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# UNIFORMLY CONVEX NORMS IN SPACES WITH UNCONDITIONAL BASIS

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Let $(E,\|\cdot\|)$ be a Banach space and let f be a non-negative function on $[0,2]$. It is known (cf. [5],[2]) that if E admits an equivalent norm, say $|||\cdot|||$, such that for $x,y \in E$

$$|||x||| = |||y||| = 1 \quad \Rightarrow \quad |||\tfrac{x+y}{2}||| \leq 1 - f(|||x-y|||) \ ,$$

then E is of cotype f in the following sense : there exist positive constants $c_1$, $c_2$ such that if $x_1,\dots,x_n \in E$ satisfy

$$\int_0^1 \| \Sigma\, x_i r_i(t)\| \, dt \leq c_1 \ ,$$

($r_i$ denoting the usual Rademacher functions), then

$$\sum_{i=1}^{n} f(\|x_i\|) \leq c_2 \ .$$

We shall prove the following partial converse to that result. (In the sequel $c_i$, $i = 1,2,\dots$ denote always some positive constants).

<u>Theorem</u> : Suppose E is of cotype f. If E is superreflexive and has an unconditional basis, then there exists an equivalent norm on E, say $|||\cdot|||$, such that if $x,y \in E$ satisfy $|||x||| = |||y||| = 1$, and $|||x-y||| \leq c_3$, then

$$|||\tfrac{x+y}{2}||| \leq 1 - c_4 \ f(|||x-y|||) \ .$$

We shall regard the elements of E as (numerical) functions defined on the set $\mathbb{N}$ of the indices of the unconditional basis. The expressions like $(|x|^p + |y|^p)^{1/p}$, involving elements of E (x,y in the latter case), are to be understood as functions on $\mathbb{N}$ obtained by applying the particular formula pointwise in the scalar sense.

The theorem being trivial if $f(t) = 0$ for each $t \in [0,c_1)$, we shall assume that it is not the case. Under this assumption we shall prove that there is a function F on $[0,\infty)$ such that $F \geq f$ on $[0,c_1]$ which has some special properties (to be specified below).

The superreflexivity of E ensures the existence of an equivalent norm on E that is p-convex for some $p > 1$ (a proof can be found in [3] or [2] ; we shall reproduce the argument later). Since the properties of F that we have mentioned hold true (perhaps with other values of the constants) when $\|\,.\,\|$ is replaced by any equivalent norm, we may assume that $\|.\|$ has already been p-convex for some $p \in [1,2]$, i.e.

$$\|(|x|^p + |y|^p)^{1/p}\|^p \leq \|x\|^p + \|y\|^p \quad , \text{ for } x,y \in E \quad .$$

It is easy to check that the assumptions of the following lemma will be fulfilled.

<u>Lemma 1</u> : Suppose E is p-convex, $1 < p \leq 2$, and F is a function on $[0,\infty]$ such that

$F(0) = 0, \quad F(1) > 0$ ;
the function $t \mapsto F(t^{1/p})$ is convex ;
the function $t \mapsto F(t)t^{-r}$ is decreasing for some $r \geq 1$ ;
if $z_1, z_2, \ldots, z_n \in E$, $n = 1,2,\ldots$, and $\|(\Sigma z_i^2)^{1/2}\| \leq 1$, then
$\Sigma F(\|z_i\|) \leq c_5$ .

Then the formula

$$|||x||| = \inf \{ t > 0 : \Sigma F(\|x_i\|/t) \leq F(1) \text{ , whenever } |x| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} \}$$

defines an equivalent p-convex norm on E such that $|||(x^2+y^2)^{1/2}||| \leq 1$ implies

$$F(|||y|||) \leq c_7(1 - |||x|||) \quad .$$

<u>Proof</u> : It is clear that $|||x||| \geq \|x\|$ for $x \in E$. On the other hand, if $c_6 = (c_5/F(1))^{1/p}$ and $|x| = (\sum_{i=1}^{n} x_i^2)^{1/2}$, then $c_6 \geq 1$, hence

$$\Sigma F(\|x_i\|/c_6\|x\|) \leq c_6^{-p} \Sigma F(\|x_i\|/\|x\|) \leq F(1) \quad .$$

This implies that $|||x||| \le c_6 \|x\|$.

