Séminaire d'analyse fonctionnelle
École Polytechnique

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Séminaire d'analyse fonctionnelle (Polytechnique) (1975-1976), exp. nº 1, p. 1-4
<http://www.numdam.org/item?id=SAF_1975-1976____A1_0>

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A PROOF OF THE HUFF-MORRIS RADON-NIKODYM THEOREM.

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Exposé No I

7 Octobre 1976

We take as our definition of the Radon-Nikodym property (RNP) the following :

A Banach space X has RNP if and only if for every probability space (S,Σ,μ) and every continuous, linear $T : L_1(S,\Sigma,\mu) \to X$ there exists a Borel measurable $\tau : S \to X$, $\int_S \|\tau(s)\| d\mu(s) < +\infty$, such that

$$T(f) = \int_S f(s)\,\tau(s)\,d\mu(s) \quad .$$

When such a τ represents an operator T we shall say T is differentiable and, of course, if no such τ exists we shall say T is non-differentiable.

Our notation is standard. We denote by (S,Σ,μ) a probability space and Σ^+ is the subset of Σ of sets of positive measure.
Our only prerequisite is the following theorem of Grothendieck [1] (using different words, of course) :

<u>Theorem</u> : Let X be a Banach space. Then X has RNP if and only if for every probability space (S,Σ,μ), every continuous $T : L_1(S,\Sigma,\mu) \to X$, and every $\delta > 0$, there exists $E \in \Sigma$, $\mu(E) < \delta$ such that $\{Tf : \|f\| = \|f \cdot \chi_{S\setminus E}\| \leq 1\}$ is relatively compact.

Our objective is to prove, using only the above, the following theorem of Huff and Morris [3] which contains several other theorems of a geometric nature (see [3] and its bibliography) :

<u>Theorem</u> : Suppose X is a Banach space that does not have RNP. Then there exists a sequence $\{x_{n,i}\}_{n=1}^{\infty}{}_{i=1}^{k(n)}$, $\varepsilon > 0$, such that $k(1) = 1$, $\|x_{n,i}\| \leq 1$, $\|x_{n,i} - x_{m,j}\| > \varepsilon$ if $n \neq m$, and for each n, i there exists pairwise disjoint sets $\sigma_{n,i} \subseteq \{j : 1 \leq j \leq k(n+1)\}$ such that $x_{n,i}$ is in the convex hull of $\{x_{n+1,j} : j \in \sigma_{n,i}\}$.

An immediate consequence of this theorem is the following :

<u>Theorem</u> : Suppose X is a Banach space that does not have RNP. Then there exist an $\varepsilon > 0$, a probability space (S,Σ,μ) (which may be assumed to be $([0,1],\mathfrak{B},\lambda)$ = Lebesgue measure on the Borel subsets of $[0,1]$),a sequence

$\{\Sigma_n\}_{n=1}^{\infty}$, $\Sigma_n \subseteq \Sigma_{n+1}$, Σ_n a finite sub-algebra of Σ, $f_n : S \to X$ Σ_n measurable and $E_n f_{n+1} = f_n$ (the conditional expectation) with $\|f_n(s) - f_m(t)\| \geq \varepsilon$ a.e. for all $n \neq m$.

The proof of the theorem follows readily from the following two lemmas :

Lemma 1 : $T : L_1(S,\Sigma,\mu) \to X$ not differentiable. Then there exists an $\varepsilon > 0$ and $A \in \Sigma^+$ such that for all $E \subseteq A$, $E \in \Sigma^+$

$$\mathcal{A}_E = \{T(\mu(F)^{-1}\chi_F) : F \subseteq E,\ F \in \Sigma^+\}$$

has no ε-net.

