Statistique et analyse des données

Marie-Françoise Delcroix
Pierre Jacob

Stability of extreme value for a multidimensional sample

Statistique et analyse des données, tome 16, nº 1 (1991), p. 1-21
<http://www.numdam.org/item?id=SAD_1991__16_1_1_0>

Statistique et Analyse des Données
1991 - Vol. 16 n° 1 p. 1 - 21

STABILITY OF EXTREME VALUE FOR A MULTIDIMENSIONAL SAMPLE

Marie-Françoise DELCROIX and Pierre JACOB
Laboratoire de Statistique et Probabilités (M_2)
Université des Sciences et Techniques de Lille Flandres-Artois

59655 VILLENEUVE D'ASCQ CEDEX (France)

Abstract.

Let $(Y_1,...,Y_n)$ *be a random sample from a continuous distribution function* F *over* $\mathbb{R}^+$. *If* Y_n^* *denotes the highest value of this sample, then the highest value of* $F(Y_1),...,F(Y_n)$ *is* $F(Y_n^*)$. *This simple remark suggests a natural definition for the highest value* $X_n^* = (R_n^*,\Theta_n^*)$ *of a random sample* $(X_1,...,X_n)$ *in* $\mathbb{R}^k$, *based upon the polar representation* $(R_1,\Theta_1),...,(R_n,\Theta_n)$ *of these variables. Precisely, if* F_θ *is the conditional distribution function of* R *given* $\Theta = \theta$, *we define the maximum value of the sample as the observation* X^* *which maximizes* $F_\Theta(R)$. *This definition is attractive because it is not based only on a classical distance in* $\mathbb{R}^k$, *but, which seems more relevant, on the probability to be at a certain distance from the origin. This notion allows us to study the stability of such extreme values. Of course, a lot of multidimensional distributions do not have stability properties. So we need a weaker notion than stability to go on. The idea is to substitute a variable* $X\varphi = (\varphi(R),\Theta)$ *for each observation* $X = (R,\Theta)$, *where* φ *is a suitable function, in order to obtain stability properties for the variable* $X\varphi$. *It consists in considering a new set of points* $E_n^\varphi = \{(\varphi(R_1),\Theta_1),...,(\varphi(R_n),\Theta_n)\}$ *instead of the initial sample. As shown in this paper, the function* φ *must be sufficiently concave.*

Key-words: sample, isobar , extreme value, stability, relative stability, asymptotic localization.

A.M.S. Subject classification : 62 G 30, 60 D 05

Manuscrit reçu le 12 juin 1991, révisé le 10 février 1992.

I. INTRODUCTION

Nowadays, theory of extreme values concerns often non identically distributed data, dependent data ([Haiman, Puri, 1990],[Haiman, Puri]) or multivariate independent identically distributed data ([Davis, Mulrow, Resnick, 1987]). However recent papers about outliers ([Gather, Rauhut, 1990],[Green, 1976],[Mathar, 1989],[Munoz-Garcia, Moreno-Rebollo, Pascual-Acosta, 1990]), give a new interest to the old notion of stability ([Geffroy, 1958,1959],[Geffroy, 1961],[Gnedenko, 1943]). We propose here a new definition for the highest value of a multidimensional sample and for the stability of this highest value. It is also possible to define outlier-resistant or outlier-prone distributions as it has been done for $\mathbb{R}^k$-valued variables in [Gather, Rauhut, 1990], [Green, 1976]. However, in the first step of this study, we examine the properties of such extreme values : in the present paper we focus our attention on the stability of the extreme value of a sample.

In this paper we consider random variables defined on a probability space $(\Omega,\mathcal{C},P)$ and with values in the Euclidian space $\mathbb{R}^k$.

For every x in $\mathbb{R}^k\backslash\{0\}$ we define a pair $(\|x\|,\frac{x}{\|x\|}) = (r,\theta)$ in $\mathbb{R}^{+*} \times S^{k-1}$, where $\|.\|$ is the Euclidean norm. The unit sphere S^{k-1} in $\mathbb{R}^k$ is endowed with the induced topology of $\mathbb{R}^k$.

For each random variable $X = (R,\Theta)$, we assume that the distribution of Θ, and for all θ, the distribution of R given $\Theta = \theta$, have a continuous density. We name F_θ the continuous conditional distribution function of R, given $\Theta = \theta$, and F_θ^{-1} its generalized inverse. For each $0 < u < 1$, we name u-level isobar - from the distribution of R given $\Theta = \theta$ - the mapping $\theta \rightsquigarrow F_\theta^{-1}(u)$. We suppose that this mapping is continuous and strictly positive ; the surface which equation is $\rho = F_\theta^{-1}(u)$ is also named an isobar.

Let $E_n = (X_1,\ldots,X_n)$ be a sample of independent random variables with the same distribution as X. For each $1 \le k \le n$ there is almost surely an unique u_k-level isobar from the distribution of R given $\Theta = \theta$ which contains (R_k,Θ_k). We define the maximum value in E_n as the point $X_n^* = (R_n^*,\Theta_n^*)$ which belongs to the upper level isobar, i.e. the isobar which level is $\max_{1\le k\le n} u_k$. Obviously, we are not able to find this maximum value of a sample from an unknown distribution, whereas it can be done with the farest point from the origin or with the fictitious point having the largest coordinates of the sample. However this kind of extreme value and, more generally, the extreme values obtained by ordering the sample according to the levels, hold more information on the conditional distributions tails and allow a statistic survey of the isobars.

In section II, we specify some properties of the distribution of the pair (R_n^*, Θ_n^*). In section III and IV, we define the notions of stability in probability and almost sure stability of the maximum value. Roughly the idea lying back of the definition is the tendency of X_n^* to be near a given surface. More precisely, X_n^* is called stable in probability (or almost surely) if there is a sequence (Γ_n) of surfaces, which equations are $\rho = g_n(\theta)$, such that $R_n^* - g_n(\Theta_n^*) \to 0$ in probability (or almost surely). For a class of distributions we precise, this phenomenon occurs and (g_n) turns out to be a sequence of isobars. This expands the notion of stability studied by J. Geffroy in [Geffroy, 1958,1959]. Actually we use several of his methods. Examples are given in section V. In section VI, the assumptions we have done throughout this paper are discussed, especially some regularity conditions for the isobars. At length, in section VII we give some properties and examples about φ-stability.

