Séminaire d'analyse fonctionnelle
École Polytechnique

R. C. James

# Quasi-reflexive Banach spaces

*Séminaire d'analyse fonctionnelle (Polytechnique)* (1975-1976), exp. nº 25, p. 1-7
<http://www.numdam.org/item?id=SAF_1975-1976____A20_0>

CENTRE DE MATHÉMATIQUES

PLATEAU DE PALAISEAU - 91120 PALAISEAU

Téléphone : 941.81.60 - Poste N°

Télex : ECOLEX 69 15 96 F

S E M I N A I R E M A U R E Y - S C H W A R T Z 1 9 7 5 - 1 9 7 6

# QUASI-REFLEXIVE BANACH SPACES

by R. C. JAMES

(Claremont, Cal.)

Exposé No XXV 8 Juin 1976

It is customary to use J to indicate any Banach space isometric to the space introduced in [2]. Thus J is isomorphic to a space that is isometric to its second dual [3], and J is quasi-reflexive of order one (i.e., the quotient of $J^{**}$ and the natural image of J in $J^{**}$ has dimension one). If J is given a norm for which J is isometric to $J^{**}$, then the $\ell_2$-product of J and $J^*$ is quasi-reflexive of order two and isometric to its first dual.

However, J and $J^*$ are not isomorphic [4, theorem 3]. The purpose of this discussion is to outline a proof of the existence of a Banach space that is quasi-reflexive of order one and isomorphic to its dual. It is not known whether there is a Banach space that is quasi-reflexive of order one and isometric to its dual.

We will be particularly interested in the norm for J given by

$$\|x\| = \sup\{\sum_{k=1}^{n} [x(p_{2k-1}) - x(p_{2k})]^2\}^{1/2} , \tag{1}$$

where the sup is over all positive integers n and all increasing (not necessarily strictly increasing sequences $\{p_k\}$ of positive integers. The space J is the completion of the space of sequences with finite support, given this norm.

In order to describe the predual I of J, some conventions are needed. A <u>bump</u> is a sequence of real number $x = \{x(i)\}$ for which there is a bounded interval and a number a such that $x(i) = a$ if i is in this interval and $x(i) = 0$ otherwise. The <u>altitude</u> of a bump is a. Two bumps are <u>disjoint</u> if the intersection of their associated intervals is empty ; they are <u>strongly disjoint</u> if these intervals are separated by at least one integer.

The space I is the completion of the normed linear space of sequences with finite support for which

$$\|x\| = \inf\{\sum_{k=1}^{n} [\![ x^k ]\!] : x = \sum_{k=1}^{n} x^k\} , \tag{2}$$

where $[\![\ ]\!]$ is the function defined by $[\![ x ]\!] = (\sum a_i^2)^{1/2}$ if x is the sum of <u>disjoint</u> bumps whose altitudes are $\{a_i\}$. Note that $[\![\ ]\!]$ is a function of x and a particular representation of x as a sum of disjoint bumps.

The spaces I and J have the properties that the natural basis $\{e_n\}$ for I is shrinking and the sequence of coefficient functionals $\{u_n\}$ is a boundedly complete basis for J (see [4, theorem 1] or [5, pg. 279], and [6, corollary 6.1, pg. 286]). If $\{\varepsilon_n\}$ is the natural basis for J with the norm (1), then $\{\varepsilon_n\}$ is shrinking and $u_n = \sum_1^n \varepsilon_i$ for each n, so

$$u_1 = \varepsilon_1 \quad \text{and} \quad u_n - u_{n-1} = \varepsilon_n \quad \text{if } n > 1 \quad . \tag{3}$$

If $x = \sum_1^\infty x(i)\ \varepsilon_i$, then $\|x\| = \sup\{ \sum_{k=1}^{n} [ \sum_{p_{2k-1}}^{p_{2k}-1} x(i) ]^2 \}^{1/2}$, where the sup is over all positive integers n and all increasing sequences $\{p_k\}$ of positive integers. These properties of I and J suggested the following theorem [4, theorem 2], which will be used later to show that a certain space B, known to be isomorphic to $B^*$, is quasi-reflexive of order one.

