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ENFLO OPERATORS ON L^1

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We present here a summary of some results obtained by P. Enflo and the author. The complete proofs can be found in [1] or [6]. We study a class of operators on L^1 called Enflo operators. The definition of an Enflo operator is given below. Theorem 1 gives a characterization of Enflo operators which could serve as an alternate definition. Theorem 2 is the main structural result concerning Enflo operators. Several corollaries are then derived.

We deal with $L^1 = L^1([0,1],\mu)$, the Banach space of equivalence classes of Lebesgue integrable real-valued functions defined on $[0,1]$. μ is Lebesgue measure. An "isomorphism" is a linear homeomorphism into. An "isomorph of L^1" is a Banach space isomorphic (i.e. linearly homeomorphic) to L^1. If two Banach spaces X and Y are isomorphic, we write $X \sim Y$. "Subspace" will mean "closed linear subspace". If E is a (Lebesgue) measurable subset of $[0,1]$, we write $|E|$ for $\mu(E)$. χ_E denotes the characteristic function of E, that is,

$$\chi_E(t) = \begin{cases} 1 & \text{if } t \in E \\ 0 & \text{if } t \notin E \end{cases} .$$

<u>Definitions</u> :

(1) A collection $\{F_i^n : i = 1,2,\dots,2^n ; n = 0,1,\dots\}$ of measurable subsets of $[0,1]$ is called a <u>tree</u> if

(a) $|F_1^o| > 0$;

(b) for any n and i, $1 \le i \le 2^n$,

$$F_{2i-1}^{n+1} \cup F_{2i}^{n+1} = F_i^n \quad ;$$

(c) $F_i^n \cap F_j^n = \emptyset$ whenever $i \ne j$, $1 \le i, j \le 2^n$;

and (d) $\max_{1 \le i \le 2^n} |F_i^n| \to 0$ as $n \to \infty$.

(2) A collection $\{F_i^n : i = 1,2,\dots,M_n ; n = 0,1,\dots\}$ of measurable subsets of $[0,1]$ is called a <u>bush</u> if

(a) $M_o = 1$ and $|F_1^o| > 0$,

(b) for any n and i, $1 \le i \le M_{n+1}$, the set F_i^{n+1} is contained is some F_j^n ;

(c) for each n, the collection $\{F_i^n : i = 1,\dots,M_n\}$ forms a partition of F_1^o ;

and (d) $\max_{1\le i\le M_n} |F_i^n| \to 0$ as $n \to \infty$.

(3) Let $T : L^1 \to L^1$ be a bounded linear operator. T is called an Enflo operator if there exist $\delta > 0$ and a bush (E_i^n), $i = 1,\dots,M_n$; $n = 0,1,\dots$ of subsets of $[0,1]$ such that

$$\frac{1}{|E_1^o|} \int \max_{1\le i\le M_n} |T(\chi_{E_i^n})| > \delta$$

for each n. If T is an Enflo operator and $\lambda > 0$, T is called an Enflo operator of constant λ if

$$\lambda \le \sup \varliminf_{n\to\infty} \frac{1}{|E_1^o|} \int \max_{1\le i\le M_n} |T(\chi_{E_i^n})| \quad ,$$

where the supremum is taken over all bushes (E_i^n) in which $|E_1^o| > 0$. (It can be shown that actually for any such bush, $\lim_{n\to\infty} \frac{1}{|E_1^o|} \int \max_{1\le i\le M_n} |T(\chi_{E_i^n})|$ exists.)

Our first major result is

Theorem 1 : Let $T : L^1 \to L^1$ be a bounded linear operator. T is an Enflo operator if and only if there exists a subspace Y of L^1 with Y isomorphic to L^1 and with $T|_Y$ an isomorphism (into).