II - PRELIMINARIES

In this section we give some results about the conditional distribution of R_n^* given $\Theta_n^* = \theta$. They will be used in the sequel of this paper.

Let $(X_1,\ldots,X_n)$ be a sample with polar representation $(R_1,\Theta_1),\ldots,(R_n,\Theta_n)$. For each $1 \le i \le n$, put :

(1) $$E_i = \{F_{\Theta_i}(R_i) = \max_{j=1}^{n} F_{\Theta_j}(R_j)\}.$$

For all θ and for all $0 \le t \le 1$, $P(F_\Theta(R) \le t/\Theta = \theta) = F_\theta(F_\theta^{-1}(t)) = t$, hence $\{F_{\Theta_j}(R_j),\ j = 1,\ldots,n\}$ is a sample from the uniform distribution over [0,1]. Now the maximum value of the sample is almost surely defined as the point X_n^* which polar representation is :

$$(R_n^*, \Theta_n^*) = \sum_{i=1}^{n} (R_i,\Theta_i)1_{E_i} .$$

PROPERTY 1.- a) Θ_n^* *and* Θ *are identically distributed.*

b) *Any* u-*level isobar from the distribution of* R *given* Θ *is also the* u^n-*level isobar from the distribution of* R_n^* *given* Θ_n^* .

∎ a) Since $P(F_\Theta(R) \le t / \Theta = \theta) = t$, for each $1 \le j \le n$ Θ_j and $F_{\Theta_j}(R_j)$ are independent. It follows easily that $\{\Theta_j ; j = 1,\dots,n\}$ and $\{F_{\Theta_j}(R_j) ; j = 1,\dots,n\}$ are independent.

Thus, for each $1 \le j \le n$, Θ_j and $\mathbf{1}_{E_j}$ are independent. Consequently, for any Borel set C of S^{k-1} :

$$(2) \qquad P(\Theta_n^* \in C) = P(\sum_{i=1}^n \Theta_i \mathbf{1}_{E_i} \in C) = \sum_{i=1}^n P(\Theta_i \in C ; E_i)$$
$$= \sum_{i=1}^n P(\Theta_i \in C)\, P(E_i) = P(\Theta \in C)$$

b) Let $\rho = F_\theta^{-1}(u)$ be an u-level isobar from the distribution of R given $\Theta = \theta$ and let B be the event $\{R_n^* \le F^{-1}_{\Theta_n^*}(u)\}$. Since $B = \cap_{i=1}^n \{F_{\Theta_i}(R_i) \le u\}$, B is independent of $\{\Theta_j,\ j = 1,\dots,n\}$. Thus for any Borel set C of S^{k-1} (2) implies :

$$P(\Theta_n^* \in C ; B) = \sum_{i=1}^n P(\Theta_i \in C ; E_i ; B) = \sum_{i=1}^n P(\Theta_i \in C)\, P(E_i ; B)$$
$$= P(\Theta_n^* \in C)\, P(B)$$

Thus Θ_n^* and $\mathbf{1}_B$ are independent ; therefore,

$$(3) \qquad P(R_n^* \le F^{-1}_{\Theta_n^*}(u) / \Theta_n^* = \theta) = P(B) = \prod_{i=1}^n P(F_{\Theta_i}(R_i) \le u) = u^n .$$

∎

COROLLARY 1.- *Let* $F^*_{n,\theta}$ *be the conditional distribution fonction of* R_n^* *given* $\Theta_n^* = \theta$. *For any* θ, $F^*_{n,\theta} = F_\theta^n$.

∎ Let S be the support of the distribution of X. Let $x = (r,\theta)$ be a point which distance from the nearest isobar is strictly positive. Taking account of the isobar's continuity, there exists an open ball $B(x,\varepsilon)$ which distance from the nearest isobar is also strictly positive. Therefore the distribution of X assigns a null mass to $B(x,\varepsilon)$. But the support S is the set of all points z in $\mathbb{R}^k$ such that $P(V) > 0$ for each open set V containing z ; hence x is not a point of S. Thus the distance between any point of S and an isobar is zero. In the sequel of the proof, we shall consider as an isobar any (uniform) limit of a decreasing sequence of isobars. S is then the union of all the isobars from the distribution of R given $\Theta = \theta$. For any open set $\mathbf{0}$ in S^{k-1} and for any pair (g,h) of isobars such that $g < h$, define

$$D(\mathbf{0},g,h) = \{x = (r,\theta) \in S : \theta \in \mathbf{0},\ g(\theta) \le r \le h(\theta)\}.$$

Clearly the class $\mathcal{U}$ of these sets is a π-system [Billingsley, 1968].
Moreover for all x in S and for all $\varepsilon > 0$ there is a set D in $\mathcal{U}$, with diameter less than ε, such that $x \in \mathring{D} \subset D$ ($\mathring{D}$ denotes the interior of D for the induced topology on S). By [Billingsley, 1968] page 14, $\mathcal{U}$ is a determining class for the separable metric space S.

Let $\tilde{X} = (\tilde{R},\tilde{\Theta})$ be a $\mathbb{R}^*$-valued random variable such that Θ and $\tilde{\Theta}$ are identically distributed and such that the distribution function of $\tilde{R}$ given $\tilde{\Theta} = \theta$ is F^n_θ. It suffices now to show that $\tilde{X}$ and X^* are identically distributed in order to obtain the corollary 1.
This follows immediatly from (3), from the following equation :

$$(4) \qquad P(\tilde{R} < g(\tilde{\Theta})/\tilde{\Theta} = \theta) = F^n_\theta(g(\theta)) = u^n \ ,$$

and from the fact $\mathcal{U}$ is a determining class.