<u>Theorem 1</u> : Suppose a Banach space X has a basic $\{e_n\}$ with coefficient functionnals $\{u_n\}$. If $\sum_1^\infty e_i$ is <u>not</u> norm-convergent and $\{u_1, u_2-u_1, u_3-u_2, \dots\}$ is a shrinking basis for $X^*$, then X is quasi-reflexive of order one.

For computational reasons, it will be useful to introduce the concept of double basis and several related concepts, some of which are extensions of familiar properties of bases.

A <u>double basis</u> for a Banach space X is a subset $\{e_n : -\infty < n < \infty\}$ such that each x in X has a unique representation as $\sum_{-\infty}^{\infty} x(i)\ e_i$ in the sense that

$$\lim_{m,n\to\infty} \|x - \sum_{i=-m}^{n} x(i)\ e_i\| = 0 \quad .$$

A <u>bimonotone double basis</u> is a double basis $\{e_n\}$ such that

$$\|\sum_p^s x(i)\ e_i\| \geq \|\sum_q^r x(i)\ e_i\| \quad \text{if} \quad p \leq q \leq r \leq s \quad .$$

A <u>shrinking double basis</u> is a double basis $\{e_n\}$ such that each of the basic sequences $\{e_n : n < 0\}$ and $\{e_n : n > 0\}$ is shrinking.

A <u>neighborly (double) basis</u> is a (double) basis $\{e_n\}$ such that $\|\sum_{-\infty}^{\infty} x(i)\, e_i\|$ is not increased if some $x(k)$ is replaced by either $x(k-1)$ or $x(k+1)$. Clearly, neighborliness implies bimonotonicity. Also, neighborliness implies what we will call <u>repetition-invariance</u>, namely, for all r,

$$\|\sum_{-\infty}^{\infty} x(i)\, e_i\| = \|\sum_{-\infty}^{r} x(i)\, e_i + \sum_{r}^{\infty} x(i)\, e_{i+1}\| \quad , \tag{4}$$

and also implies <u>translation-invariance</u> for double basis, namely,

$$\|\sum_{-\infty}^{\infty} x(i)\, e_i\| = \|\sum_{-\infty}^{\infty} x(i-1) e_i\| \quad , \tag{5}$$

since it is possible to transform $\sum_{-\infty}^{\infty} x(i)\, e_i$ into the vector of the second member of (4) or (5) and back again by successive replacements of components, each being replaced by one of the two neighboring components. [Note that this process cannot be completed for (5) and a basis $\{e_n : n \geq 1\}$ unless $x(1) = 0$.]

An <u>inversion-invariant</u> (double) basis is a (double) basis such that, whenever x has finite support, there is an n such that

$$\|\sum x(i)\, e_i\| = \|\sum x(n-i)\, e_i\|$$

and, when $\{e_n\}$ is a basis (and <u>not</u> a double basis), $x(i) = 0$ if $i \geq n$. The importance of inversion-invariance at several places in proofs in [4] is due to the fact that if $\{e_n\}$ is a double basis with coefficient functionals $\{u_n\}$ and if $x = \sum_{-\infty}^{\infty} x(i)\, e_i$ and $f = \sum_{-\infty}^{\infty} f(i)(u_i - u_{i-1})$, then

$$(x,f) = \sum_{-\infty}^{\infty} [f(i) - f(i+1)]x(i) = \sum_{-\infty}^{\infty} f(i)[x(i) - x(i-1)] \quad .$$

For a while, let us concentrate our attention on the linear space of functions (double sequences) defined on the set of <u>all</u> integers and having finite support, and on norms we might give this space (and then complete the space). For example, suppose we let

$$I_1(x) = 2^{1/2}\|x\|_I \quad \text{and} \quad J_1(x) = 2^{-1/2}\|x\|_J \quad ,$$

where $\|\ \|_I$ and $\|\ \|_J$ are the norms described in (2) and (1), respectively, applied to double sequences with finite support. By the same methods used for the spaces I and J, it follows that the natural basis is shrinking for each of the norms $I_1$ and $J_1$. If x is the sum of disjoint bumps with altitudes $\{a_n\}$, then

$$J_1(x) \leq 2^{-1/2}(\Sigma 4a_n^2)^{1/2} = 2^{1/2}(\Sigma\ a_n^2)^{1/2} \quad .$$

Since the unit ball for the norm $I_1$ is the closure of the convex span of such x's for which $2^{1/2}(\Sigma\ a_n^2)^{1/2} = 1$, it follows that

$$J_1(x) \leq I_1(x) \quad , \tag{6}$$

for all double sequences x with finite support.

