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On Equitorsion Geodesic Mappings of General Affine Connection Spaces

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ABSTRACT - In the papers [19], [20] several Ricci type identities are obtained by using non-symmetric affine connection. In these identities appear 12 curvature tensors, 5 of which being independent [21], while the rest can be expressed as linear combinations of the others.

In the general case of a geodesic mapping f of two non-symmetric affine connection spaces GA_N and $G\overline{A}_N$ it is impossible to obtain a generalization of the Weyl projective curvature tensor. In the present paper we study the case when GA_N and $G\overline{A}_N$ have the same torsion in corresponding points. Such a mapping we name "equitorsion mapping".

With respect to each of mentioned above curvature tensors we have obtained quantities $\underset{\theta}{\mathcal{E}}{}^i_{jmn}$ $(\theta = 1, \cdots, 5)$, that are generalizations of the Weyl tensor, i.e. they are invariants based on f. Among $\underset{\theta}{\mathcal{E}}$ only $\underset{5}{\mathcal{E}}$ is a tensor. All these quantities are interesting in constructions of new mathematical and physical structures.

1. Introduction.

Geodesic mappings of Riemannian spaces and of affine spaces and their generalizations were investigated by many authors, for examle N. S. Sinjukov [34], J. Mikeš [12]-[18], M. Prvanović [25]-[28, 29], S. Minčić [23], [24], [38]-[41], M. Stanković [35]-[42] and many others.

Consider two N-dimensional differentiable manifolds GA_N and $G\overline{A}_N$ and differentiable mapping $f : GA_N \to G\overline{A}_N$. We can consider these manifolds in the *common by this mapping system of local coordinates.*

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Namely, if $f : M \in GA_N \to \overline{M} \in G\overline{A}_N$ and if $(\mathcal{U}, \varphi)$ is local chart around the point M it will be $\varphi(M) = x = (x^1, \cdots, x^N) \in E^N$ (Euclidean N-space). In this case, we define for the coordinate mapping in the $G\overline{A}_N$ the mapping $\overline{\varphi} = \varphi \circ f^{-1}$, and then

$$(1) \qquad \overline{\varphi}(\overline{M}) = (\varphi \circ f^{-1})(f(M)) = \varphi(M) = x = (x^1, \cdots, x^N) \in E^N,$$

that is the points M and $\overline{M} = f(M)$ have the same local coordinates. If connection coefficients $L^i_{jk}(x)$ and $\overline{L}^i_{jk}(x)$, for the connection introduced in GA_N and $G\overline{A}_N$ respectively, are non-symmetric in lower indices, we call GA_N and $G\overline{A}_N$ *general affine connection spaces.*

Although the notion of non-symmetric affine connection is used in several works before A. Einstein, for example in [2] (Eisenhart, 1927), [8] (Hayden, 1932), the use of non-symmetric connection became especially actual after appearance the works of Einstein, relating to create the Unified Field Theory (UFT).

Einstein was not satisfied with his General Theory of Relativity (GTR, 1916), and from 1923. to the end of his life (1955), he worked on various variants of UFT. This theory had the aim to unite the gravitation theory, to which is related GTR, and the theory of electromagnetism.

One says that reciprocal one valued mapping $f : GA_N \to G\overline{A}_N$ is *geodesic*, [23], [24] if geodesics of the space GA_N pass to geodesics of the space $G\overline{A}_N$. In the corresponding points $M(x)$ and $\overline{M}(x)$ we can put

$$(2) \qquad \overline{L}^i_{jk}(x) = L^i_{jk}(x) + P^i_{jk}(x), \quad (i, j, k = 1, ..., N),$$

where $P^i_{jk}(x)$ is *the deformation tensor* of the connection L of GA_N according to the mapping $f : GA_N \to G\overline{A}_N$.

A necessary and sufficient condition that the mapping f be geodesic [23] is that the deformation tensor P^i_{jk} from (2) has the form

$$(3) \qquad P^i_{jk}(x) = \delta^i_{(j}\,\psi_{k)}(x) + \xi^i_{jk}(x),$$

where

$$(4) \qquad \psi_i = \frac{1}{N+1} P^p_{\underline{ip}} = \frac{1}{N+1}(\overline{L}^p_{\underline{ip}} - L^p_{\underline{ip}}),$$

$$(5) \qquad \xi^i_{jk} = P^i_{\underset{\vee}{jk}} = \frac{1}{2}(P^i_{jk} - P^i_{kj}) = \frac{1}{2} P^i_{[jk]}$$

where $\underline{ij}$ denotes a symmetrization, $\underset{\vee}{ij}$-antisymmetrization, $[i \ldots j]$ denotes an antisymmetrization without division with respect to the indices i, j, and also $(i \ldots j)$ denotes a symmetrization without division with respect to indices i, j.

