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THE RADON-NYKODYM PROPERTY

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Exposé N° 0

A Banach space, X, is said to have the Radon-Nikodym property (RNP) if, for every measure $\mu:(S, \Sigma) \to X$ having finite total variation on the σ - algebra, Σ, and being absolutely continuous with respect to a scalar measure λ, there is a Bochner integrable $f:S \to X$ such that for every $E \in \Sigma$, $\mu(E) = \int_E f \, d\lambda$. J. von Neumann [13] (see also [3]) showed that Hilbert spaces have (RNP). Clarkson [5] showed that uniformly convex spaces and ℓ_1 have (RNP), but that c_0 and $L_1([0, 1])$ fail the property. Dunford and Morse [9] showed that spaces having boundedly complete bases have the property (see §1, below). Following these lines, and the work of Dunford, Pettis and Phillips, by 1940 the following result was known: If X is reflexive, or a separable dual space, then X has the Radon-Nikodym property. Section 1 here is devoted to the current status of these characterizations.

In 1967, Rieffel [15] gave a geometric condition on a space X which is sufficient for X to have the RNP. If A is a subset of a Banach space X, then A is <u>dentable</u> if for every $\epsilon > 0$, there exists $x \in A$ such that $x \notin \overline{co}\,(A \setminus S_\epsilon(x))$ [here $co(B)$ is the convex hull of B, $\overline{co}(B)$ its closure and $S_\epsilon(x)$ is the ball of radius ϵ about x]. The space X is dentable if every bounded subset of X is dentable. Rieffel showed that dentable spaces have the RNP. In fact, spaces with the RNP are dentable, and even more, as we shall see in the second section of this exposition.

For the most part, I shall present only sketches of proofs. I wish to thank all of my friends who are allowing me to mention their results which have not yet been published, in particular, Professors James, Stegall, Lindenstrauss, Phelps, Huff, Pelczynski, Figiel and Johnson. I also owe special thanks to J. Diestel for his historical exposition of the RNP [8].

§ 1. Spaces which embed into separable conjugates.

In this section we are interested in pursuing the extensions of the theorems of von Neumann, Birkhoff and Dunford-Morse mentioned above, with the hope of finding a characterization of spaces with the RNP in terms of certain embeddings. Toward this end, we mention the following result of Uhl's [17] which says that the RNP is a separably determined property. Theorem: A B - space X has the RNP if and only if each separable subspace of X has the RNP.

A geometric proof of this result was recently given by Maynard [12], and will be sketched in the next section. The main extension of the results above is also due to Uhl [17]. Theorem: A space X has the RNP if every separable subspace of X embeds in a separable dual space.

It is now possible to prove this result from the (easy) Dunford-Morse argument. Recall that a biorthogonal system (y_i, g_i) is said to be a boundedly complete basis for Y if it is a basis and if the boundedness of a sequence $(\sum_{i=1}^{n} a_i y_i)_{n=1}^{\infty}$ implies the convergence of the series $\sum a_i x_i$. It is well known that a space with boundedly complete basis is isomorphic to a dual space.

<u>Proof of theorem</u>: First we differentiate suitable $\mu:(S, \Sigma) \to Y$ with boundedly complete basis. This is the Dunford-Morse proof: Notice that for $n = 1, 2, \ldots$, the scalar measures $\mu_n(E) = g_n(\mu(E))$ are finite and absolutely continuous with respect to λ. Hence, for each n there is a scalar function $f_n : S \to \mathbb{R}$ such that for $E \in \Sigma$, $\mu_n(E) = \int_E f_n \, d\lambda$. Now define a sequence of functions $h_n : S \to Y$ by $h_n(\cdot) = \sum_{k=1}^{n} f_k(\cdot) y_k$.

Since (y_n) is a basis, there is a constant $K \geq 1$ such that, for $E \in \Sigma$, $\|\int_E h_n \, d\lambda\| = \|\sum_{k=1}^{n} \mu_n(E)\, y_k\| \leq K \|\mu(E)\|$. Using the fact that (y_n) is boundedly complete, and the dominated convergence theorem, one sees that $h(\cdot) = \lim h_n(\cdot)$ (λ - a.e.) is the desired derivative of μ.

To complete the proof, we need the following result [7]: If W embeds into a separable dual space, then W embeds into a space with boundedly complete basis. This result if not difficult, but a proof would require too much space for this exposition. This completes the proof.