The previous results state that both the distribution of R given Θ and the distribution of R^*_n given Θ^*_n have the same set of isobars. Hence we will deal only with the formers. In the sequel any u-level isobar from the distribution of R given Θ is labelled as u-level isobar.

We assume in the remainder of this paper that for all θ the mapping F_θ is strictly increasing and thus bijective. Fix a point θ_1 in S^{k-1} and provide the axis $(0\theta_1)$ with the unit vector $\overrightarrow{0\theta_1}$. For every point w on the positive half axis $0\theta_1^+$, there is an unique isobar containing w, which level is denoted by u(w). Let $\rho = g(\theta,w)$ be the equation of this u(w)-level isobar (note that $g(\theta_1,w) = w$). Moreover the mapping $w \leadsto u(w)$ from R^{+*} into]0,1[is increasing and bijective. The following condition (H) will be used in most theorems.

(H) there exists $0 < \alpha_1 \leq \beta_1 < +\infty$ such that for all θ in S^{k-1} and for all $w > 0$: $\alpha_1 \leq \frac{\partial g}{\partial w}(\theta,w) \leq \beta_1$

An immediate consequence is given by the next property :

PROPERTY 2.- *For all* $\varepsilon > 0$, *there exists* $\eta > 0$, *and for all* $w > 0$, *there exists two isobars* $h^{\varepsilon}(\theta,w)$ and $\tilde{h}^{\varepsilon}(\theta,w)$ *such that for every* θ :

$$g(\theta,w) - \varepsilon < \tilde{h}^{\varepsilon}(\theta,w) < g(\theta,w) - \eta < g(\theta,w) + \eta < h^{\varepsilon}(\theta,w) < g(\theta,w) + \varepsilon \tag{5}$$

(*we lay stress on the fact that* η *does not depend upon* w).

∎ By the mean value theorem, we obtain :

$$g(\theta,w) + \ell\alpha_1 \leq g(\theta,w+\ell) \leq g(\theta,w) + \ell\beta_1 \qquad \text{if } \ell > 0$$

and

$$g(\theta,w) + \ell\beta_1 \leq g(\theta,w+\ell) \leq g(\theta,w) + \ell\alpha_1 \qquad \text{if } \ell < 0$$

It suffices to choose $\ell = \frac{\varepsilon}{\beta_1}$ (resp. $\ell = -\varepsilon/\beta_1$) and to put

$$\eta = \varepsilon\alpha_1 / \beta_1 \tag{6}$$

$$h^{\varepsilon}(\theta,w) = g(\theta,w + \varepsilon/\beta_1) \tag{7}$$

$$\tilde{h}^{\varepsilon}(\theta,w) = g(\theta,w - \varepsilon/\beta_1). \tag{8}$$

∎

REMARK 1.- The level $u(w + \varepsilon/\beta_1)$ (resp. $u(w - \varepsilon/\beta_1)$) of $h^{\varepsilon}(\theta,w)$ (resp. of $\tilde{h}^{\varepsilon}(\theta,w)$) is a increasing function of w or of u(w).

REMARK 2.- Actually, (5) is a key-property, but H is somewhat easier to handle. H was suggested to us by an unpublished work of Geffroy on a closely related topic. Most details can be found in the thesis of Lecoutre [Lecoutre, 1982]. Geffroy considered the case of an unimodal density in $\mathbb{R}^2$, decreasing in any direction. The sample was ordered according to the level-lines of the bivariate density of the isobars. Anyway, (H) is not necessary as shown in section VI.

REMARK 3.- For a gaussian sample of $\mathbb{R}^2$-vectors with covariance matrix $\begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix}$, the hypothesis (H) is satisfied. We point out that the isobars are also the level-lines of the bivariate density. Their polar equations are $g(\theta,w) = w\varphi(\theta)$ where $\varphi(\theta) = \frac{1}{\sqrt{2}\sigma}\left(\frac{\cos^2\theta}{2\sigma^2} + \frac{\sin^2\theta}{2\tau^2}\right)^{-1/2}$.

III - STABILITY IN PROBABILITY OF $X_n^* = (R_n^*,\Theta_n^*)$.

By property 1, the distributions of (R_n^*,Θ_n^*) and of (R,Θ) do have the same set of isobars. So we can propose the following definition.

1. DEFINITION 1.- $(X_n^*)_n$ *is stable in probability if and only if there is a sequence* $(g_n)_n$, *of isobars satisfying*

$$R_n^* - g_n(\Theta_n^*) \xrightarrow{P} 0. \tag{9}$$

2. PROPOSITION 1.- *Suppose that (H) holds. If* $R_n^* - g_n(\Theta_n^*) \xrightarrow{P} 0$ (*where* $(g_n)_n$ *is a sequence of isobars), then for all* $\varepsilon > 0$:

$$\min_\theta P(R_n^* < g_n(\theta) + \varepsilon / \Theta_n^* = \theta) \to 1 \text{ and} \tag{10}$$

$$\max_\theta P(R_n^* < g_n(\theta) - \varepsilon / \Theta_n^* = \theta) \to 0 \tag{11}$$

▮ Let $\varepsilon > 0$; by property 2, there is $\eta > 0$ and two sequences of isobars $(h_n^\varepsilon)_n$ and $(\tilde{h}_n^\varepsilon)_n$ such that for all θ and for all n,

$$g_n(\theta) - \varepsilon < \tilde{h}_n^\varepsilon(\theta) < g_n(\theta) - \eta < g_n(\theta) + \eta < h_n^\varepsilon(\theta) < g_n(\theta) + \varepsilon, \tag{12}$$