The proof of Theorem 1 depends upon the next Theorem :

Theorem 2 : Suppose T is an Enflo operator of constant λ, and $0 < \varepsilon < 1/2$. Then there exist a measurable subset E_o of $[0,1]$ and a σ-algebra $\mathcal{a}$ of subsets of E_o such that

1. $\mu|\mathcal{a}$ is purely non-atomic, and hence $L^1(\mu|\mathcal{a})$ is isometric to L^1 (see [2]) ;

2. $T|L^1(\mu|\mathcal{a})$ is an isomorphism ; for $f \in L^1(\mu|\mathcal{a})$,

$$\|Tf\| \ge \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2 \lambda \|f\| \quad ;$$

3. The image $T(L^1(\mu|\mathcal{a}))$ is complemented ; it is the range of a projection of norm at most

$$\frac{\|T\|}{\lambda} \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2 \quad .$$

In fact, there is a tree (A_i^n), $i = 1,\dots,2^n$; $n = 0,1,\dots$ of measurable subsets of $E_o = A_1^o$ with

(a) $$(1-\varepsilon)\frac{|E_o|}{2^n} \le |A_i^n| \le (1+\varepsilon)\frac{|E_o|}{2^n}$$

and there is a tree (F_i^n), $i = 1,\dots,2^n$; $n = 0,1,\dots$ of measurable subsets of a set $F_1^o \subset [0,1]$ such that for each n and i, $1 \le i \le 2^n$,

(b) $$(1-\varepsilon)\lambda\frac{|E_o|}{2^n} \le \int_{F_1^o} |T\chi_{A_i^n}| < (1+\varepsilon)\int_{F_i^n} |T\chi_{A_i^n}| \ .$$

Remark : Conclusions 1, 2, and 3 follow, using facts about relative disjointness (see [5]), from the existence of the trees (A_i^n) and (F_i^n) satisfying (a) and (b).

Remark : Note that Theorem 2, conclusions 1 and 2, asserts a strong form of the direct implication claimed in Theorem 1.

Corollary 1 : Let $Z = L^1$. If X is a subspace of Z and $X \sim L^1$, then there exists a subspace Y of X with $Y \sim L^1$ and Y complemented in Z.

Proof : Let $T : L^1 \to Z$ be any isomorphism onto X, apply Theorem 1 and then Theorem 2.

Q.E.D.

A consequence of Corollary 1 together with the decomposition method of Pełczynski [4] is that if a complemented subspace X of L^1 contains a subspace isomorphic to L^1, then $X \sim L^1$.

Before stating the next corollary, we make the following

Remark : If $T_1 + T_2$ is on Enflo operator, then either T_1 or T_2 must be an Enflo operator. For

(*) $$\int \max_i |(T_1+T_2)E_i^n| \le \int \max_i |T_1E_i^n| + \int \max_i |T_2E_i^n|$$

for each n and for any bush (E_i^n). If neither T_1 nor T_2 is an Enflo operator, then as $n \to \infty$ the limit of the right-hand side, and hence the left-hand side, of (*) is 0. Since this is true for all bushes, $T_1 + T_2$ cannot be an Enflo operator.

Now we can state the next corollary, an alternate proof of which is given in [3].

<u>Corollary 2</u> : L^1 is primary ; i.e., if $L^1 \sim X \oplus Y$, then either $X \sim L^1$ or $Y \sim L^1$ (or both).

<u>Proof</u> : Consider X and Y as complementary subspaces in L^1, with projections P onto X and I-P onto Y. Then, since these two operators sum to the identity operator, which is certainly an Enflo operator, one of them (let us say P) is an Enflo operator.

Hence by theorem 2 the range of P, which is X, must contain a complemented subspace isomorphic to L^1. Pełczynsli's decomposition method [4] then implies that $X \sim L^1$.

Q.E.D.

<u>Corollary 3</u> : Let $T : L^1 \to L^1$ be a bounded linear operator. If there exists a subspace Y isomorphic to L^1 with $T|_Y$ an isomorphism, then there exists a subspace Z isometric to L^1 with $T|_Z$ an isomorphism.

<u>Proof</u> : Combine Theorems 1 and 2.

Q.E.D.

Recall that if $T : L^1 \to L^1$ is a bounded linear operator, its absolute value, $|T|$, is the operator on L^1 defined for $f \geq 0$, $f \in L^1$ by

$$(|T|f)(t) = \sup \left\{ \sum_{i=1}^{m} |Tf_i(t)| \;\middle|\; \sum_{i=1}^{m} f_i = f,\ f_i \geq 0 \right\}$$

for all t in [0,1], and defined for general $f \in L^1$ by linearity, writing f as the difference of two non-negative functions.

<u>Proposition</u> : Let $T : L^1 \to L^1$ be a bounded linear operator. T is an Enflo operator if ond only if $|T|$ is an Enflo operator.

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