If $\|\ \|$ is a norm for which the natural basis $\{e_n\}$ is basic and if $\{u_n\}$ is the sequence of coefficient functionals, then a new norm $|||\ |||$ is determined by $\|\ \|$ if we let

$$\left|\left|\left| \sum_{-\infty}^{\infty} x(i)\ e_i \right|\right|\right| = \left\| \sum_{-\infty}^{\infty} x(i)(u_i - u_{i-1}) \right\| \quad , \tag{7}$$

for all double sequences $\{x(i)\}$ with finite support. This process is extremely critical for what we will be doing. Clearly, it is closely related to the relation (3) between the shrinking bases for I and J. The use in the next theorem of the process described in (7) is the backbone of our entire construction. This theorem is lemma 5 of [4].

<u>Theorem 2</u> : Let X be the linear space of double sequences with finite support, let $I_n$ be a norm for X, and let $J_n$ be the norm determined from $I_n$ by use of (7). Also, let

$$I_{n+1} = [1/2(I_n^2 + J_n^2)]^{1/2} \quad ,$$

and $J_{n+1}$ be determined from $I_{n+1}$ by use of (7). If there is a number $\overline{M}$ for which $I_n$ and $J_n$ have properties (A) through (C) of the following discussion, and if $J_n \leq I_n$, then $I_{n+1}$ and $J_{n+1}$ have properties (A) through (C) for the same number $\overline{M}$, and

$$J_n \leq J_{n+1} \leq I_{n+1} \leq I_n \quad .$$

We know that $J_1 \le I_1$. By the same methods used for the spaces I and J, it can be shown that $J_1$ is determined from $I_1$ by use of (7). Thus if $I_1$ and $J_1$ have properties (A) through (C), we will have two sequences of norms for which

$$\lim_{n\to\infty} I_n(x) = \lim_{n\to\infty} J_n(x) = \|x\| \quad , \tag{9}$$

for all double sequences x with finite support.

(A) The natural basis $\{e_n\}$ is neighborly. It is easy to check that $\{e_n\}$ is neighborly for each of the norms $I_1$ and $J_1$. Also, it is clear from (8) that $\{e_n\}$ is neighborly for $I_{n+1}$ if it is for both $I_n$ and $J_n$. That $\{e_n\}$ is neighborly for $J_{n+1}$ if it is neighborly for $I_{n+1}$ is not so clear, but the proof is short (see [4, lemma 4]). The property of neighborliness was introduced both because it assures that $\{e_n\}$ is basic for the norm defined by (9), and because it implies both repetition-invariance and translation-invariance. These properties are useful in several proofs.

(B) The natural basis $\{e_n\}$ is inversion-invariant. It is trivial that $\{e_n\}$ is inversion-invariant for the norms $I_1$ and $J_1$. To prove that $\{e_n\}$ is inversion-invariant for $I_{n+1}$ if it is for both $I_n$ and $J_n$, one needs to use the translation-invariance of $\{e_n\}$ for $I_n$ and $J_n$. Inversion-invariance of $\{e_n\}$ for $J_{n+1}$ then follows easily from inversion-invariance for $I_{n+1}$.

(C) The natural basis $\{e_n\}$ is "strongly somewhat Euclidean", in the sense that there is a number M such that

$$\|z\| \le M\Big[\sum_1^s \|z^k\|^2\Big]^{1/2} \tag{10}$$

if $z = \sum_1^s z^k$ and the supports of the $z^k$'s are in intervals each two of which are separated by at least one integer n for which $z(n) = 0$, and M also satisfies

$$\frac{1}{M}\Big[\sum_1^s \|x^k\|^2\Big]^{1/2} \le \|z\| \tag{11}$$

if $z = \sum_{-\infty}^{\infty} (z_i)\, e_i$, $\{p_k\}$ is a strictly increasing sequence of integers, and $\{x^k\}$ is a sequence sucht that $x^k(i) = 0$ if $i \le p_{2k-1}$ or $i > p_{2k}$, and