It follows that for each fixed θ :

$$(R_n^* < g_n(\theta) + \eta) \subset (R_n^* < h_n^\varepsilon(\theta)).$$

Since $R_n^* - g_n(\Theta_n^*) \xrightarrow{P} 0$,

(13) $$\lim_{n\to+\infty} P(R_n^* < h_n^\varepsilon(\Theta_n^*)) = 1.$$

Furthermore, if $v_{\varepsilon,n}$ is the level of h_n^ε :

(14) $$\begin{aligned} P(R_n^* < h_n^\varepsilon(\Theta_n^*)) &= \int_{S^{k-1}} P(R_n^* < h_n^\varepsilon(\theta)/\Theta_n^* = \theta)\ P_{\Theta_n^*}(d\theta) \\ &= \int_{S^{k-1}} v_{\varepsilon,n}\ P_{\Theta_n^*}(d\theta) = v_{\varepsilon,n} \end{aligned}$$

And for every θ, $v_{\varepsilon,n} = P(R_n^* < h_n^\varepsilon(\theta)/\Theta_n^* = \theta) < P(R_n^* < g_n(\theta) + \varepsilon/\Theta_n^* = \theta)$.
We deduce (10) from (13) and (14). The proof of (11) can be treated in the same way. ∎

This proposition provides a criterion of stability in probability.

3. THEOREM 1.- *Let* $(g_n)_n$ *be a sequence of isobars* ;
i) *if* (H) *holds and if* $R_n^* - g_n(\Theta_n^*) \xrightarrow{P} 0$, *then for all* $0 < a < b < 1$:

(15) $$\lim_{n\to\infty} \sup_{\theta\in S^{k-1}} [(F_{n,\theta}^*)^{-1}(b) - (F_{n,\theta}^*)^{-1}(a)] = 0$$

ii) *conversely, if* (15) *holds, then* $X_n^* = (R_n^*,\Theta_n^*)$ *is stable in probability.*

∎ Let $\varepsilon > 0$. In view of the proposition 1, for n large enough, $b < P(R_n^* < g_n(\theta) + \varepsilon/\Theta_n^* = \theta)$ for any θ. Then for every θ, $(F_{n,\theta}^*)^{-1}(b) < g_n(\theta) + \varepsilon$. Similarly, $(F_{n,\theta}^*)^{-1}(a) > g_n(\theta) - \varepsilon$ for any θ. Thus (15) follows. Conversely, for $\varepsilon > 0$, $(F_{n,\theta}^*)^{-1}(1-\varepsilon) - (F_{n,\theta}^*)^{-1}(\varepsilon)$ converges to 0 uniformly. We choose by a diagonal method a sequence (ε_n) which decreases to 0, such that :

$$(F_{n,\theta}^*)^{-1}(1-\varepsilon_n) - (F_{n,\theta}^*)^{-1}(\varepsilon_n)$$

converges to 0 uniformly
Putting $h_n(\theta) = (F_{n,\theta}^*)^{-1}(1-\varepsilon_n)$ and $g_n(\theta) = (F_{n,\theta}^*)^{-1}(\varepsilon_n)$, it follows that :

$$P(R_n^* < h_n(\theta) \mid \Theta_n^* = \theta) = 1 - \varepsilon_n \quad \text{and}$$

$$P(R_n^* < g_n(\theta) \mid \Theta_n^* = \theta) = \varepsilon_n \quad \text{for all } \theta ,$$

so that

$$P(g_n(\Theta_n^*) < R_n^* < h_n(\Theta_n^*)) > 1 - 2\varepsilon_n .$$

This completes the proof of the theorem 1. ∎

The corollary 1 gives for each $0 < y < 1$: $(F^*_{n,\theta})^{-1}(y) = F_\theta^{-1}(y^{1/n})$ ([Geffroy, 1958,1959], page 70) and this entails the next theorem. The proof is the same as for theorem 20 in [Geffroy, 1958, 1959].

4. THEOREM 2.-

i) *If* (H) *holds and if* $R_n^* - g_n(\Theta_n^*) \xrightarrow{P} 0$, *then for all* $0 < \alpha < \beta$:

(16) $$\lim_{n \to +\infty} \sup_{\theta \in S^{k-1}} \mid F_\theta^{-1}(1 - \alpha/n) - F_\theta^{-1}(1 - \beta/n) \mid = 0.$$

ii) *If* (16) *holds then* $(X_n^*)_n$ *is stable in probability.*

REMARK 4.-

a) $F_\theta^{-1}(1\text{-}\beta/n)$ and $F_\theta^{-1}(1\text{-}\alpha/n)$ are meaningful for $\beta/n < 1$ and for $\alpha/n < 1$.

b) For all θ and for all $0 < t < 1$, put $G_\theta^{-1}(t) = F_\theta^{-1}(1-t)$ (where $G_\theta = 1 - F_\theta$). We have just proved that if $(X_n^*)_n$ is stable in probability, the "highest" values of the sample are then stable "about" the isobars $\gamma_n(\theta) = G_\theta^{-1}(\frac{1}{n})$, for n large enough.

5. Gnedenko's theorem ([Geffroy, 1958,1959],[Gnedenko, 1943]) gives a simple criterion of stability in Probability in our context.

THEOREM 3.- *Suppose that* (H) *holds, then* $(X_n^*)_n$ *is stable in Probability if and only if for some* θ_1 :

(17) $$\lim_{x \to +\infty} \frac{G_{\theta_1}(x)}{G_{\theta_1}(x\text{-}h)} = 0 \text{ , for all } h > 0$$

(Then (17) is true for all θ*).*

Put W_n^* the intersection of $0\theta_1^+$ with the isobar containing X_n^* (see section II). For all $w > 0$,

$$P(W_n^* \leq w) = P(R_n^* \leq g(\Theta_n^*, w)),$$

where g is the isobar which equation is $g(\theta,w)$.
Since g is an isobar, $P(R_n^* \leq g(\theta,w) / \Theta_n^* = \theta)$ does not depend upon θ, then

$$\begin{aligned} P(W_n^* \leq w) &= P(R_n^* \leq g(\theta_1,w) \mid \Theta_n^* = \theta_1) \\ &= P(R_n^* \leq w \mid \Theta_n^* = \theta_1) \\ &= F_{\theta_1}^n(w). \end{aligned}$$