In GA_N one can define four kinds of covariant derivatives [19, 20]. For example, for a tensor a^i_j in GA_N we have

$$a^i_{\underset{1}{j|m}} = a^i_{j,m} + L^i_{pm}a^p_j - L^p_{jm}a^i_p, \quad a^i_{\underset{2}{j|m}} = a^i_{j,m} + L^i_{mp}a^p_j - L^p_{mj}a^i_p,$$

$$a^i_{\underset{3}{j|m}} = a^i_{j,m} + L^i_{pm}a^p_j - L^p_{mj}a^i_p, \quad a^i_{\underset{4}{j|m}} = a^i_{j,m} + L^i_{mp}a^p_j - L^p_{jm}a^i_p.$$

Denote by $\underset{\theta}{|}, \underset{\overline{\theta}}{|}$ a covariant derivative of the kind θ in GA_N and $G\overline{A}_N$ respectively.

While at the Riemannian space (the space of GTR) the connection coefficients are expressed by virtue of symmetric basic tensor g_{ij}, at Einstein's works from UFT (1950-1955) the connection between these magnitudes is determined by equations

$$g_{\underset{+-}{ij};m} \equiv g_{ij,m} - \Gamma^p_{im}g_{pj} - \Gamma^p_{mj}g_{ip} = 0, \quad (g_{ij,m} = \frac{\partial g_{ij}}{\partial x^m}). \tag{6}$$

The equation (6) signifies that the index i one treats in the sense of the first kind of derivative $(\underset{1}{|})$, and j in the sense of the second one $(\underset{2}{|})$.

Proceeding at that sense, Einstein in [1], 1950, for covariant curvature tensor in his theory obtains a Bianchi-type identity:

$$R_{\underset{-+-+}{iklm};n} + R_{\underset{-+++}{ikmn};l} + R_{\underset{-+--}{iknl};m} = 0, \tag{7}$$

where $R_{iklm} = g_{si}R^s{}_{klm}$.

In the case of the space GA_N we have five independent curvature tensors [21] (in [21] $\underset{5}{R}$ is denoted by $\underset{2}{\tilde{R}}$):

$$\begin{aligned}
&\underset{1}{R}^i_{jmn} = L^i_{j[m,n]} + L^p_{j[m}L^i_{pn]}, \quad \underset{2}{R}^i_{jmn} = L^i_{[mj,n]} + L^p_{[mj}L^i_{n]p},\\
&\underset{3}{R}^i_{jmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + L^p_{nm}L^i_{[pj]},\\
&\underset{4}{R}^i_{jmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + L^p_{mn}L^i_{[pj]},\\
&\underset{5}{R}^i_{jmn} = \frac{1}{2}(L^i_{j[m,n]} + L^i_{[mj,n]} + L^p_{jm}L^i_{pn} + L^p_{mj}L^i_{np} - L^p_{jn}L^i_{mp} - L^p_{nj}L^i_{pm}).
\end{aligned} \tag{8}$$

By virtue of the geodesic mapping $f : GA_N \to G\overline{A}_N$ we obtain tensors $\underset{\theta}{\overline{R}}^i_{jmn}$, $(\theta = 1, ..., 5)$, where for example

$$\underset{1}{\overline{R}}^i_{jmn} = \overline{L}^i_{j[m,n]} + \overline{L}^p_{j[m}\overline{L}^i_{pn]}. \tag{9}$$

U. P. Singh ([31], 1968) for a special nonsymmetric connection of M. Prvanović ([27], 1959) uses 2 kinds of covariant derivatives and obtains 3 curvature tensors by forming Ricci identities for a vector.

F. Graif [4] gives geometric interpretations of two kinds parallel displacement, based on non-symmetric connection. M. Prvanović [26], using non-symmetry of the connection, obtains geometric interpretations of of the tensors $\underset{1}{R}, \dots, \underset{4}{R}$, given in (8), and S. Minčić in [22] gives also corresponding interpretations of $\underset{5}{R}$ from (8).

An application of more kinds of covariant derivative at L_N makes possible to express more concise some results. For example, let us consider infinitesimal deformations, defined by

$$\bar{x}^i = x^i + \varepsilon z^i(x), \quad x = (x^1, \dots, x^N), \quad i = 1, \dots, N, \tag{10}$$

where ε is an infinitesimal parameter, $z^i(x)$ a vector field. As it is known, a deformed geometric object $\bar{\mathcal{A}}(x)$ (e. g. a tensor, a connection) of the object $\mathcal{A}(x)$ is

$$\bar{\mathcal{A}} = \mathcal{A} + \varepsilon \mathcal{L}_z \mathcal{A}, \tag{11}$$

where $\mathcal{L}_z\mathcal{A}$ is the Lie derivative of $\mathcal{A}$ in the direction of the field $z^i(x)$. Then, e. g. for a tensor t^{ij}_{kl} ([44] (K. Yano, 1949), [45](K. Yano, 1957) we have:

$$\mathcal{L}_z t^{ij}_{kl} = t^{ij}_{kl,p} z^p - z^i_{,p} t^{pj}_{kl} - z^j_{,p} t^{ip}_{kl} + z^p_{,k} t^{ij}_{pl} + z^p_{,l} t^{ij}_{kp}, \tag{12}$$

