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Signs in Weight Spectral Sequences, Monodromy-Weight Conjectures, Log Hodge Symmetry and Degenerations of Surfaces.

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ABSTRACT - In this paper we study signs in various weight spectral sequences. After these studies, we prove that the p-adic, the l-adic and the ∞-adic monodromy filtrations and the weight filtrations on the first log cohomologies of proper simple normal crossing log surfaces do not necessarily coincide. Conversely we prove that the log hard Lefschetz conjecture for the first log l-adic cohomology of a projective simple normal crossing log variety implies the coincidence of the two filtrations on the first log l-adic cohomology of it. We also study the log Hodge symmetry.

1. Introduction.

This paper is a continuation of my previous paper [Nakk3]. The paper [Nakk3] is mainly about the p-adic weight spectral sequence of a proper SNCL(=simple normal crossing log) variety over a log point s by the use of log de Rham-Witt complexes ([Mo], [Nakk3]) (SNCL variety is also called a strict semistable log scheme over s); this paper is about the l-adic and the ∞-adic weight spectral sequences of it ([Nak3], [FN]) as well as the p-adic weight spectral sequence of it. This paper except the Introduction consists of two parts.

In the Part I of this paper, we study signs in various weight spectral sequences and we discuss fundamental related topics.

In [Mo] Mokrane has constructed the p-adic weight spectral sequence of a proper SNCL variety over the log point of a perfect field of characteristic $p > 0$. Though some points in the construction are incomplete

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and mistaken, these have been completed and corrected in [Nakk3]. It has been thought that, in [St2], Steenbrink has constructed an analogue in a case where the base field is the complex number field. However the $\mathbb{Z}$-structure and the $\mathbb{Q}$-structure of the cohomological mixed Hodge complex in [loc. cit.] depend on the choice of local charts a priori. In [FN] (cf. [KwN]), Fujisawa and Nakayama have constructed a cohomological mixed $\mathbb{Q}$-Hodge complex in an intrinsic way, and they have proved that their cohomological mixed $\mathbb{Q}$-Hodge complex is isomorphic to Steenbrink's one over $\mathbb{Q}$. In [Nak3] Nakayama has constructed the l-adic weight spectral sequence of a proper SNCL variety over a log point.

Unfortunately, except the ∞-adic weight spectral sequence in [GN] and the p-adic weight spectral sequence in [Nakk3], there are some non-good choices, incomplete parts or mistakes in signs in the weight spectral sequences. Especially we correct the description of the boundary morphism between the E_1-terms of the l-adic weight spectral sequence of Rapoport-Zink ([RZ]).

Let κ be a perfect field of characteristic $p > 0$. Let s_p be the log point whose underlying scheme is $\mathrm{Spec}\,\kappa$. Let V be a complete discrete valuation ring of mixed characteristics with residue field κ. Endow $\mathrm{Spf}\,V$ and $\mathrm{Spec}\,V$ with the canonical log structures and denote the resulting log (formal) schemes by $(\mathrm{Spf}\,V)_{\mathrm{can}}$ and $(\mathrm{Spec}\,V)_{\mathrm{can}}$, respectively. Let X be a proper SNCL variety over s_p and denote by $\overset{\circ}{X}$ the underlying scheme of X. In the Part II of this paper, we study different and common points between the following three sets of objects with respect to the (non)coincidence of the monodromy filtration and the weight filtration on the log l-adic and the p-adic cohomologies of X:

(1) $\{X \mid$ there exists a formal proper strict semistable family $\mathcal{X}$ over $(\mathrm{Spf}\,V)_{\mathrm{can}}$ with canonical log structure such that $X \simeq \mathcal{X} \times_{(\mathrm{Spf}\,V)_{\mathrm{can}}} s_p\}$,

(See (6.2) below for the definition of the canonical log structure on the underlying formal scheme $\overset{\circ}{\mathcal{X}}$ over $\mathrm{Spf}\,V$.)