Let W_i be the intersection of $0\theta_1^+$ with the isobar containing $X_i = (R_i,\Theta_i)$, $(i=1,\ldots,n)$. For all $w > 0$,

$$P(W_i \leq w) = P(R_i \leq w / \Theta_i = \theta_1) = F_{\theta_1}(w) \ . \tag{18}$$

The random variable W_n^* is then the maximum of n $\mathbb{R}^+$-valued i.i.d. variables, $W_1,\ldots,W_n$, having a common distribution function F_{θ_1}. From Gnedenko's theorem, $(W_n^*)_n$ is stable in Probability if on only if :

$$\lim_{x \to +\infty} \frac{G_{\theta_1}(x)}{G_{\theta_1}(x-h)} = 0 \ , \text{ for all } h > 0$$

Now it suffices to prove that $(W_n^*)_n$ is stable in probability if and only if $(X_n^*)_n$ is stable in probability.

Let $\varepsilon > 0$, suppose that $(W_n^*)_n$ is stable in Probability ; there exists a sequence $(a_n)_n$ such that $W_n^* - a_n \xrightarrow{P} 0$. Let ℓ_n be the isobar containing a_n and $\tilde{h}_n^\varepsilon$, h_n^ε the isobars satisfying (5) for all θ :

$$\ell_n(\theta) - \varepsilon < \tilde{h}_n^\varepsilon(\theta) < \ell_n(\theta) - \eta < \ell_n(\theta) < \ell_n(\theta) + \eta < h_n^\varepsilon(\theta) < \ell_n(\theta) + \varepsilon \tag{19}$$

(η does not depend upon n).
Then,

(20) $\{|W_n^* - a_n| \leq \eta\} \subset \{\tilde{h}_n^\varepsilon(\Theta_n^*) \leq R_n^* \leq h_n^\varepsilon(\Theta_n^*)\} \subset \{|R_n^* - \ell_n(\Theta_n^*)| \leq \varepsilon\}$

so that
$$R_n^* - \ell_n(\Theta_n^*) \xrightarrow{P} 0$$

Conversely, if there exists a sequence of isobars $(g_n)_n$ such that $R_n^* - g_n(\Theta_n^*) \xrightarrow{P} 0$, note a_n the intersection of $0\theta_1^+$ with g_n. For $\varepsilon > 0$, there exists $\eta > 0$ and for all n, there exists $\tilde{h}_n^\varepsilon$ and h_n^ε satisfying (19) for all θ ; then,

(21) $\{|R_n^* - g_n(\Theta_n^*)| \leq \eta\} \subset \{\tilde{h}_n^\varepsilon(\Theta_n^*) \leq R_n^* \leq h_n^\varepsilon(\Theta_n^*)\} \subset \{|W_n^* - a_n| \leq \varepsilon\}$

and this achieves the proof.

IV - ALMOST SURE STABILITY OF $(X_n^*)_n$.

1.- DEFINITIONS.

a) DEFINITION 2.- *The sequence* $(X_n^*)_n$ *is almost surely stable if and only if there is a sequence* $(g_n)_n$ *of isobars such that* :

(22)
$$R_n^* - g_n(\Theta_n^*) \xrightarrow{a.s.} 0.$$

By theorem 2, if (X_n^*) is almost surely stable, $\gamma_n(\theta) = G_\theta^{-1}(\frac{1}{n})$ is a convenient g_n-sequence. Thus, in the remainder, $\gamma_n(\theta)$ will denote the isobar $G_\theta^{-1}(\frac{1}{n})$ and Γ_n the set of points $\{(\rho,\theta) ; \rho \leq \gamma_n(\theta)\}$, $(n \geq 2)$.

For $\varepsilon > 0$, put :

(23)
$$\Gamma_n^\varepsilon = \{(\rho,\theta) : \rho \leq \gamma_n(\theta) + \varepsilon\}$$

(24)
$$\Gamma_n^{-\varepsilon} = \{(\rho,\theta) : \rho \leq \gamma_n(\theta) - \varepsilon\}$$

Then, $(X_n^*)_n$ is almost surely stable if and only if for all $\varepsilon > 0$,

$$P\{\text{Lim inf}\,(X_n^* \in \Gamma_n^{\varepsilon} - \Gamma_n^{-\varepsilon})\} = 1 \tag{25}$$

2.- As it has been done with Gnedenko's theorem, it is possible to prove the next theorem, with theorems 49 and 50 of [Geffroy, 1958,1959],

THEOREM 4.- *If* (H) *holds, then* $(X_n^*)_n$ *is almost surely stable if there exists* θ_1 *such that*

$$\lim_{x \to +\infty} \frac{G_{\theta_1}(x-h)}{G_{\theta_1}(x)\,\text{Log}\,G_{\theta_1}(x)} = +\infty, \text{ for all } h > 0. \tag{26}$$

(If (26) *is true for* θ_1, *then* (26) *is true for all* θ*).*

∎ As for theorem 3, put W_n^* the intersection of $0\theta_1^+$ with the isobar containing X_n^*. If $(W_n^*)_n$ is almost surely stable, then there exists a sequence $(a_n)_n$ such that for all $\varepsilon > 0$:

$$P\{\text{Lim inf}(a_n - \varepsilon \le W_n^* \le a_n + \varepsilon)\} = 1.$$

Let ℓ_n be the isobar containing a_n. For all $\varepsilon > 0$, there exists $(h_n^{\varepsilon})_n$ and $(\bar{h}_n^{\varepsilon})_n$ two sequences of isobars and $\eta > 0$ such that for all θ (19) holds. The inclusions (20) and (21) show that under (H), $(X_n^*)_n$ is almost surely stable if and only if $(W_n^*)_n$ is almost surely stable. Now, [Geffroy, 1958, 1959] gives (26) as a necessary and sufficient condition for $(W_n^*)_n$ to be almost surely stable, and this achieves the proof.

