check that $h(k,s) = U(k)(s)$ defines a unique (equivalence class) function on $K \times S$ and that $\int_K \int_S |h(k,s)|\,d\mu(k)\,d\nu(s) < +\infty$. Similarly it is easy to check that the function $s \to h(.,s)$ is a strongly measurable function from S to $L_1(K,\mu)$. Define $\tau : S \to C(M)$ by $\tau(s) = V(h(.,s))$; τ is strongly measurable and $\int \|\tau(s)\|\,d\nu(s) < +\infty$. We shall show τ represents the operator $I^{**}T^{**}U^*$ and $\tau(s) \in I(Y)$ ν-a.e. Let $\xi \in C(M)^*$ and $\psi \in L_\infty(S,\nu)$; then $\langle I^{**}T^{**}U^*\psi,\xi\rangle$ $= \langle \psi,UT^*I^*\xi\rangle = \langle \psi,UR^*JV^*\xi\rangle = \langle \int_K (V^*\xi)(k)\,U(k)\,d\mu(k),\psi\rangle$ $= \int_S \int_K (V^*\xi)(k)\,\psi(s)\,h(k,s)\,d\mu(k)\,d\nu(s) = \int_S \psi(s)\,\langle h(.,s),V^*\xi\rangle\,d\nu(s)$ $= \int_S \psi(s)\,\langle \tau(s),\xi\rangle\,d\nu(s) = \langle \int \psi(s)\,\tau(s)\,d\nu(s),\xi\rangle$. Thus τ represents $I^{**}T^{**}U^*$.

Since T is weakly compact $I^{**}T^{**} = IT^{**}$; that is $T^{**}U^{*}\psi \in Y$ for all $\psi \in L_\infty(S,\nu)$ Suppose $z_o \in C(M)$ and $\delta > 0$ such that $\|z_o - Iy\| > \delta$ for all $y \in Y$. Let $E = \tau^{-1}\{z : \|z - z_o\| \leq \delta\}$. If $\nu(E) > 0$, then $IT^{**}U^{*}(\nu(E)^{-1}\chi_E) \in \{z : \|z - z_o\| \leq \delta\}$ which is a contradiction. Therefore $\tau(s) \in I(Y)$ ν-a.e. Thus $I^{-1}\tau$ represents the operator $T^{**}U^{*}$ or T is 1-Radonifying.

Corollary 2 : Let X, Y, Z be Banach spaces ; $X \not\supseteq \ell_1$, Z wcg, and $Y \subseteq X^{*}$. If $T : X^{*} \to Z$ is an operator that is an isomorphism on Y then Y has the Radon-Nikodym property (RNP).

There are other obvious variations on Corollary 2.

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