Now, if $|z| = (|x|^p + |y|^p)^{1/p}$, where $x,y \in E \setminus \{0\}$ and $|||x|||^p + |||y|||^p = 1$, then, for any function a on $\mathbb{N}$ with $|a| \le 1$, we have

$$F(\| az \|) = F(\| (|ax|^p + |ay|^p)^{1/p} \|)$$

$$\le F((\|ax\|^p + \|ay\|^p)^{1/p})$$

$$= F((|||x|||^p (\|ax\| / |||x|||)^p + |||y|||^p (\|ay\| / |||y|||)^p)^{1/p})$$

$$\le |||x|||^p F(\|ax\| / |||x|||) + |||y|||^p F(\|ay\| / |||y|||).$$

Hence, given any sequence $a_1, \ldots, a_n$ of such functions that satisfies $\sum_{i=1}^{n} a_i^2 = 1$, applying the latter estimate for $i = 1,2,\ldots,n$ and adding up these inequalities we obtain

$$\sum_{i=1}^{n} F(\|a_i z\|) \le (|||x|||^p + |||y|||^p) F(1) = F(1) .$$

The system $(a_i)$ being arbitrary, we have established that $|||.|||$ is p-convex, hence, a fortiori, it is a norm on E.

To check the last statement assume $|||(x^2+y^2)^{1/2}||| \le 1$, $x \ne 0$, $|x| = (\sum_{i=1}^{n} x_i^2)^{1/2}$. Since $\sum_{i=1}^{n} F(\|x_i\|) + F(\|y\|) \le F(1)$, we have

$$\sum_{i=1}^{n} F(\|x_i\| / (1 - F(\|y\|)/F(1))^{1/r}) \le [1 - F(\|y\|)/F(1)]^{-1} \sum_{i=1}^{n} F(\|x_i\|) \le F(1) .$$

Therefore

$$|||x||| \le (1 - F(\|y\|)/F(1))^{1/r} \le 1 - r^{-1} F(\|y\|)/F(1)$$

and finally

$$F(|||y|||) \le F(c_6 \|y\|) \le c_6^r F(\|y\|) \le c_6^r \, r \, F(1)(1 - |||x|||) = c_7 (1 - |||x|||) .$$

This completes the proof of the lemma. We also need the following simple facts.

Lemma 2 : Given real numbers p, t, s, with $1 \le p \le 2$. Let

$$z = [\tfrac{1}{2}(|t|^p + |s|^p)]^{1/p}, \quad w = (z^2 - (\tfrac{t-s}{2})^2)^{1/2} .$$

Then

$$|\tfrac{t+s}{2}| \le (2-p)z + (p-1)w .$$

Proof : By the homogeneity, it suffices to consider the case $z = 1$. Then $x = (2^{-1/p}t, 2^{-1/p}s)$ and $y = (2^{-1/p}s, 2^{-1/p}t)$ are norm one vectors in $\ell_p^2$. Recall that the modulus of convexity of $\ell_p$ satisfies $\delta_{\ell_p}(\varepsilon) \ge \frac{p-1}{8}\varepsilon^2$ (a short proof can be found in [2 ], Proposition 24).
Thus we have

$$z - |\tfrac{t+s}{2}| = 1 - \|\tfrac{x+y}{2}\| \ge \delta_{\ell_p}(\|x-y\|)$$

$$= \delta_{\ell_p}(|t-s|) \ge \frac{p-1}{2}(\frac{t-s}{2})^2$$

$$\ge (p-1)[1-(1-(\tfrac{t-s}{2})^2)^{1/2}] = (p-1)(z-w) ,$$

which is equivalent to the statement of the lemma.