(2) $\{X \mid$ there exists an algebraic proper strict semistable family $\mathcal{X}$ over $(\mathrm{Spec}\,V)_{\mathrm{can}}$ with canonical log structure such that $X \simeq \mathcal{X} \times_{(\mathrm{Spec}\,V)_{\mathrm{can}}} s_p\}$,

(3) $\{X \mid \overset{\circ}{X}$ is projective over $\kappa\}$.

Let s_∞ be the log point whose underlying scheme is $\mathrm{Spec}\,\mathbb{C}$. Set $\varDelta := \{t \in \mathbb{C} \mid |t| < 1\}$, endow $\varDelta$ with the canonical log structure defined by the origin of $\varDelta$ and denote the resulting log analytic space by $\varDelta_{\mathrm{can}}$. We also have three analogous sets of objects over s_∞ and $\varDelta_{\mathrm{can}}$ by replacing "formal proper strict semistable family over $(\mathrm{Spf}\,V)_{\mathrm{can}}$" (resp. "algebraic proper strict semistable family over $(\mathrm{Spec}\,V)_{\mathrm{can}}$") in (1) (resp. (2)) by "analytic proper strict semistable family over $\Delta_{\rm can}$" (resp. "analytic proper strict semistable family which is obtained by the analytification of an algebraic proper strict semistable family over a smooth algebraic curve over $\mathbb{C}$").

Then we have the following table for the (non)coincidence of the monodromy filtration and the weight filtration:

(1.0.1)

coincidence	l-adic	p-adic	∞-adic
(1)	No in general	No in general	No in general
(2)	Yes if dim $\overset{\circ}{X} \leq 2$	Yes if dim $\overset{\circ}{X} \leq 2$	Yes
(3)	Yes if dim $\overset{\circ}{X} \leq 2$	Yes if dim $\overset{\circ}{X} \leq 1$	Yes

As to (2) and (3) in the general l-adic and p-adic cases, the coincidences have not yet been proved, which are, what is called, the l-adic and the p-adic monodromy-weight conjectures, though some affirmative results have been obtained ([D4], [RZ], [Fa], [Fu1], [Nak3], [It1], [It2], [Nakk2], [Nakk3], ...).

In this paper we give the answers for (1) in the table above for the l-adic and the p-adic cases for the case dim $\overset{\circ}{X} = 2$ for certain X's which have appeared in [Ue]: the monodromy filtration and the weight filtration on the 1st and the 3rd log l-adic etale and log crystalline cohomologies of X do not coincide. These X's are counter-examples of [Mo, 6.2.4] (in the ∞-adic case, the analogous noncoincidence for one of them is well-known ([Cl])). They are also counter-examples of Chiarellotto's conjecture ([Ch]).

Rapoport and Zink have given the answer for (2) for the l-adic case in the case dim $\overset{\circ}{X} \leq 2$ ([RZ]); Mokrane has given the answer for (2) for the p-adic case in the case dim $\overset{\circ}{X} \leq 2$ for the 0, 2, 4-th log crystalline cohomologies ([Mo]); we give the answer for the 1st and the 3rd log crystalline cohomologies (see also (6.8) (4) below).

The answer for (3) for the 0, 2, 4-th log l-adic cohomologies in the case dim $\overset{\circ}{X} \leq 2$ is given by the same proof as that in [RZ] (cf. [Mo]); we deduce the coincidence for (3) for the 1st and the 3rd log l-adic cohomologies of X from the log l-adic hard Lefschetz conjecture for the 1st log l-adic cohomology (see (9.5) below for the precise statement of the conjecture). Recently Kajiwara has proved the log l-adic hard Lefschetz conjecture for the 1st log l-adic cohomology ([Ka]). Consequently we know the answer for (3) for the l-adic case in the case dim $\overset{\circ}{X} \leq 2$. Though the answer for (3) for the p-adic case in the case dim $\overset{\circ}{X} = 2$ has not yet been given, we would like to discuss it in the future (In the case dim $\overset{\circ}{X} = 1$, we can check the coincidence directly as in [Mo] and [Nakk3] or we can reduce the coincidence to that for (2) by using log deformation theory in [Kk1].).