Using the covariant derivatives of one kind, for example the first, instead of partial derivatives, we have

$$\begin{aligned} \mathcal{L}_z t^{ij}_{kl} = {} & t^{ij}_{kl\underset{1}{|}p} z^p - z^i_{\underset{1}{|}p} t^{pj}_{kl} - z^j_{\underset{1}{|}p} t^{ip}_{kl} + \\ & + z^p_{\underset{1}{|}k} t^{ij}_{pl} + z^p_{\underset{1}{|}l} t^{ij}_{kp} + 2(L^i_{\underset{\vee}{ps}} t^{sj}_{kl} + L^j_{\underset{\vee}{ps}} t^{is}_{kl} + L^s_{\underset{\vee}{kp}} t^{ij}_{sl} + L^s_{\underset{\vee}{lp}} t^{ij}_{ks}) z^p, \end{aligned} \tag{13}$$

from where we see that Lie derivative is a tensor. But, using more kinds of covariant derivative, we obtain in the considered case [43] (Lj. Velimirović, S. Minčić, M. Stanković):

$$\mathcal{L}_z t^{ij}_{kl} = t^{ij}_{kl\underset{\lambda}{|}p} z^p - z^i_{\underset{\mu}{|}p} t^{pj}_{kl} - z^j_{\underset{\mu}{|}p} t^{ip}_{kl} + z^p_{\underset{\nu}{|}k} t^{ij}_{pl} + z^p_{\underset{\nu}{|}l} t^{ij}_{kp}, \tag{14}$$

where $(\lambda, \mu, \nu) \in \{(1,2,2), (2,1,1), (3,4,3), (4,3,4)\}$ i.e. in (14) we have 4 manners of the presenting of Lie derivative, that are more concise than in (13).

In the case of geodesic mapping $f : A_N \to \overline{A}_N$ of the symmetric affine connection spaces A_N and $\overline{A}_N$ we have an invariant geometric object (Weyl projective tensor)

$$(15)\quad W^i_{jmn} = R^i_{jmn} + \frac{1}{N+1}\delta^i_j R_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m} R_{jn]} + \frac{1}{N^2-1}\delta^i_{[m} R_{n]j},$$

where R^i_{jmn} is Riemann-Cristoffel's curvature tensor of the space A_N, and R_{jm} is Ricci tensor.

The object W^i_{jmn} is a tensor and it is called Weyl tensor, or a tensor of projective curvature [34]. Having a geodesic mapping of two general affine connection spaces, we can not find a generalization of Weyl tensor as an invariant of geodesic mapping in general case. For that reason we define a special geodesic mapping.

A mapping $f : GA_N \to G\overline{A}_N$ is *equitorsion geodesic mapping* if the torsion tensor of the spaces GA_N and $G\overline{A}_N$ are equal. Then from (2), (3) and (5)

$$(16)\qquad \overline{L}^h_{\underset{\vee}{ij}} - L^h_{\underset{\vee}{ij}} = \xi^h_{ij}(x) = 0,$$

where $\underset{\vee}{ij}$ denotes an antisymmetrization with respect to i,j.

2. ET-projective parameter of the first kind.

Using (3), (9) and (16) we get a relation between the first kind of curvature tensors (8) of the spaces GA_N and $G\overline{A}_N$

$$\underset{1}{\overline{R}}{}^i_{jmn} = \underset{1}{R}{}^i_{jmn} + \delta^i_j \underset{1}{\psi}_{[mn]} + \delta^i_{[m} \underset{1}{\psi}_{jn]} + 2L^i_{\underset{\vee}{mn}}\psi_j + 2L^p_{\underset{\vee}{mn}}\psi_p\delta^i_j\,,$$

where we denote

$$(17)\qquad \underset{\theta}{\psi}_{mn} = \psi_{m\underset{\theta}{|}n} - \psi_m\psi_n, \quad \theta = 1,2.$$