∎

Let f_θ denote the density of F_θ.

COROLLARY 2.- *If* (H) *is fulfilled, if there exists* θ_1 *such that*

$$\lim_{x \to +\infty} \frac{f_{\theta 1}(x)}{G_{\theta 1}(x)\,\text{Log}\,|\text{Log}\,G_{\theta 1}(x)|} = +\infty \tag{27}$$

then $(X_n^*)_n$ *is almost surely stable.*

For the proof, see [Geffroy, 1958,1959].

V - EXAMPLES

In this section, we suppose that $k = 2$ and we use the polar coordinates in $\mathbb{R}^2$.

1. EXAMPLE 1.- In this first example, $F_\theta(x) = (1 - e^{-\alpha(\theta)x^m}) \mathbb{1}\{x > 0\}$ where $m > 0$ and α is a continuous, strictly positive function over $[0,2\pi]$ such that $\alpha(0) = \alpha(2\pi)$. Then $f_\theta(x) = m\,\alpha(\theta)\, x^{m-1}\, e^{-\alpha(\theta)x^m}\, \mathbb{1}\{x < 0\}$.

For a fixed θ_1 and for every $w > 0$, the u(w)-level isobar $g(\theta,w)$ is defined by

$$g(\theta,w) = \left(\frac{\alpha(\theta_1)}{\alpha(\theta)}\right)^{1/m} w.$$

So that (H) is fulfilled.

However (27) is satisfied only for $m > 1$ since

$$\frac{f_\theta(x)}{G_\theta(x) \operatorname{Log} |\operatorname{Log} G_\theta(x)|} = \frac{m\,\alpha(\theta)\, x^{m-1}}{\operatorname{Log}(\alpha(\theta).x^m)}.$$

Then $(X_n^*)_n$ is almost surely stable for $m > 1$.

2. EXAMPLE 2. Assume now that $m = 1$, so that $F_\theta(x) = (1 - e^{-\alpha(\theta)x}) \mathbb{1}\{x > 0\}$ (exponential distribution). Then the conditions of Gnedenko's theorem are not satisfied and $(X_n^*)_n$ is not almost surely stable. As mentionned in the introduction, this suggests a weaker notion than stability : the φ-stability. Before studying the φ-stability we examine more deeply the condition (H).

VI - SOME REMARKS ABOUT (H).

First, it can be shown that (H) is not a necessary condition for the stability of $(X_n^*)_n$; for this, it suffices to consider the next example :

1. EXAMPLE 3.- Suppose that $(X_1,\ldots,X_n)$ is a sample form a common two-dimensional distribution :

(28) $$F_\theta(x) = (1 - e^{-x^{\alpha(\theta)}})\,\mathbb{1}\,\{x>0\}\,,$$

where α is continuous over $[0,2\pi]$, $\alpha(0)=\alpha(2\pi)$ and $\alpha(\theta)>0$ for all θ.
The isobars containing $w>0$ are defined by :

$$g(\theta,w) = w^{\frac{\alpha(0)}{\alpha(\theta)}}\,.$$

a) Clearly (H) is not fulfilled for this example. In fact, even (4) does not hold : Choose $\alpha(\theta)=\pi+\theta$ over $[0,\pi]$, then for $0<\theta\le\pi$,

(29) $$\frac{1}{2} \le \frac{\pi}{\pi+\theta} < 1$$

Suppose (4) true ; then for all $\varepsilon>0$ there exists $\eta>0$ and for all $w>0$ there exists $v(w)$ such that for all θ :

(30) $$w^{\pi/\pi+\theta} < w^{\pi/\pi+\theta} + \eta < (v(w))^{\pi/\pi+\theta} < v^{\pi/\pi+\theta} + \varepsilon\,.$$

Particularly,

$$w < v(w) < w + \varepsilon\,.$$

Therefore,

$$w^{\pi/\pi+\theta} - (v(w))^{\pi/\pi+\theta} < w^{\pi/\pi+\theta}\,[(1+\frac{\varepsilon}{w})^{\pi/\pi+\theta} - 1]$$

and, for w large enough, for all $\theta>0$:

$$w^{\pi/\pi+\theta} - (v(w))^{\pi/\pi+\theta} < w^{\pi/\pi+\theta}\,(\frac{\varepsilon}{w}\,\frac{\pi}{\pi+\theta} + 0\,(\frac{1}{w}))$$

From (29) we deduce :

$$\lim_{n\to+\infty}\,(w^{\pi/\pi+\theta} - (v(w))^{\pi/\pi+\theta}) = 0\,,$$

and this contradicts (30).

b) However $(X_n^*)_n$ is almost surely stable.

Let $\beta(\theta) = 1/\alpha(\theta)$ and suppose that β has a maximum $\beta(0) = \beta < 1$. The equation of the isobars is :

$$g(\theta,w) = w^{\beta(\theta)/\beta} \quad ; \tag{31}$$

since for all θ $\beta(\theta) \leq \beta < 1$,

$$\sup_{\theta} |w^{\beta(\theta)/\beta} - v^{\beta(\theta)/\beta}| = |w - v| \tag{32}$$

for w and v large enough, then

$$\sup_{\theta} |g(\theta,w) - g(\theta,v)| = |g(0,w) - g(0,v)| \tag{33}$$

Now, put W_n^* the intersection of the isobar containing X_n^* and the positive half-axis $\theta = 0$; $(W_n^*)_n$ is almost surely stable because (27) holds.

Hence, for all $\varepsilon > 0$:

$$P\left\{\text{Lim inf}\,[a_n - \varepsilon < W_n^* < a_n + \varepsilon\,]\right\} = 1,$$

with $a_n = G_0^{-1}\left(\frac{1}{n}\right) = (\text{Log}\, n)^{\beta}$.