There is some evidence that the above condition is both necessary and sufficient. First we observe the following: If Z is a separable subspace of X^*, then there is a separable subspace Y of X such that Z is isometric to a subspace of Y^*. Simply choose a sequence (y_n) in the ball of X such that for $z \in Z$, $\|z\| = \sup z(y_n)$ and let $Y = \overline{\text{span}}(y_n)$. Recall that X is said to be <u>weakly compactly generated</u> (WCG) [1] if there is a weakly compact set $K \subset X$ such that $X = \overline{\text{span}}\, K$.

<u>Lemma</u>: If X^* is WCG, then every separable subspace of X^* embeds in a separable dual.

<u>Proof</u>: We show that if $Y \subset X$ is separable, then Y^* is separable.

First notice that Y^* is a quotient of X^*, and hence is WCG. Let K_1 be weakly compact $\subset Y^*$ such that $Y^* = \overline{\text{span}}\, K_1$. Since K_1 is weakly compact, the topologies $\sigma(Y^*, Y)$ and $\sigma(Y^*, Y^{**})$ agree on K_1, and, by separability of Y, both are separable. Hence $\overline{\text{span}}\, K_1$ is $\sigma(Y^*, Y^{**})$ (therefore $\|\cdot\|$) separable.

This lemma shows that weakly compactly generated conjugate spaces have the RNP. Using a similar argument with an appeal to the Bishop-Phelps theorem [4], one can show that if X has a Fréchet differentiable norm, then every separable subspace of X^* embeds into a separable dual.

The complete answer to the question of what dual spaces have the RNP has been obtained recently by Stegall [16]: <u>X^* has the RNP if and only if each separable subspace of X^* embeds into a separable dual.</u> The device used to prove this is Stegall's

<u>Theorem</u>: If X is separable and X^* is non-separable, then for each $\epsilon > 0$, there is a weak homeomorph, Δ, of the Cantor set in the sphere of X^* and a sequence $(x_{n,i}) \subset X$ with $\|x_{n,i}\| < 1 + \epsilon$ such that if $T : X \to C(\Delta)$ is the canonical evaluation operator, then

$$\sum_{n=0}^{\infty} \sum_{i=0}^{2^n - 1} \|Tx_{n,i} - 1_{A_{n,i}}\| < \epsilon \text{, where } (A_{n,i})_{n=0,\ i=0}^{\infty\ \ 2^n - 1}$$

is the canonical generating system for the Borel sets in Δ.

It is relatively easy to see that such a Δ cannot exist in a space with the RNP.

The major problem left open, then, is: If X has the RNP, does each separable subspace of X embed in a separable dual? In view of Stegall's results, this can be restated as: <u>If X is separable and has the RNP, does X embed into a dual space which has the RNP?</u>

§ 2. Geometric characterizations of spaces with the RNP.

In [15], Rieffel showed that X has the RNP if X is a dentable space. Maynard [12] showed that the result becomes necessary and sufficient if "dentable" is replaced by "s-dentable." A set A is said to be s-dentable if for each $\epsilon > 0$ there is $x \in A$ such that $x \notin \sigma(A \setminus S_\epsilon(x))$. Here $\sigma(B) = \{ \sum_{i=1}^{\infty} \lambda_i b_i \mid \lambda_i \geq 0 , \sum \lambda_i = 1 , b_i \in B \}$ so that in general, $co(B) \subset \sigma(B) \subset \overline{co}(B)$. A space X is s-dentable if each bounded subset of X is s-dentable.

Maynard observed that a set is s-dentable if and only if each of its countable subsets is s-dentable. Thus, since he also showed that X has the RNP if and only if X is s-dentable, we see that Uhl's theorem in the previous section follows.

In [6] it is shown that a space is dentable if and only if it is s-dentable. To prove this, we need the following lemma of Rieffel's [15] whose proof is straightforward.

Lemma: If $\overline{co}$ A is dentable, then A is dentable.

Using Maynard's and Rieffel's theorem, we can now prove

Theorem [6]: X has the RNP if and only if X is dentable.

Proof: The implication "dentability implies RNP" is Rieffel's theorem. For the other direction, suppose that X is not a dentable space, and that A is a bounded, non-dentable subset of X. Let $x \in X$ such that $x + A$ and $-x - A$ are separated. Then, if $C = \overline{co}(x + A, -x - A)$, C is closed, convex, symmetric, and if C is dentable, the same must be true of the set $\{x + A\} \cup \{-x-A\}$, by Rieffel's lemma. It is easy to see that this forces $x + A$ or $-x-A$ to be dentable, which is absurd.