Contracting with respect to i and n, from (2) one gets

$$\underset{1}{\overline{R}}{}^q_{jmq} = \underset{1}{R}{}^q_{jmq} + \delta^q_j \underset{1}{\psi}_{[mq]} + \delta^q_m \underset{1}{\psi}_{jq} - \delta^q_q \underset{1}{\psi}_{jm} + 2L^q_{\underset{\vee}{mq}}\psi_j + 2L^p_{\underset{\vee}{mq}}\psi_p\delta^q_j\,, \quad \text{i.e.}$$

$$\underset{1}{\overline{R}}_{jm} = \underset{1}{R}_{jm} + \underset{1}{\psi}_{[mj]} + \underset{1}{\psi}_{jm} - N\underset{1}{\psi}_{jm} + 2L^q_{\underset{\vee}{mq}}\psi_j + 2L^p_{\underset{\vee}{mj}}\psi_p,$$

from where

$$(18)\qquad \underset{1}{\overline{R}}_{jm} = \underset{1}{R}_{jm} - \underset{1}{\psi}_{[jm]} + (1-N)\underset{1}{\psi}_{jm} + 2L^q_{\underset{\vee}{mq}}\psi_j + 2L^p_{\underset{\vee}{mj}}\psi_p.$$

Here $\underset{1}{\overline{R}}_{jm}$ and $\underset{1}{R}_{jm}$ are the first kind Ricci tensors of the spaces $G\overline{A}_N$ and GA_N respectively. From (18) we obtain

$$\underset{1}{\overline{R}}_{[jm]} = \underset{1}{R}_{[jm]} - 2\underset{1}{\psi}_{[jm]} + (1-N)\underset{1}{\psi}_{[jm]} + 2\underset{\vee}{L}^q_{[mq}\psi_{j]} + 4\underset{\vee}{L}^p_{mj}\psi_p. \tag{19}$$

From (19) and (4) one obtains

$$\begin{aligned}\underset{1}{\overline{R}}_{[jm]} = \underset{1}{R}_{[jm]} - (N+1)\underset{1}{\psi}_{[jm]} &+ \frac{2}{N+1}\underset{\vee}{L}^q_{mq}(\overline{L}^p_{\underline{jp}} - L^p_{\underline{jp}})\\ &- \frac{2}{N+1}\underset{\vee}{L}^q_{jq}(\overline{L}^p_{\underline{mp}} - L^p_{\underline{mp}}) + \frac{4}{N+1}\underset{\vee}{L}^p_{mj}(\overline{L}^q_{\underline{pq}} - L^q_{\underline{pq}}),\end{aligned}\tag{20}$$

from where

$$\begin{aligned}(N+1)\underset{1}{\psi}_{[jm]} = \underset{1}{R}_{[jm]} - \underset{1}{\overline{R}}_{[jm]} &+ \frac{2}{N+1}\underset{\vee}{L}^q_{mq}(\overline{L}^p_{\underline{jp}} - L^p_{\underline{jp}})\\ &- \frac{2}{N+1}\underset{\vee}{L}^q_{jq}(\overline{L}^p_{\underline{mp}} - L^p_{\underline{mp}}) + \frac{4}{N+1}\underset{\vee}{L}^p_{mj}(\overline{L}^q_{\underline{pq}} - L^q_{\underline{pq}}).\end{aligned}\tag{21}$$

Because of (18), (4) and (21), we have

$$\begin{aligned}\underset{1}{\overline{R}}_{jm} = \underset{1}{R}_{jm} &- \frac{1}{N+1}\big[\underset{1}{R}_{[jm]} - \underset{1}{\overline{R}}_{[jm]} + \frac{2}{N+1}\underset{\vee}{L}^q_{mq}(\overline{L}^p_{\underline{jp}} - L^p_{\underline{jp}})\\ &- \frac{2}{N+1}\underset{\vee}{L}^q_{jq}(\overline{L}^p_{\underline{mp}} - L^p_{\underline{mp}}) + \frac{4}{N+1}\underset{\vee}{L}^p_{mj}(\overline{L}^q_{\underline{pq}} - L^q_{\underline{pq}})\big]\\ &- (N-1)\underset{1}{\psi}_{jm} + \frac{2}{N+1}\underset{\vee}{L}^q_{mq}(\overline{L}^p_{\underline{jp}} - L^p_{\underline{jp}}) + \frac{2}{N+1}\underset{\vee}{L}^p_{mj}(\overline{L}^q_{\underline{pq}} - L^q_{\underline{pq}}),\end{aligned}$$

that is

$$\begin{aligned}(N-1)\underset{1}{\psi}_{jm} = \underset{1}{R}_{jm} - \underset{1}{\overline{R}}_{jm} &- \frac{1}{N+1}\big[\underset{1}{R}_{[jm]} - \underset{1}{\overline{R}}_{[jm]} + \frac{2}{N+1}\underset{\vee}{L}^q_{mq}(\overline{L}^p_{\underline{jp}} - L^p_{\underline{jp}})\\ &- \frac{2}{N+1}\underset{\vee}{L}^q_{jq}(\overline{L}^p_{\underline{mp}} - L^p_{\underline{mp}}) + \frac{4}{N+1}\underset{\vee}{L}^p_{mj}(\overline{L}^q_{\underline{pq}} - L^q_{\underline{pq}})\big]\\ &+ \frac{2}{N+1}\underset{\vee}{L}^q_{mq}(\overline{L}^p_{\underline{jp}} - L^p_{\underline{jp}}) + \frac{2}{N+1}\underset{\vee}{L}^p_{mj}(\overline{L}^q_{\underline{pq}} - L^q_{\underline{pq}}),\end{aligned}$$

