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A SHORT PROOF OF DVORETZKY'S THEOREM

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The theorem of Dvoretzky [2] states that for any $\varepsilon > 0$ and any positive integer K there exists an $N = N(K,\varepsilon) < \infty$ such that every normed space $(X, \| \ \|)$ with dim $X > N$ contains a K-dimensional subspace E which is ε-Euclidean (i.e. there exists a Euclidean norm $|.|$ on E such that $\|x\| \le |x| \le (1+\varepsilon)\|x\|$ for $x \in E$).

Two proofs of the theorem have already been presented on this seminar (cf.[1], [4], [3]). I am going to show another one based on an idea of Szankowski's [6] but simpler in details . Only the obvious modifications (viz. considering the complex Stiefel and Grassmann manifolds) are needed to obtain a proof of the complex version of the theorem.

In the sequel let K be a fixed integer greater than 1.

We shall need the following consequence of the Dvoretzky-Rogers lemma (cf. [2], [1]).

(D-R) For every integer $n \ge K$ and every normed space $(X, \| \ \|)$ with dim $X \ge 4n^2$ there exists an operator $I : \mathbb{R}^n \to X$ such that

$$\frac{1}{2} \|x\|_{\ell_\infty^n} \le \|Ix\| \le \|x\|_{\ell_2^n} \quad \text{for } x \in \mathbb{R}^n .$$

Let $F = I(\mathbb{R}^n)$ and let $\|x\|_2 = \|I^{-1}x\|_{\ell_2^n}$ for $x \in F$. Since any norm on F can be approximated (uniformly on the unit ball) by smooth ones, we may assume that $\|.\|$ is smooth on F. Thus for each $x \in F \setminus \{0\}$ there is a unique $T_x \in F^*$ such that

$$T_x(x) = \|x\| \|T_x\|_{F^*} = 1 .$$

Clearly, T_x depends continuously on x, and $\|x\| T_x$ is simply the Gâteaux derivative of the norm $\|.\|$ at x.

For any linear subspace $E \subseteq F$ with dim $E \ge 2$, let S_E denote the unit sphere $\{x \in E : \|x\|_2 = 1\}$ and let Σ_E denote the Stiefel manifold of all ordered pairs $\langle x,y \rangle \in S_E \times S_E$ such that $y \in x^\perp = \{f \in F : (I^{-1}f, I^{-1}x)_{\ell_2^n} = 0\}$

The normalized $\| \ \|_2$-rotation invariant measures on S_E and Σ_E will be denoted by λ_E and σ_E respectively.

Our basic invariant characterizing the closeness of $\| \ \|$ to $\| \ \|_2$ on E is defined as follows

$$v(E) = \int_{\Sigma_E} T_x(y)^2 d\sigma_E(x,y).$$

We shall check the following facts :

1) There exists a subspace $E \subseteq F$ such that $\dim E = K$ and $v(E) \le v(F)$;

2) $d(E) = \int_{S_E \times S_E} (\|x\| - \|z\|)^2 \, d\lambda_E(x)\, d\lambda_E(z)/\sup_{S_E} \|x\|^2 \le (\pi/2)^2 v(E)$;

3) $b = 1 - \inf_{S_E} \|x\|/\sup_{S_E} \|x\| \le C d(E)^{1/(K+1)}$, where C depends only on K.

It follows from 1, 2, 3 that $b \le C_1 v(F)^{1/(K+1)}$, where C_1 is another constant. Since $\varepsilon \le b/(1-b)$, the proof will be complete, if we also establish :

4) There exists a sequence $(C_n)_{n=2}^{\infty}$ tending to zero such that the F yielded by (D-R) satisfies $v(F) \le C_n$.

<u>Proofs</u> : 1) is an immediate consequence of the formula

$$v(F) = \int_{\Sigma_F} T_x(y)^2 d\sigma_F(x,y)$$

$$= \int_{\Gamma} d\gamma(E) \int_{\Sigma_E} T_x(y)^2 d\sigma_E(X,y) = \int_{\Gamma} v(E)\, d\gamma(E),$$

where γ is the normalized $\|\cdot\|_2$-rotation invariant measure on the Grassmann manifold Γ of all K-dimensional linear subspaces of F. (The second equality is valid when $T_x(y)^2$ is replaced by any function $f(x,y)$ defined and continuous on Σ_F ; it follows from the uniqueness of a normalized invariant measure on Σ_F).