As to the ∞-adic case for (3), M. Saito's proof in [SaM] has given the answer (cf. [St2]); one can deduce the answer for (2) in the ∞-adic case from that for (3) by using Chow's lemma and the semistable reduction theorem.

Over s_∞, we give the following table for the log Hodge symmetry:

(1.0.2)

log Hodge symmetry	∞-adic
(1)	No in general
(2)	Yes
(3)	Yes

Here we replace "algebraic proper strict semistable family over $(\mathrm{Spec}\, V)_{\mathrm{can}}$" in (2) by "analytic proper strict semistable family over Δ_{can} whose fiber for any $t \in \Delta$ is algebraic".

Other results in this paper are the following.

We prove the l-adic and p-adic local invariant cycle theorems for the 1st log l-adic and p-adic cohomologies of a projective SNCL variety without using the log l-adic and p-adic hard Lefschetz conjectures. These include an affirmative answer to Chiarellotto's conjecture ([Ch]) for the 1st log p-adic cohomology of it. We also give a new proof of Fontaine's conjecture (=Coleman-Iovita's theorem ([CI])) on the criterion of the good reduction of an abelian variety over a local field of mixed characteristics.

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NOTATION. (1) For a log scheme (resp. log analytic space) X in the sense of Fontaine-Illusie-Kato ([Kk1], (resp. [KN])), we denote by $\overset{\circ}{X}$ the underlying scheme (resp. underlying analytic space) of X and by $\mathcal{M}_X$ the log structure of X.

(2) Following Friedman ([Fr]), for a morphism $X \longrightarrow S$ of log schemes, we denote by $\Lambda^i_{X/S}$ ($= \omega^i_{X/S}$ in [Kk1]) the sheaf of the relative logarithmic differential forms on X/S of degree i ($i \in \mathbb{N}$).

(3) (S)NC(L)=(simple) normal crossing (log).

CONVENTIONS. We make the following conventions about signs (cf. [BBM], [Co]).

Let $\mathcal{A}$ be an exact additive category.

(1) For a complex $(E^\bullet, d^\bullet)$ of objects in $\mathcal{A}$ and for an integer n, $(E^\bullet\{n\}, d^\bullet\{n\})$ denotes the following complex:

$$\underset{}{\cdots \longrightarrow} \underset{q-1}{E^{q-1+n}} \overset{d^{q-1+n}}{\longrightarrow} \underset{q}{E^{q+n}} \overset{d^{q+n}}{\longrightarrow} \underset{q+1}{E^{q+1+n}} \overset{d^{q+1+n}}{\longrightarrow} \cdots.$$

Here the numbers under the objects above in $\mathcal{A}$ mean the degrees.

For a morphism $f\colon (E^\bullet, d^\bullet_E) \longrightarrow (F^\bullet, d^\bullet_F)$ of complexes, $f\{n\}$ denotes a natural morphism $(E^\bullet\{n\}, d^\bullet_E\{n\}) \longrightarrow (F^\bullet\{n\}, d^\bullet_F\{n\})$ induced by f. This operation is well-defined in derived categories: for a morphism $f\colon (E^\bullet, d^\bullet_E) \longrightarrow (F^\bullet, d^\bullet_F)$ in the derived category $\mathrm{D}^\star(\mathcal{A})$ ($\star = \mathrm{b}, +, -,$ nothing) of complexes of objects in $\mathcal{A}$, there exists a naturally induced morphism $f\{n\}\colon (E^\bullet\{n\}, d^\bullet_E\{n\}) \longrightarrow (F^\bullet\{n\}, d^\bullet_F\{n\})$ in $\mathrm{D}^\star(\mathcal{A})$.