from where

$$\begin{aligned}(N^2-1)\underset{1}{\psi}_{jm} = (N\underset{1}{R}_{jm} + \underset{1}{R}_{mj}) - (N\underset{1}{\overline{R}}_{jm} + \underset{1}{\overline{R}}_{mj}) &+ \frac{2N}{N+1}\underset{\vee}{L}^q_{mq}(\overline{L}^p_{\underline{jp}} - L^p_{\underline{jp}})\\ + \frac{2}{N+1}\underset{\vee}{L}^q_{jq}(\overline{L}^p_{\underline{mp}} - L^p_{\underline{mp}}) &+ 2\underset{\vee}{L}^p_{mj}(\overline{L}^q_{\underline{pq}} - L^q_{\underline{pq}})\frac{N-1}{N+1}.\end{aligned}\tag{22}$$

Substituting (22) at (2), by help of (16), it is easy to prove that

$$\underset{1}{\overline{\mathcal{E}}}{}^i_{jmn} = \underset{1}{\mathcal{E}}{}^i_{jmn}, \tag{23}$$

where we have denoted

$$\begin{aligned}\underset{1}{\mathcal{E}}{}^i_{jmn} = {} & \underset{1}{R}{}^i_{jmn} + \frac{N}{N+1}\delta^i_j \underset{1}{R}{}_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m}\underset{1}{R}{}_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}\underset{1}{R}{}_{n]j}\\ & + \frac{2}{(N+1)^2(N-1)} L^p_{\underline{[mp}}\delta^i_{n]} \underset{\vee}{L^q_{jq}} - \frac{2}{(N+1)^2} L^p_{\underline{[mp}}\delta^i_j \underset{\vee}{L^q_{n]q}}\\ & + \frac{2N}{(N+1)(N^2-1)} L^p_{\underline{jp}}\delta^i_{[n}\underset{\vee}{L^q_{m]q}}\\ & - \frac{4}{N+1} L^p_{\underline{jp}}\underset{\vee}{L^i_{mn}} - \frac{2}{(N+1)^2} L^q_{\underline{pq}}\big[\delta^i_{(j}\underset{\vee}{L^p_{nm)}} - \delta^i_n \underset{\vee}{L^p_{mj}} + (N+1)\delta^i_j \underset{\vee}{L^p_{mn}}\big]\end{aligned} \tag{24}$$

Obviously, the magnitude $\underset{1}{\mathcal{E}}{}^i_{jmn}$ is not a tensor and we call it the ET**-projective parameter of the first kind** for the mapping $f : GA_N \to G\overline{A}_N$. So, we have

THEOREM 1. *The ET-projective parameter of the first kind* $\underset{1}{\mathcal{E}}{}^i_{jmn}$ (24) *is an invariant of the equitorsion mapping* $f : GA_N \to G\overline{A}_N$. □

3. ET-projective parameter of the second kind.

For the second kind of curvature tensors $\underset{2}{R}$ and $\underset{2}{\overline{R}}$ (8) of the spaces GA_N and $G\overline{A}_N$ respectively, we get the relation

$$\underset{2}{\overline{R}}{}^i_{jmn} = \underset{2}{R}{}^i_{jmn} + \delta^i_j \underset{2}{\psi}{}_{[mn]} + \delta^i_{[m}\underset{2}{\psi}{}_{jn]} + 2\underset{\vee}{L^i_{nm}}\psi_j + 2\underset{\vee}{L^p_{nm}}\psi_p\delta^i_j\,, \tag{25}$$

Analogously to previous case, we get

$$\underset{2}{\overline{\mathcal{E}}}{}^i_{jmn} = \underset{2}{\mathcal{E}}{}^i_{jmn}, \tag{26}$$

where is

$$\begin{aligned}\underset{2}{\mathcal{E}}{}^i_{jmn} = {} & \underset{2}{R}{}^i_{jmn} + \frac{N}{N+1}\delta^i_j \underset{1}{R}{}_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m}\underset{2}{R}{}_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}\underset{2}{R}{}_{n]j}\\ & + \frac{2}{(N+1)^2(N-1)} L^p_{\underline{[mp}}\delta^i_{n]} \underset{\vee}{L^q_{qj}} - \frac{2}{(N+1)^2} L^p_{\underline{[mp}}\delta^i_j \underset{\vee}{L^q_{qn]}}\\ & + \frac{2N}{(N+1)(N^2-1)} L^p_{\underline{jp}}\delta^i_{[n}\underset{\vee}{L^q_{qm]}}\\ & - \frac{4}{N+1} L^p_{\underline{jp}}\underset{\vee}{L^i_{nm}} - \frac{2}{(N+1)^2} L^q_{\underline{pq}}\big[\delta^i_j\underset{\vee}{L^p_{mn}} + \delta^i_{(m} \underset{\vee}{L^p_{jn)}} + (N+1)\delta^i_j \underset{\vee}{L^p_{nm}}\big]\end{aligned} \tag{27}$$