Proof : Suppose not. Let $\delta_n > 0$. Then for all $A \in \Sigma^+$ there exists an $E \subseteq A$, $E \in \Sigma^+$ such that $\mathcal{A}_E$ has a δ_n-net. Thus, there exists a sequence $\{E_{n,i}\}_{i=1}^{\infty}$ of pairwise disjoint elements of Σ^+ such that $\mu[\bigcup_{i=1}^{\infty} E_{n,i}] = 1$ and $\mathcal{A}_{E_{n,i}}$ has a δ_n net for each (n,i). Choose $k(n)$ such that $\mu[\bigcup_{i=1}^{k(n)} E_{n,i}] > 1 - \delta_n$. Let $A_n = \bigcup_{i=1}^{k(n)} E_{n,i}$. It is clear that $\mathcal{A}_{A_n}$ has a δ_n-net. For $\delta > 0$, choose $\{\delta_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \delta_n < \delta$, $\delta_n > 0$. Let $A = \bigcap_{n=1}^{\infty} A_n$, then $\mu(A) > 1-\delta$ and $\mathcal{A}_A$ has a δ_n-net for every δ_n i.e. $\mathcal{A}_A$ is relatively compact. Contradiction. Thus the lemma is proved.

Lemma 2 : Let $T : L_1(S,\Sigma,\mu) \to X$ such that for all $E \in \Sigma^+$, $\mathcal{A}_E$ has no ε-net. Then for all $f \in L_1(S,\Sigma,\mu)$, $\|f\| = 1$, $f \geq 0$ a.e., all $\delta > 0$, and all $y_o = Tf, y_1, \ldots, y_m \in X$, there exist $f_1, \ldots, f_n \in L_1(S,\Sigma,\mu)$, $\|f_i\| = 1$, $f_i \geq 0$ a.e., $\lambda_i > 0$, $\sum_{i=1}^{n} \lambda_i = 1$, $\|y_j - Tf_i\| > \varepsilon$ for all $1 \leq j \leq m$ an all $1 \leq i \leq n$ and $\|Tf - \sum_{i=1}^{n} \lambda_i Tf_i\| < \delta$.

Proof : We may assume $f = \sum_{k=1}^{\ell} s_k \mu(A_k)^{-1} \chi_{A_k}$ where $\{A_k\}_{k=1}^{\ell}$ is a pairwise disjoint collection in Σ^+, $\sum_{k=1}^{\ell} s_k = 1$, $s_k > 0$. Since $\mathcal{A}_{A_k}$ has no ε-net, choose a maximal collection of disjoint elements $\{E_{k,p}\}_{p=1}^{\infty}$ in Σ^+, $\Sigma_{k,p} \subseteq A_k$, $\|T(\mu(E_{k,p})^{-1} \chi_{E_{k,p}}) - y_j\| > \varepsilon$ for all p and all j, $1 \le j \le m$. By maximality, we have that $\mu[A_k \setminus \bigcup_{p=1}^{\infty} E_{k,p}] = 0$.

Choose q such that $(1+\delta/2)\mu(\bigcup_{p=1}^{q} E_{k,p}) > \mu(A_k)$.

Then $\left\| Tf - \sum_{k=1}^{\ell} \sum_{p=1}^{q} s_k \frac{\mu(E_{k,p})}{\sum_{p=1}^{q} \mu(E_{k,p})} T(\mu(E_{k,p})^{-1} \chi_{E_{k,p}} \right\|$ is less than δ.

Proof (of theorem) : Suppose X does not have RNP. Then there exists $T : L_1(S,\Sigma,\mu) \to X$ and $\varepsilon > 0$ satisfying Lemma 1. By Lemma 2, $K = \{Tf : f \ge 0$ a.e. $\|f\| = 1\}$ has the following property : for any choice $y_1,\dots,y_m \in X$:

$$K \subseteq \overline{co}\ [K \setminus \sum_{j=1}^{m} B(y_j,\varepsilon)]$$

$(B(y,\varepsilon)) = \{z : \|z-y\| < \varepsilon\}$; co denotes the connex hull ; $\overline{co}$ denotes the closed connex hull). Using a trick of Davis and Phelps [2] and an elementary computation shows that if we denote

$$K_1 = K + B(0,\varepsilon/2) \quad (K_1 \text{ is open, convex})$$

then for any choice of $y_1,\dots,y_m \in X$

$$K_1 = co[K_1 \setminus \bigcup_{j=1}^{m} B(y_j,\varepsilon/2)] \quad .$$

Given such a set K_1 it is completely straightforward to construct the desired sequence.

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