Lemma 3 : Suppose $(E, |||\ \ |||)$ is p-convex, $1 \le p \le 2$, and h is a function such that whenever $u, v \in E$ and $|||(u^2+v^2)^{1/2}||| \le 1$ one has $h(|||v|||) \le 1 - |||u|||$. Let x,y be vectors in E with $|||x|||, |||y||| \le 1$.
Then

$$|||\tfrac{x+y}{2}||| \le 1 - (p-1)\, h(\tfrac{1}{2}\|x-y\|) .$$

Proof : Let

$$z = [\tfrac{1}{2}(|x|^p + |y|^p)]^{1/p}, \quad w = (z^2 - |\tfrac{x-y}{2}|^2)^{1/2} .$$

Since, by the p-convexity, $|||z||| \leq 1$, our assumption on h yields

$$h\left(\tfrac{1}{2}|||x-y|||\right) \leq 1 - |||w||| \quad .$$

By Lemma 2, $\left|\frac{x+y}{2}\right| \leq (2-p)z + (p-1)w$. Using the triangle inequality we get the desired estimate

$$\begin{aligned} \left|\left|\left|\frac{x+y}{2}\right|\right|\right| = \left|\left|\left| \left|\frac{x+y}{2}\right| \right|\right|\right| &\leq |||(2-p)z + (p-1)w||| \\ &\leq (2-p)|||z||| + (p-1)|||w||| \\ &\leq 2-p + (p-1)\left(1 - h\left(\tfrac{1}{2}|||x-y|||\right)\right) . \end{aligned}$$

The theorem follows now immediately. By Lemma 1, we can put $h(t) = c_7^{-1} F(t)$ and it remains to note that, if $t \leq c_1$, then

$$(p-1)\, c_7^{-1} F(\tfrac{1}{2}t) \geq c_8 F(t) \geq c_8 f(t)$$

where $c_8 = 2^{-r}(p-1)\, c_7^{-1}$ .

It remains to construct the function F. This done in a number of steps.

We know that

$$\int \|\Sigma x_i r_i\| \leq c_1 \quad \Rightarrow \quad \Sigma f(\|x_i\|) \leq c_2 \quad .$$

By the principle of contraction, if we let $f_1(u) = \sup\{f(t) : 0 \leq t \leq u\}$ for $u \in [0, c_1]$ and $f_1(u) = f_1(c_1)$ for $u > c_1$, then $f_1 \geq f$ on $[0, c_1]$, $f_1$ is non-decreasing and still

$$\int \|\Sigma x_i r_i\| \leq c_1 \quad \Rightarrow \quad \Sigma f_1(\|x_i\|) \leq c_2 \quad .$$

Now, since $f_1(t) > 0$ for some $t < c_1$, the space E does not contain $\ell_\infty^n$ uniformly (the latter follows also from the super-reflexivity of E), and hence, by Théorème 4 and Corollaire 1 of [6], we have

(i) there is a $q < \infty$ such that

$$\int \| \Sigma x_i r_i \| \leq c_1 \quad \Rightarrow \quad \Sigma \| x_i \|^q \leq c_9 \quad ;$$

(ii) there is a $c_{10}$ so that

$$\| (\Sigma x_i^2)^{1/2} \| \leq c_{10} \quad \Rightarrow \int \| \Sigma x_i r_i \| \leq c_1 \quad .$$

Given $A > 1$ let $\varphi(A)$ denote the l.u.b. of the sums $\sum_{i=1}^{n} f_1(A \|x_i\|)$ where the sequence $x_1, x_2, \dots, x_n \in E$ satisfies $\| (\sum_{i=1}^{n} x_i^2)^{1/2} \| \leq c_{10}$. We shall prove that

$$\varphi(A) \leq c_{11} A^q \quad .$$

To this end pick $x_1, \dots, x_n \in E$ such that

$$\| ( \Sigma x_i^2)^{1/2} \| \leq c_{10} \qquad \text{and} \qquad \Sigma f(A \|x_i\| ) \geq \tfrac{1}{2} \varphi(A) \quad ,$$

and define inductively the sequence

$$0 = s_o < s_1 < \dots < s_k \leq n$$

of integers so that, for $j = 1, 2, \dots, k$ ,

$$\| ( \sum_{s_{j-1}+1}^{s_j - 1} x_i^2)^{1/2} \| < c_{10}/A \quad , \quad \| ( \sum_{s_{j-1}+1}^{s_j} x_i^2)^{1/2} \| \geq c_{10}/A \ ,$$