$x^k(i) = z(i) - z(p_{2k-1})$ if $p_{2k-1} \le i \le p_{2k}$. The inequality (11) is formally stronger than if the $x^k$'s were replaced by the $z^k$'s used in (10). If $I_n$ and $J_n$ have property (C) for the same number $\overline{M}$, then it follows easily from (8) that $I_{n+1}$ has property (C) for this $\overline{M}$. Also, (10) for $I_{n+1}$ implies (11) for $J_{n+1}$, and (11) for $I_{n+1}$ implies (10) for $J_{n+1}$ (for this, repetition-invariance is needed : see lemma 3 of [4]). Property (C) was introduced because a property was needed that would be inherited by the norm $\|\ \|$ defined by (9) and which also would imply the natural basis $\{e_n\}$ for $\|\ \|$ is shrinking. It is easy to show that $J_1$ has property (C) for $M = 2^{1/2}$. Intuitively, it seems natural that $I_1$ also should have property (C). In fact, $I_1$ does have property (C) and M can be $2(1+2^{1/2})$, but this is not as easy to establish (see the proof of lemma 6 of [4]).

It follows from theorem 2 that, for the norm $\|\ \|$ defined by (9), $\{e_n\}$ is a basis that is inversion-invariant, neighborly, and satisfies (10) and (11) with $M = 2(1+2^{1/2})$. Because of (10), $\{e_n\}$ is shrinking. Also, if $\{u_n\}$ are the coefficient functionals, then, for all double sequences $\{x(i)\}$ with finite support,

$$\|\Sigma\, x(i)\, e_i\| = \|\Sigma\, x(i)(u_i - u_{i-1})\| \quad . \tag{12}$$

Now let B be the space spanned by the basic sequence $\{e_n : n \ge 1\}$. This basis for B is shrinking, and (12) implies that $\{u_1, u_2-u_1, u_3-u_2, \dots\}$ is a shrinking basis for $B^*$. Thus it follows from theorem 1 that B is quasi-reflexive of order one. Because of (12), the map T defined by

$$T[\sum_1^\infty x(i)\, e_i] = x(1)u_1 + \sum_2^\infty x(i)(u_i - u_{i-1})$$

is an isometry of $\{x : x \in B \text{ and } x(1) = 0\}$ onto the closure of the linear span of $\{u_i - u_{i-1} : i \ge 2\}$. As a map of B onto $B^*$, $\|T\| \le 2$ and $\|T^{-1}\| \le 3$ (see the proof of theorem 4 of [4]). Although $\|\ \|$ is not known explicitly , several other interesting properties of B are known (see [4, theorem 4]) :

(i) The basic $\{e_n\}$ is inversion-invariant, translation-invariant, and neighborly, with $1 \le \|e_n\| \le (3/2)^{1/2}$.

(ii) $[\Sigma\, a_n^2]^{1/2} \le \|x\| \le 2^{1/2}[\Sigma\, a_n^2]^{1/2}$ if x is the sum of strongly disjoint bumps with altitudes $\{a_n\}$.

(iii) If $z = \sum_1^s z^k$ and the supports of the $z^k$'s are in intervals each two of which are separated by at last one integer n for which $z(n) = 0$, then

$$[2(1+2^{1/2})]^{-1}[\Sigma \|z^k\|^2]^{1/2} \leq \|z\| \leq 2(1+2^{1/2})[\Sigma \|z^k\|^2]^{1/2} .$$

Finally, $\{e_n\}$ being neighborly implies the sequence of coefficient functionals is equal-signs-additive, which with [1, theorem 4] implies the existence of a Banach space X that is quasi-reflexive of order one, isomorphic to $X^*$, and isometric to $X^{**}$.

## REFERENCES


[1] A. Brunel and L. Sucheston, Equal signs additive sequences in Banach spaces, J. Funct. Analysis 21, to appear (March, 1976).

[2] R. C. James, Bases and reflexivity of Banach spaces, Ann. of Math. 53 (1950), 518-527.

[3] R. C. James, A non-reflexive Banach space isometric with its second conjugate space, Proc. Nat. Acad. Sci. U.S.A. 37 (1951), 174-177.

[4] R. C. James, Banach spaces quasi-reflexive of order one, to appear in Studia Math. 60, No 3.

[5] J. Lindenstrauss, On James's paper "Separable Conjugate Spaces", Israel J. Math. 9 (1971), 279-284.

[6] I. Singer, Bases in Banach spaces I, Springer, Berlin, 1970.


-----