The magnitude $\underset{2}{\mathcal{E}}{}^i_{jmn}$ is not a tensor and we call it **ET-projective parameter of the second kind**. In this case we have

THEOREM 2. *The ET-projective parameter of the second kind* $\underset{2}{\mathcal{E}}{}^i_{jmn}$ (27) *is an invariant of the equitorsion mapping* $f : GA_N \to G\overline{A}_N$. □

4. ET-projective parameter of the third kind.

In the case of the third kind of curvature tensors $\underset{3}{R}$ and $\underset{3}{\overline{R}}$ (8) of the spaces GA_N and $G\overline{A}_N$ we get

$$\underset{3}{\overline{R}}{}^i_{jmn} = \underset{3}{R}{}^i_{jmn} + \delta^i_{(j}\underset{2}{\psi}_{m)n} - \delta^i_{(j}\underset{1}{\psi}_{n)m} + 2\psi_{(n}\underset{\vee}{L}{}^i_{m)j} \tag{28}$$

Since

$$\underset{2}{\psi}_{mn} = \underset{1}{\psi}_{mn} + 2\underset{\vee}{L}{}^p_{mn}\psi_p,$$

from (28) one obtains

$$\underset{3}{\overline{R}}{}^i_{jmn} = \underset{3}{R}{}^i_{jmn} + \delta^i_j\underset{1}{\psi}_{[mn]} + \delta^i_{[m}\underset{1}{\psi}_{jn]} + 2\delta^i_{(j}\underset{\vee}{L}{}^p_{m)n}\psi_p + 2\psi_{(n}\underset{\vee}{L}{}^i_{m)j}. \tag{29}$$

Contracting in (29) over i, n, we obtain

$$\underset{3}{\overline{R}}_{jm} = \underset{3}{R}_{jm} - \underset{1}{\psi}_{[jm]} - (N-1)\underset{1}{\psi}_{jm} + 2\psi_q\underset{\vee}{L}{}^q_{mj} + 2\psi_m\underset{\vee}{L}{}^p_{pj}. \tag{30}$$

Alternating the last equation with respect to j, m, we have

$$\underset{3}{\overline{R}}_{[jm]} = \underset{3}{R}_{[jm]} - 2\,\underset{1}{\psi}_{[jm]} - (N-1)\underset{1}{\psi}_{[jm]} + 4\,\psi_q\underset{\vee}{L}{}^q_{mj} + 2\psi_{[m}\underset{\vee}{L}{}^p_{pj]}.$$

By using (4), we get

$$(N+1)\,\underset{1}{\psi}_{jm} = \underset{3}{R}_{[jm]} - \underset{3}{\overline{R}}_{[jm]} + \frac{4}{N+1}\underset{\vee}{L}{}^p_{mj}(\overline{L}{}^q_{\underline{pq}} - L^q_{\underline{pq}})$$
$$+\frac{2}{N+1}\underset{\vee}{L}{}^p_{pj}(\overline{L}{}^q_{\underline{mq}} - L^q_{\underline{mq}} - \frac{2}{N+1}\underset{\vee}{L}{}^p_{pm}(\overline{L}{}^q_{\underline{jq}} - L^q_{\underline{jq}}), \tag{31}$$

By virtue of (4), (31) and (30) we obtain

$$\begin{aligned}\overline{\underset{3}{R}}_{jm} = \underset{3}{R}_{jm} &- \frac{1}{N+1}\Big[\underset{3}{R}_{[jm]} - \overline{\underset{3}{R}}_{[jm]} + \frac{4}{N+1}\underset{\vee}{L}^p_{mj}(\overline{L}^q_{\underline{pq}} - L^q_{\underline{pq}})\\ &+ \frac{2}{N+1}\underset{\vee}{L}^p_{pj}(\overline{L}^q_{\underline{mq}} - L^q_{\underline{mq}}) - \frac{2}{N+1}\underset{\vee}{L}^p_{pm}(\overline{L}^q_{\underline{jq}} - L^q_{\underline{jq}})\Big] - (N-1)\underset{1}{\psi}_{jm}\\ &+ \frac{2}{N+1}\underset{\vee}{L}^p_{mj}(\overline{L}^q_{\underline{pq}} - L^q_{\underline{pq}}) + \frac{2}{N+1}\underset{\vee}{L}^p_{pj}(\overline{L}^q_{\underline{mq}} - L^q_{\underline{mq}}),\end{aligned}$$

i.e.