and

$$\| ( \sum_{s_k+1}^{n} x_i^2)^{1/2} \| < c_{10}/A \quad .$$

Using (ii) and the definitions we get easily

$$\Sigma f_1(A \|x_i\| ) \leq (2k+1) c_2 \leq 3k \, c_2 \quad .$$

Let $y_j = \left( \sum_{s_{j-1}+1}^{s_j} x_i^2 \right)^{1/2}$, $j = 1,2,\dots,k$. Then

$$\left\| \left( \sum_{j=1}^{k} y_j^2 \right)^{1/2} \right\| \le \left\| \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2} \right\| \le c_{10} ,$$

hence $\int \| \Sigma\, y_j r_j \| \le c_1$, whence

$$c_3 \ge \sum_{j=1}^{k} \|y_j\|^q \ge k(c_{10}/A)^q \quad .$$

Thus we get the promised estimate

$$\varphi(A) \le 6c_2 k \le 6c_2 c_3 c_{10}^{-q} A^q = c_{11} A^q \quad .$$

Now fix an $r > q$ and let

$$f_2(t) = \sum_{n=0}^{\infty} 2^{-rn} f_1(2^n t) \ .$$

Then, whenever $\|(\Sigma\, x_i^2)^{1/2}\| \le c_{10}$, one has

$$\sum_i f_2(\|x_i\|) = \sum_i \sum_m 2^{-rm} f_1(2^m \|x_i\|)$$

$$\le \sum_m 2^{-rm} \varphi(2^m) \le c_{12} < \infty \quad .$$

Now let $f_3(t) = \sup_{u > t} f_2(u)\,(t/u)^r$, since for all s

$$f_2(2s) = 2^r(f_2(s) - f_1(s)) \le 2^r f_2(s) \quad ,$$

we obtain that, whenever $0 < t \le 2^k t \le u < 2^{k+1} t$ ,

$$f_2(u) \le f_2(2^{k+1} t) \le f_2(t)(2^r)^{k+1} \le 2^r (u/t)^r f_2(t) \ .$$

Consequently, $f_1(t) \le f_2(t) \le f_3(t) \le 2^r f_2(t)$ and

$$\|(\Sigma x_i^2)^{1/2}\| \le c_{10} \Rightarrow \Sigma f_3(\|x_i\|) \le 2^r c_{12} .$$

Observe that $f_3(t)t^{-r}$ is a decreasing function of t.

Let $f_4(t) = \sup_{u \ge 1} u f_3(t/\sqrt{u})$ and let $f_5(t) = \sup_{m \ge 1} m f_3(t/\sqrt{m})$ (m running over the positive integers). If $m \le u < m+1$, then

$$u f(t/\sqrt{u}) \le (m+1) f(t/\sqrt{m}) \le 2m f(t/\sqrt{m}) \le 2 f_5(t) .$$

On the other hand, if $\|(\sum_{i=1}^{n} x_i^2)^{1/2}\| \le c_{10}$ and $m_1, \dots, m_n$ are positive integers, then letting $y_{ij} = m_i^{-1/2} x_i$, for $j = 1, 2, \dots, m_i$, we get

$$\sum_i m_i f_3(\|x_i\| / \sqrt{m}) = \sum_{i,j} f_3(\|y_{ij}\|) \le 2^r c_{12} .$$

It follows easily that $\|(\Sigma x_i^2)^{1/2}\| \le c_{10}$ implies

$$\Sigma f_4(\|x_i\|) \le 2^{r+1} c_{12} = c_{13} .$$

Clearly, $f_4 \ge f_3$, $f_4(t)/t^2 \nearrow$, $f_4(t)/t^r \searrow$ .