Hence, C is non-dentable. Now let B be the unit ball of X and $U = \overline{B + C}$. Let $\epsilon > 0$ such that for $x \in C$, $x \in \overline{co}(C \setminus S_\epsilon(x))$, and let $u = b + c \in B + C$. Then, $c \in \overline{co}(C \setminus S_\epsilon(c))$, so $u \in \overline{co}\ ((b + C) \setminus S_\epsilon(b + c)) \subset \overline{co}\ ((B + C) \setminus S_\epsilon(u))$, so that $B + C$ is non-dentable. Again using Rieffel's lemma, U is non-dentable. U is a convex body in X, so its gauge ρ is a norm on X equivalent to the original. Thus, we may assume that the unit ball B of X is non-dentable. Let $\epsilon > 0$ such that $\|x\| \leq 1$ implies that $x \in \overline{co}(B \setminus S_\epsilon(x))$. Let $\|x\| < 1 - \frac{\epsilon}{4}$. Then there is $\lambda > 0$ such that $\|\lambda x\| < 1$, $\|x - \lambda x\| > \frac{\epsilon}{4}$ and $\|x + \lambda x\| > \frac{\epsilon}{4}$. Thus, $x \in co(B \setminus \overline{S_{\epsilon/4}(x)})$. If $1 > \|x\| > 1 - \frac{\epsilon}{4}$, then $S_{\epsilon/4}(x) \subset S_\epsilon(\frac{x}{\|x\|})$, so that $\frac{x}{\|x\|} \in \overline{co}(B \setminus S_{\epsilon/4}(x))$. For small ϵ, 0 is an interior point of $\overline{co}(B \setminus S_{\epsilon/4}(x))$, so the entire segment $[0, \frac{x}{\|x\|})$ is in the interior of that set. In particular, $x \in co(B^\circ \setminus S_{\epsilon/4}(x))$, where B° denotes the interior of the unit ball. Thus, the interior of the ball is non-s-dentable, so the space X is non-s-dentable. The other direction is trivial, and we have shown that X is dentable if and only if X is s-dentable. Using Maynard's theorem, the proof is complete.

It must be noted that the previous theorem has recently been proved by R. Huff [10] directly using an improvement of Maynard's argument. I shall not sketch that proof here in order to have space for the next remarkable result of R. R. Phelps.

A Banach space X is said to have the Krein-Milman property if every non-empty, closed, bounded, convex subset $A \subset X$ is the closed convex hull of its extreme points. Lindenstrauss [11] showed that ℓ_1 has the Krein-Milman property, and has recently noted that his argument together with the embedability of separable duals into spaces with boundedly complete basis (above) can be used to prove the beautiful theorem of Bessaga and Pelczynski [2]: If X embeds in a separable dual, then X has the Krein-Milman property. This has led several people to ask what the relation between the Krein-Milman and Radon-Nikodym properties is (e.g. [8]). One difficulty here is the fact that it is apparently unknown whether or not the Krein-Milman property is separably determined. Recently, Lindenstrauss has shown that the RNP implies the Krein-Milman property. A proof of this will appear in [14]. Now we shall outline the proof of this stronger result of Phelps [14].

Theorem: A space X has the RNP if and only if every nonempty, closed, bounded, convex subset of X is the closed convex hull of its strongly exposed points.

Before we prove this, we need some definitions and a lemma. For a convex set A, say that x is a denting point of A if for every $\epsilon > 0$, $x \notin \overline{\mathrm{co}}(A \setminus S_\epsilon(x))$. The point is strongly exposed if there is a functional f and a number α such that $\{u \mid f(u) = \alpha\} \cap A = \{x\}$ and if $(y_n) \subset A$ has $f(y_n) \to \alpha$ implies that $\|y_n - x\| \to 0$. We shall call a set of the form $\{f(u) \geq \beta\} \cap A$ a slice of A if there is $z \in A$ with $f(z) > \beta$. The next lemma contains the characterizations of denting points and strongly exposed points used in the proof of the theorem. Part (d) is due to E. Bishop who communicated the result in a more general form to R. Phelps in 1967.

Lemma: Let A be a closed, bounded, convex and nonempty in the Banach space X. Then

a) A is dentable if and only if for every $\epsilon > 0$ there is a slice S of A having diameter less than ϵ.