Let $\varepsilon > 0$ and denote by $\ell_n^{-\varepsilon}(\theta)$ (resp. $\ell_n(\theta)$, $\ell_n^{\varepsilon}(\theta)$) the isobar containing $a_n - \varepsilon$ (resp. a_n , $a_n + \varepsilon$). Since $\ell_n^{-\varepsilon}$ and ℓ_n^{ε} are isobars,

$$\begin{aligned} 1 &= P\left\{\text{Lim inf}\,[a_n - \varepsilon < W_n^* < a_n + \varepsilon]\right\} \\ &= P\left\{\text{Lim inf}\,\ell_n^{-\varepsilon}(\Theta_n^*) < R_n^* < \ell_n^{\varepsilon}(\Theta_n^*)\right\} \end{aligned}$$

Moreover from (33) ,

$$\lim_{n\to\infty} \sup_{\theta} |\,\ell_n^{-\varepsilon}(\theta) - \ell_n(\theta)| = \varepsilon$$

and

$$\lim_{n\to\infty} \sup_{\theta} |\ell_n^{\varepsilon}(\theta) - \ell_n(\theta)| = \varepsilon\,.$$

Then,

$$R_n^* - \ell_n(\Theta_n^*) \overset{p.s.}{\rightarrow} 0$$

and $(X_n^*)_n$ is almost surely stable.

2.- From this example we deduce another sufficient condition for the stability of $(X_n^*)_n$. Suppose there exists θ_1 such that for all $w > 0$ and for all $\eta > 0$

(K) $$\sup_\theta \; (g(\theta,w+\eta) - g(\theta,w-\eta)) = g(\theta_1,w+\eta) - g(\theta 1,w-\eta) .$$

Let W_n^* be the intersection of the isobar containing X_n^* and the positive half-axis $0\theta_1^+$.

As for example 3 we can prove the next theorem.

THEOREM 5. *If* (K) *holds and if* $(W_n^*)_n$ *is almost surely stable, then* $(X_n^*)_n$ *is almost surely stable.*

3.- Conditions (H) and (K) differ in kind : (H) is an uniformity condition whereas (K) uses the existency of a direction θ_1, which can be interpreted as the less favourable direction.

Clearly (H) and (K) are not equivalent (see example 3). Of course if $g(\theta,w) = k(\theta)w + \ell(\theta)$, then (H) involves (K). Remark also that under (K) theorem 3 is still true.

VII - φ-STABILITY

1.- Let φ be a function over $\mathbb{R}^+$, positive, C^1, increasing and bijective. In this section we consider the variables $X_1^\varphi = (\varphi(R_1),\Theta_1),\dots,X_n^\varphi = (\varphi(R_n),\Theta_n)$.

Put for all θ, f_θ^φ the conditional density of $\varphi(R)$ given $\Theta = \theta$ and $F_\theta^\varphi = 1 - G_\theta^\varphi$ the conditional distribution function of $\varphi(R)$ given $\Theta = \theta$.

For all $t > 0$,

(34) $$F_\theta^\varphi(t) = F_\theta(\varphi^{-1}(t))$$

(35) $$f_\theta^\varphi(t) = (\varphi^{-1}(t))' \cdot f_\theta(\varphi^{-1}(t)).$$

Put $X_\theta^{\varphi^*} = ((\varphi(R_n))^*, \Theta_n^*)$ the maximum value of $(X_1^\varphi, \ldots, X_n^\varphi)$; then, $X_n^{\varphi^*} = (\varphi(R_n^*), \Theta_n^*)$.

Clearly, if $(g_n)_n$ is a sequence of u-level isobars for the distribution of (R,Θ), then $(\varphi(g_n))_n$ is a sequence of u-level isobars for the distribution of $(\varphi(R),\Theta)$ and conversely.

This remark entails a definition which generalizes the definition of relative stability given by Geffroy in [Geffroy, 1958,1959] and Gnedenko in [Gnedenko, 1943].

DEFINITION 3.- *The sequence* $(X_n^*)_n$ *is called* φ*-stable in probability (almost surely) if there exists a sequence of isobars* $(g_n)_n$ *(for the distribution of* (R,Θ)*) such that* :

(36) $$\varphi(R_n^*) - \varphi(g_n(\Theta_n^*)) \underset{\text{a.s.}}{\overset{P}{\to}} 0.$$

REMARK 6.- When $\varphi(x) = \widehat{\text{Log}}\,(x) = \text{Max}(0, \text{Log}x)$, φ -stability reduces to relative stability [Geffroy, 1958,1959], [Gnedenko, 1943], [Green,1976] ; (36) can be written :

(37) $$\frac{R_n^*}{g_n(\Theta_n^*)} \underset{\text{a.s.}}{\overset{P}{\to}} 1$$

2.- All the results of sections III and IV can be used for $(X_1^\varphi, X_2^\varphi, \ldots, X_n^\varphi)$:

a) It is easily seen that the equation of the isobar containing $v > 0$ on the half-axis $0\theta_1^+$ is :

(38) $$g^\varphi(\theta, v) = \varphi(g(\theta, \varphi^{-1}(v)))$$

b) Condition (H) becomes $(H)^\varphi$; for example, if $\varphi = \widehat{\text{Log}}$, $(H)^\varphi$ reduces to :

$$(\hat{H}) \qquad \alpha_1 \leq \frac{w \dfrac{\partial g(\theta,w)}{\partial w}}{g(\theta,w)} \leq \beta_1$$

c) Theorems 3 and 4 can be used with G_θ^φ, f_θ^φ, ... instead of G_θ^φ, f_θ.

d) Of course, if $(X_n^*)_n$ is stable, stability properties subsit for $(X_n^{\varphi *})_n$ if φ is a concave function, [Gather and Rauhut, 1990].

3.- EXAMPLES

a) Example 4 : Cauchy's distribution.