$$\begin{aligned}(N^2-1)\underset{1}{\psi}_{jm} &= (N\underset{3}{R}_{jm} + \underset{3}{R}_{mj}) - (N\overline{\underset{3}{R}}_{jm} + \overline{\underset{3}{R}}_{mj}) + 2\underset{\vee}{L}^p_{mj}(\overline{L}^q_{\underline{pq}} - L^q_{\underline{pq}})\frac{N-1}{N+1}\\ &+ 2\underset{\vee}{L}^p_{pj}(\overline{L}^q_{\underline{mq}} - L^q_{\underline{mq}})\frac{N}{N+1} + \frac{2}{N+1}\underset{\vee}{L}^p_{pm}(\overline{L}^q_{\underline{jq}} - L^q_{\underline{jq}}).\end{aligned} \tag{32}$$

Substituting (32) into (29) using the condition (4) we obtain

$$\overline{\underset{3}{\mathcal{E}}}^i_{jmn} = \underset{3}{\mathcal{E}}^i_{jmn}, \tag{33}$$

where

$$\begin{aligned}\underset{3}{\mathcal{E}}^i_{jmn} = \underset{3}{R}^i_{jmn} &+ \frac{1}{N+1}\delta^i_j\underset{3}{R}_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m}\underset{3}{R}_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}\underset{3}{R}_{n]j}\\ &+ \frac{2}{(N-1)(N+1)^2}L^q_{\underline{jq}}\delta^i_{[n}\underset{\vee}{L}^p_{pm]} + \frac{2}{(N+1)^2}L^q_{\underline{[mq}}\delta^i_j L^p_{pn]}\\ &+ \frac{2N}{(N+1)^2(N-1)}L^q_{\underline{[mq}}\delta^i_{n]}\underset{\vee}{L}^p_{pj} - \frac{2}{(N+1)}L^q_{\underline{(mq}}\underset{\vee}{L}^i_{n)j}\\ &- \frac{2}{(N+1)^2}L^q_{\underline{pq}}[2(N-1)\delta^i_j\underset{\vee}{L}^p_{nm} + \delta^i_{[m}\underset{\vee}{L}^p_{n]j} + (N+1)\delta^i_m\underset{\vee}{L}^p_{jn}]\end{aligned} \tag{34}$$

The magnitude $\underset{3}{\mathcal{E}}^i_{jmn}$ is not a tensor and we call it **ET-projective parameter of the third kind**. In this case we have

THEOREM 3. *The ET-projective parameter of the third kind* $\underset{3}{\mathcal{E}}^i_{jmn}$ (34) *is an invariant of the equitorsion mapping* $f : GA_N \to G\overline{A}_N$. □

5. ET-projective parameter of the fourth kind.

For the curvature tensors of the fourth kind $\underset{4}{R}$ and $\overline{\underset{4}{R}}$ (8) of the space GA_N and $G\overline{A}_N$ respectively, we get

$$\overline{\underset{4}{R}}^i_{jmn} = \underset{4}{R}^i_{jmn} + \delta^i_{(j}\underset{2}{\psi}_{m)n} - \delta^i_{(j}\underset{1}{\psi}_{n)m} + 2\psi_{(m}\underset{\vee}{L}^i_{n)j}, \tag{35}$$

from where

$$\underset{4}{\overline{\mathcal{E}}}{}^i_{jmn} = \underset{4}{\mathcal{E}}{}^i_{jmn}, \tag{36}$$

where

$$\begin{aligned}\underset{4}{\mathcal{E}}{}^i_{jmn} = {} & \underset{4}{R}{}^i_{jmn} + \frac{1}{N+1}\delta^i_j \underset{4}{R}{}_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m} \underset{4}{R}{}_{jn]} + \frac{1}{N^2-1}\delta^i_{[m} \underset{4}{R}{}_{n]j} \\ & + \frac{2}{(N-1)(N+1)^2} L^q_{\underline{jq}}\delta^i_{[n} L^p_{\underset{\vee}{pm]}} - \frac{2}{(N+1)^2} L^q_{\underline{(mq}}\delta^i_j L^p_{pn)} \\ & + \frac{2N}{(N+1)^2(N-1)} L^q_{\underline{(mq}}\delta^i_{n)} L^p_{\underset{\vee}{pj}} - \frac{2}{(N+1)} L^q_{\underline{[mq}} L^i_{\underset{\vee}{n]j}} \\ & - \frac{2}{(N+1)^2} L^q_{\underline{pq}}[2(N-1)\delta^i_j L^p_{\underset{\vee}{nm}} + \delta^i_{[m} L^p_{\underset{\vee}{n]j}} + (N+1)\delta^i_m L^p_{\underset{\vee}{jn}}]\end{aligned} \tag{37}$$