2) For any $x, z \in S_E$ with $z \ne \pm x$, let $a_{x,z}(t)$, $0 \le t \le 2\pi$, be the arc-length parametrization of the great circle of S_E starting at x and passing through z, -x , - z back to x. We have

$$4\left|\|x\| - \|z\|\right| \le \int_0^{2\pi} \left|\frac{d}{dt} \|a_{x,z}(t)\|\right| dt \le \sqrt{2\pi}\left(\int_0^{2\pi} \left(\frac{d}{dt} \|a_{x,z}(t)\|\right)^2 dt\right)^{1/2}$$

hence

$$\int_{S_E \times S_E} (\|x\| - \|z\|)^2 d\lambda(x)\, d\lambda(z) \le (\pi/8) \int_{S \times S} \int_0^{2\pi} (\frac{\partial}{\partial t} \|a_{x,z}(t)\|)^2 dt\, d\lambda(x)\, d\lambda(z)$$

$$= (\pi/8) \int_0^{2\pi} du \int_{S \times S} (\frac{\partial}{\partial t} \|a_{x,z}(t)\| |_{t=u})^2 d\lambda(x)\, d\lambda(z)$$

$$= (\pi/2)^2 \int_{S \times S} (\frac{\partial}{\partial t} \|a_{x,z}(t)\| |_{t=0})^2 d\lambda(x)\, d\lambda(z)$$

$$= (\pi/2)^2 \int_{S \times S} [(D\| \ \|)(x) (\frac{\partial a_{x,z}(t)}{\partial t} |_{t=0})]^2 d\lambda(x)\, d\lambda(z$$

$$= (\pi/2)^2 \int_S d\lambda_S(x) \int_{S \cap x^\perp} (D\| \ \|)(x)(y)^2 d\lambda_{S \cap x^\perp}(y)$$

$$= (\pi/2)^2 \int_{\Sigma_E} \|x\|^2 T_x(y)^2 d\sigma_E(x)$$

$$\le (\pi/2)^2 \sup_{S_E} \|x\|^2\, v(E).$$

3) Let us write $P(\varphi(x))$ instead of $\lambda_E(\{x \in S_E : \varphi(x)\})$. Let $a = \sup_{x \in E} \|x\|$ and let $t \in (0,1)$ be fixed.

If $P(\|x\| \ge a(1 - \frac{1}{2}b)) \ge \frac{1}{2}$, we pick an x_o such that $\|x_o\| = a(1-b) = \inf_{S_E} \|x\|$. (Otherwise we would take x_o with $\|x_o\| = a$, and proceed analogously).

Observe that there is an $s > 0$, depending only on K, such that $P(\|x - x_o\| \ge s b^{K-1})$ for $b \le 1$. Thus we have

$$a^2 d(E) \ge (\tfrac{1}{2} abt)^2\, P(\|x\| - \|x_o\| \le \tfrac{1}{2} ab(1-t)) P(\|y\| \ge a(1 - \tfrac{1}{2} b))$$

$$\ge (\tfrac{1}{2} abt)^2 2P(\|x - x_o\| \le \tfrac{1}{2} b(1-t)) \cdot \tfrac{1}{2}$$

$$\ge \tfrac{1}{4} a^2 b^{K+1} s t^2 (1-t)^{K-1},$$

which implies the desired inequality.

To get 4) observe first that

$$v(F) = \int_{S_F} d\lambda_F(x) \int_{S_{F\cap x^\perp}} (T_x(y))^2 \, d\lambda_{F\cap x^\perp}(y)$$

$$\leq \int_{S_F} d\lambda_F(x) \frac{?}{n-1} (\|T_x\|_2^* \leq \frac{1}{n-1} \int_{S_F} \|x\|^{-2} d\lambda_F(x)$$

(recall that $\|T_x\|_Z^* \leq \|T_x\|_{F^*} = \|x\|^{-1}$).

$$= \frac{1}{n-1} \int_{S_{\mathbb{R}^n}} \|Ix\|^{-2} d\lambda_{\mathbb{R}^n}(x) \leq \frac{4}{n-1} \int_{S_{\mathbb{R}^n}} \|x\|_{\ell_\infty^n}^{-2} d\lambda_{\mathbb{R}^n}(x) .$$

The following short reasoning was shown to the author by D. Burkholder. Let $X_1, X_2, \ldots$ be independent normalized Gaussian variables on a probability space (Ω, Σ, P). Then one has

$$\frac{1}{n} \int_{S_{\mathbb{R}^n}} (\max_{1\leq i\leq n} |x_i|)^{-2} d\lambda_{\mathbb{R}^n}(x) = \frac{1}{n} \int_\Omega (\max_{1\leq i\leq n} \frac{|X_i(\omega)|}{(\sum_{i=1}^n X_i(\omega)^2)^{1/2}})^{-2} dP(\omega)$$

$$= \frac{1}{n} \int_\Omega \frac{\sum_{i=1}^n X_i(\omega)^2}{\max_{1\leq i\leq n} X_i(\omega)^2} dP(\omega) = \int_\Omega \frac{X_1(\omega)^2}{\max_{1\leq i\leq n} X_i(\omega)^2} dP(\omega) \overset{\text{def}}{=} b_n .$$

The fact that the b_n's tend to zero is a well-known consequence of the unboundedness of the X_i's and the Lebesgue dominated convergence theorem. This is sufficient to establish 4) and complete the proof.

It is easy to prove that in fact $b_n = O((\operatorname{Log} n)^{-1})$ which yields an estimate $N(K,\varepsilon) \leq \exp(C_2 \varepsilon^{-K-1})$ for small $\varepsilon > 0$. This bound can be considerably improved by using the p-th powers instead of squares in the definition of v(E) (p being a large number depending on n) and more careful estimates of the appearing integrals. The result seems to be slightly stronger than those found in [2], [5] and [6] .

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