Suppose E_n is a sample from a multidimensional distribution :

$$f_\theta(x) = \frac{2}{\pi} \frac{\lambda(\theta)}{x^2 + \lambda^2(\theta)} \mathbb{1}\{x > 0\} ;$$

λ is a continuous and positive function.

Then

$$F_\theta(x) = \frac{2}{\pi} \text{Arctg } (\frac{x}{\lambda(\theta)}) \mathbb{1}\{x > 0\} .$$

Since Cauchy's distribution does not have moment of any order, $(X_n^*)_n$ is not stable [Geffroy, 1958, 1959].

However, for all θ and for x large enough,

$$F_\theta(x) \sim 1 - \frac{2}{\pi} \frac{\lambda(\theta)}{x} .$$

Then the conditions of Gnedenko's theorem are satisfied by G_θ^φ if and only if :

$$(39) \qquad \lim_{x \to +\infty} \frac{\varphi^{-1}(x-h)}{\varphi^{-1}(x)} = 0, \text{ for all } h > 0 .$$

It suffices to choose $\varphi(x) = \sqrt{\text{Log } x}$ or $\varphi(x) = \text{Log Log } x$ which is more concave than $\sqrt{\text{Log } x}$. Moreover for this two functions, (H^{φ}) holds for v large enough. Therefore $(X_n^*)_n$ is φ-stable in probability. Note that $\varphi_\alpha(x) = (\text{Log } x)^\alpha$, $(0 < \alpha < 1)$ is also a suitable function, but $(X_n^*)_n$ is not relatively stable.

b) **Example 5 : exponential distribution.**

Suppose E_n is a sample from a multidimensional distribution :

$$F_\theta(x) = (1 - e^{-\alpha(\theta)x}) \ \mathbb{1}\{x > 0\} \ ;$$

α is a continuous and positive function.

Similarly, for an exponential distribution, the condition for φ is :

$$\lim_{x \to +\infty} \varphi^{-1}(x-h) - \varphi^{-1}(x) = -\infty , \quad \text{for all } h > 0 . \tag{40}$$

The functions $\varphi(x) = \sqrt{x}$ or $\varphi(x) = \widehat{\text{Log}}\,(x)$ are suitable and $(H)^{\varphi}$ is also satisfied.

c) **Example 6.**

More generally, consider Example 1 with $0 < m \leq 1$. Choose $\varphi(x) = x^{1/2m}$, then $F_\theta^{\varphi}(x) = (1 - e^{-\alpha(\theta)x^2}) \ \mathbb{1}\{x > 0\}$ and $(X_n^{\varphi *})_n$ is almost surely stable, as it has been shown in Example 1.

d) **Example 7.**

Suppose E_n is a sample from a multidimensional distribution.

$$F_\theta(x) = (1 - \frac{\lambda(\theta)}{x^{\alpha(\theta)}}) \ \mathbb{1}\{x > 0\} \ ;$$

λ and α are continuous and positive functions. (For $\alpha(\theta) \equiv 1$, we obtain asymptotically a Cauchy's distribution). Choose $\varphi(x) = \sqrt{\text{Log } x}$; then ,

$$F_\theta^{\varphi}(x) = (1 - \lambda(\theta) \ e^{-\alpha(\theta)x^2}) \ \mathbb{1}\{x > 0\}$$

and Example 1 shows that $(X_n^{\varphi *})_n$ is almost surely stable.

VIII - CONCLUDING REMARKS

In view of the previous results, we are led to consider the geometric aspect of the stability : in a further paper we plan to describe how the sample set of points lies in a shape limited by the isobars γ_n . Geffroy [1958,1959], Fisher [1966] were the first to deal with this subject in the case of a convex shape (see also [Davis, Mulrow, Resnick, 1987]), as for the typical example of a gaussian sample. Using the isobars it is possible to study the case of a non necessarily convex shape.

Moreover it would be interesting to also examine the statistical aspect of the problem : for example to give a functional estimate of γ_n for an unknown sample.

BIBIOGRAPHIE

Billingsley, P., (1968), Convergence of Probability Measures. *Wiley.*

Fisher, L., (1966), The convex hull of a sample, *Bull. Am. Math. Soc. 72, 555-558* .

Gather, U. and Rauhut, B., (1990), The outlier behaviour of probability distributions, *Jounal of statistical planning and Inference 26, 237-252.*

Geffroy, J., (1958,1959), Contribution à la théorie des valeurs extrêmes, *Pub. Inst. Stat. Univ. Paris, 1958 (fasc. 4) et 1959 (fasc. 1).*

Geffroy, J., (1961), Localisation asymptotique du polyèdre d'appui d'un échantillon laplacien à k dimension, *Publ. Inst. Stat. Univ. Paris, 1961, n° 59.*

Gnedenko, B.V., (1943), Sur la distribution limite du terme maximum d'une série aléatoire. *Ann. Math. 44, 423-453.*

Green, R.F., (1976), Outlier-prone and outlier-resistant distributions, *J. Amer. Statist. Assoc. 71, 502-505.*

Haiman, G., Puri, M., (1990), A strong invariance principle concerning the J-uper order statistics for stationary m-dependent sequences, *Journal of Statistical Planning and inference.*

Haiman, G., Puri, M., A strong invariance principle concerning the J-upper ordre of statistics for stationary gaussian sequences" (à paraître dans Annals of Probability).

Mathar, R., (1989), The outlier-behaviour of distributions and the decay power of optimal tests at decreasing level. *Statistics 20 (2), 247-254.*

Muñoz-Garcia, J., Moreno-Rebollo, J.L. and Pascual-Acosta, A., (1990), Outliers : A formal approach. *International Statistical Review, 58, 3, 215-226.*

Lecoutre, J.P., (1982), Contribution à l'estimation non-paramétrique de la régression. *Thèse , Université Pierre et Marie Curie, Paris.*

Davis, R., Mulrow, E., Resnick, S., (1987), The convex hull of a random sample in $\mathbb{R}^2$. *Comm. statist. - stochastic models ; 3(1), 1-27.*