The magnitude $\underset{4}{\mathcal{E}}{}^i_{jmn}$ is not a tensor and we call it **ET-projective parameter of the fourth kind**. In this case we have

THEOREM 4. *The ET-projective parameter of the fourth kind* $\underset{4}{\mathcal{E}}{}^i_{jmn}$ (37) *is an invariant of the equitorsion mapping* $f : GA_N \to G\overline{A}_N$. □

6. ET-projective curvature tensor.

For the curvature tensors $\underset{5}{R}$ and $\underset{5}{\overline{R}}$ (8) of the space GA_N and $G\overline{A}_N$ respectively, we find the relation

$$\underset{5}{\overline{R}}{}^i_{jmn} = \underset{5}{R}{}^i_{jmn} + \frac{1}{2}(\underset{1}{\psi}{}_{[mn]} + \underset{2}{\psi}{}_{[mn]}) + \frac{1}{2}\delta^i_{[m} \underset{1}{\psi}{}_{jn]} + \frac{1}{2}\delta^i_{[m} \underset{2}{\psi}{}_{jn]}. \tag{38}$$

Denoting

$$\underset{12}{\psi}{}_{jn} = \frac{1}{2}(\underset{1}{\psi}{}_{jn} + \underset{2}{\psi}{}_{jn}), \tag{39}$$

we can write (38) in the form

$$\underset{5}{\overline{R}}{}^i_{jmn} = \underset{5}{R}{}^i_{jmn} + \delta^i_j \underset{12}{\psi}{}_{[mn]} + \delta^i_{[m} \underset{12}{\psi}{}_{jn]}. \tag{40}$$

Eliminating $\underset{12}{\psi}{}_{mn}$ from (40) by analogous procedure as in the previous cases, one obtains

$$\underset{5}{\overline{\mathcal{E}}}{}^i_{jmn} = \underset{5}{\mathcal{E}}{}^i_{jmn}, \tag{41}$$

where

$$(42)\quad \underset{5}{\mathcal{E}}{}^i_{jmn} = \underset{5}{R}{}^i_{jmn} + \frac{1}{N+1}\delta^i_j \underset{5}{R}{}_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m}\underset{5}{R}{}_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}\underset{5}{R}{}_{n]j}.$$

In contrast to the previous cases, when $\underset{\theta}{\mathcal{E}}{}^i_{jmn}$ $(\theta = 1, ..., 4)$ are not tensors, the magnitude $\underset{5}{\mathcal{E}}{}^i_{jmn}$ is a tensor. We call it **ET-projective curvature tensor**. We see that the following is valid

THEOREM 5. *The ET-projective tensor* $\underset{5}{\mathcal{E}}{}^i_{jmn}$ (42) *is an invariant of the equitorsion mapping* $f : GA_N \to G\overline{A}_N$. □

7. Conclusion.

In the case of generalized Riemannian space (GR_N) the connection coefficients are defined by means of a non-symmetric basic tensor [3], [19], [20], [32] and they are non-symmetric too. ET-projective parameters $\underset{\theta}{E}{}^i_{jmn}$ and the projective tensor $\underset{5}{E}{}^i_{jmn}$, obtained as invariants of a map $f : GR_N \to G\overline{R}_N$ are particular cases of obtained here parameters $\underset{\theta}{\mathcal{E}}{}^i_{jmn}$ $(\theta = 1, \cdots 4)$ and of the tensor $\underset{5}{\mathcal{E}}{}^i_{jmn}$. For example,

$$\begin{aligned}\underset{1}{E}{}^i_{jmn} &= \underset{1}{R}{}^i_{jmn} + \frac{N}{N+1}\delta^i_j \underset{1}{R}{}_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m}\underset{1}{R}{}_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}\underset{1}{R}{}_{n]j}\\ &\quad - \frac{4}{N+1}L^p_{\underline{jp}}\underset{\vee}{L}{}^i_{mn} - \frac{2}{(N+1)^2}L^q_{\underline{pq}}\big[\delta^i_j \underset{\vee}{L}{}^p_{nm} + \delta^i_{[m}\underset{\vee}{L}{}^p_{n]j} + (N+1)\delta^i_j \underset{\vee}{L}{}^p_{mn}\big]\end{aligned}$$

When GA_N (GR_N) reduces to the Riemannian space, the magnitudes $\underset{\theta}{\mathcal{E}}(\underset{\theta}{E})$, $(\theta = 1, \cdots 5)$ reduce to the Weyl tensor [34]

$$W^i{}_{jmn} = R^i_{jmn} + \frac{1}{N-1}\delta^i_{[m}R_{jn]}.$$

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