A point $x \in A$ is

b) a denting point of A if for all $\epsilon > 0$ there is a slice S of A, diam $S < \epsilon$, with $x \in S^+ = \{u | f(u) > \alpha\}$,

c) a strongly exposed point if there is a functional g and a sequence β_n of numbers such that $\text{diam}(\{g(u) \geq \beta_n\} \cap A) \to 0$ and such that $x \in \{g(u) > \beta_n\}$ for each n.

d) The set A has a strongly exposed point if there is a sequence of slices S_n of A with diam $S_n \to 0$, $S_{n+1} \subset S_n$ and such that the determining functionals g_n (for S_n) are a norm-Cauchy sequence.

Proof: We shall prove only (a). The proofs of (b) and (c) are also easy, but the proof of (d) is more delicate, and will appear in [14]. Suppose A is dentable. Let $\epsilon > 0$ and $x \in A$ such that $x \notin \overline{\text{co}}(A \backslash S_\epsilon(x))$. Then there is a functional f and α such that $f(x) > \alpha > \sup\{f(u) | u \in \overline{\text{co}}(A \backslash S_\epsilon(x))\}$. The slice $\{f(u) \geq \alpha\} \cap A$ is contained in $S_\epsilon(x)$, and therefore has diameter less than 2ϵ. The other direction is also immediate.

Proof of theorem: We shall prove first that each closed, bounded, convex, nonempty set A in a dentable space has a denting point. The rest of the proof follows by careful use of a lemma of Bishop and Phelps [4] together with parts (c) and (d) of the above lemma. We use parts (a) and (b) of the lemma. According to (b), given a slice S_1 of A and

$\varepsilon > 0$ we need to find a slice S_2 of A, $S_2 \subset S_1$, with diam $S_2 < \varepsilon$. Suppose that $S_1 = \{f(u) \geq 0\} \cap A$ and let $z \in S_1$ with $f(z) > 0$. Let $D = \{f(u) = 0\} \cap A$. If $D = \emptyset$, there is nothing to prove due to part (a) of the lemma, so assume $D \neq \emptyset$. For each $x \in D$, define an involution of the space X through $\{f(u) = 0\}$ by $T_x(y) = y - 2\frac{f(y)}{f(z)}(z - x)$. Then, it is easy to see that $\{T_x\}_{x \in D}$ is a norm bounded set (say by M). Consider the set $K = \overline{co}\{S_1 \cup \cup\{T_x S_1 | x \in D\}\}$. It is bounded, closed, convex and nonempty, hence by (a) of the lemma, has a slice Σ of diameter less than δ, where $\delta < \min\{\varepsilon, \frac{\varepsilon}{M}, f(z)\}$. Suppose that $\Sigma = \{g(u) \geq \beta\} \cap K$. If $\Sigma \cap D \neq \emptyset$, then for some $x \in D$, either the segment $[z,x]$ or $[x, T_x z]$ is in Σ, but both segments have length greater than δ which is impossible. Next, for some $w \in S_1$ or $w \in T_x S_1$ for some $x \in D$, we have $\sup_{u \in K} g(u) > g(w) > \beta$. In the first case, let $S_2 = \Sigma \cap S_1$, and in the second, let $S_2 = (T_x(\Sigma)) \cap S_1$. It is easy to verify the desired properties, completing the proof of the existence of denting points.

In order to find strongly exposed points, we show that each slice, $S = \{f(x) \geq 0\} \cap A$ has a slice $S_1 = \{g(x) \geq \beta\} \cap A$ with diam $S_1 < \varepsilon$ and $\|f - g\| < \varepsilon$. To see this, let $K = co(S, \lambda B \cap \{f(x) = 0\})$, where λ is large and B denotes the ball of the space. By the first part of the proof, there is a slice S_2 of K of diameter $< \delta$ which misses $\lambda B \cap \{f(x) = 0\}$. With $S_2 = \{g(x) \geq \beta\} \cap K$, let $S_1 = S_2 \cap S$. Normalizing g and f, the Bishop-Phelps lemma [4] shows that for suitable choices of δ and λ, diam $S_1 < \varepsilon$ and $\|f - g\| < \varepsilon$. The existence of strongly exposed points follows from (c) and (d) of the lemma above.

In examining the relationship, then, between the Krein-Milman and Radon-Nikodym properties, the following problems remain open: 1. If X has the KMP, does X have the RNP? 2. If each separable subspace of X has the KMP, does X ? 3. If X has the KMP, does every closed bounded convex set have a strongly exposed (even denting) point?

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