Now let $\varphi$ denote the lower convex envelope of the function g, where $g(t) = f_4(\sqrt{t})$. Then

$$f_4(t) \ge g(t^2) \ge \varphi(t^2) \ge g(\tfrac{1}{2}t^2) \ge f_4(2^{-1/2}t) \ge 2^{-r/2} f_4(t) .$$

(The third inequality can be proved as follows. Suppose an s does not satisfy $\varphi(s) \ge g(s)$. Then there exist $0 < u < s < v$ such that

$$\varphi(s) = \frac{v-s}{v-u} g(u) + \frac{s-u}{v-u} g(v) .$$

If $u < \frac{1}{2}s$, then $g(v)(s-u)/(v-u) \ge g(v)\,\frac{1}{2}\,s/v \ge g(\frac{1}{2}s)$, the other summand being non-negative. In the opposite case one simply has $g(v) \ge g(u) \ge g(\frac{1}{2}s)$ ).

Let us define

$$F(t) = 2^{r/2} \sup_{u>t} (t/u)^r \varphi(u^2) \quad .$$

Clearly, F satisfies $F \le 2^{r/2} f_4$, $F(t)/t^r \searrow$, and F is a convex function of $\sqrt{t}$. Consider an arbitrary sequence $x_1, \dots, x_n \in E$ with $|||(\sum_{i=1}^{n} x_i^2)^{1/2}||| \le 1$. Let $y_i = cx_i$, where $c = \min(c_{10}, 1)$. Then

$$\begin{aligned} \Sigma\, F(\|x_i\|) &\le 2^{r/2}\, \Sigma\, f_4(c^{-1}\|y_i\|) \\ &\le 2^{r/2} c^{-r}\, \Sigma\, f_4(\|y_i\|) \\ &\le 2^{r/2} c^{-r} c_{13} = c_5 \quad . \end{aligned}$$

Thus F satisfies all the assumptions of Lemma 1.

For the sake of completeness let us recall how one can introduce a p-convex norm on E. Since E is superreflexive, there are $q, L < \infty$ such that every operator T from $c_o$ to $E^*$ has its q-absolutely summing norm $\le L\|T\|$. It follows easily that if $x_1, \dots, x_n \in E^*$, then

$$(\sum_{i=1}^{n} \|x_i\|^q)^{1/q} \le L \|(\sum_{i=1}^{n} |x_i|^q)^{1/q}\| \quad .$$

Given a finite sequence $a = (a_1, \dots, a_n)$ of functions on $\mathbb{N}$ such that $\sum_{i=1}^{n} |a_i|^q = 1$ we set for $x \in E^*$

$$\|x\|_a = (\sum_{i=1}^{n} \|a_i x\|^q)^{1/q} \quad ,$$

$$|||x||| = \sup_a \|x\|_a \quad .$$

Plainly, $|||\cdot|||$ is a norm on $E^*$ (being the supremum of the norms $\|\cdot\|_a$) that satisfies $\|\cdot\| \le |||\cdot||| \le L\|\cdot\|$. Moreover, for any $x, y \in E^*$ one has

$$|||x|||^q + |||y|||^q \le |||(|x|^q + |y|^q)^{1/q}|||^q \quad .$$

It is a standard exercise on duality to check that the norm on E dual to $|||\cdot|||$ is p-convex, with $p = q/(q-1)$. This completes the proof.

Remark 1 : The example of $\ell_1$ (which is of type f, where $f(t) = t^2$, but not uniformly convexifiable) shows that it is necessary to assume the superreflexivity of E. The other assumption can be weakened, but not just dropped. For instance, it is enough to assume that E be a complemented subspace of a Banach lattice (the proof combines the renorming techniques applied above with those used in [4]). On the other hand, (after this talk was given) G. Pisier has constructed an example of a superreflexive Banach space that is of cotype $t^p$ but does not admit an equivalent p-uniformly convex norm for some $p < \infty$ .

Remark 2 : The methods employed above are mostly taken from [2], where mainly the renormings related to properties of disjointly supported elements were considered. The results can easily be dualized to relate the "type" and the moduli of uniform smoothness of superreflexive spaces with local unconditional structure.

Remark 3 : Let us just mention (without proof) an application of the theorem. W.J. Davis has constructed in [1] a uniformly convex space Y with a symmetric basis that contains the space E as a complemented subspace. Now, Y can be shown to admit the moduli of convexity not worse than those of E.

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