(2) For a complex $(E^\bullet, d^\bullet)$ of objects in $\mathcal{A}$ and for an integer n, $(E^\bullet[n], d^\bullet[n])$ denotes the following complex as usual: $(E^\bullet[n])^q := E^{q+n}$ with boundary morphisms $d^\bullet[n] = (-1)^n d^{\bullet+n}$.

For a morphism $f\colon (E^\bullet, d^\bullet_E) \longrightarrow (F^\bullet, d^\bullet_F)$ of complexes, $f[n]$ denotes a natural morphism $(E^\bullet[n], d^\bullet_E[n]) \longrightarrow (F^\bullet[n], d^\bullet_F[n])$ induced by f without change of signs. This operation is well-defined in derived categories as in (1).

(3) ([BBM, 0.3.2], [Co, (1.3.2)]) For a short exact sequence

$$0 \longrightarrow (E^\bullet, d^\bullet_E) \overset{f}{\longrightarrow} (F^\bullet, d^\bullet_F) \overset{g}{\longrightarrow} (G^\bullet, d^\bullet_G) \longrightarrow 0$$

of bounded below complexes of objects in $\mathcal{A}$, let $\mathrm{MC}(f) := (E^\bullet[1], d^\bullet_E[1]) \oplus (F^\bullet, d^\bullet_F)$ be the mapping cone of f. We fix an isomorphism "$(E^\bullet[1], d^\bullet_E[1]) \oplus (F^\bullet, d^\bullet_F) \ni (x, y) \longmapsto g(y) \in (G^\bullet, d^\bullet_G)$" in the derived category $\mathrm{D}^+(\mathcal{A})$.

Let $\mathrm{MF}(g) := (F^\bullet, d^\bullet_F) \oplus (G^\bullet[-1], d^\bullet_G[-1])$ be the mapping fiber of g. We fix an isomorphism "$(E^\bullet, d^\bullet_E) \ni x \longmapsto (f(x), 0) \in (F^\bullet, d^\bullet_F) \oplus (G^\bullet[-1], d^\bullet_G[-1])$" in the derived category $\mathrm{D}^+(\mathcal{A})$.

(4) ([BBM, 0.3.2], [Co, (1.3.3)]) In the situation (3), the boundary morphism $(G^\bullet, d_G^\bullet) \longrightarrow (E^\bullet[1], d_E^\bullet[1])$ in $\mathrm{D}^+(\mathcal{A})$ is the following composite morphism

$$(G^\bullet, d_G^\bullet) \overset{\sim}{\longleftarrow} \mathrm{MC}(f) \overset{\text{proj.}}{\longrightarrow} (E^\bullet[1], d_E^\bullet[1]) \overset{(-1)\times}{\longrightarrow} (E^\bullet[1], d_E^\bullet[1]).$$

More generally, we use only the similar boundary morphism for a triangle in a derived category; we do not use the classical boundary morphism in e.g., [Ha1] of a triangle.

(5) Assume that $\mathcal{A}$ has enough injectives. Let $\mathcal{F}\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor of abelian categories. Then, in the situation (3), the boundary morphism $\partial\colon R^q\mathcal{F}((G^\bullet, d_G^\bullet)) \longrightarrow R^{q+1}\mathcal{F}((E^\bullet, d_E^\bullet))$ of cohomologies is, by definition, the induced morphism by the morphism $(G^\bullet, d_G^\bullet) \longrightarrow (E^\bullet[1], d_E^\bullet[1])$ in (4). By taking injective resolutions $(I^\bullet, d_I^\bullet)$, $(J^\bullet, d_J^\bullet)$ and $(K^\bullet, d_K^\bullet)$ of $(E^\bullet, d_E^\bullet)$, $(F^\bullet, d_F^\bullet)$ and $(G^\bullet, d_G^\bullet)$, respectively, which fit into the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (I^\bullet, d_I^\bullet) & \longrightarrow & (J^\bullet, d_J^\bullet) & \longrightarrow & (K^\bullet, d_K^\bullet) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & (E^\bullet, d_E^\bullet) & \longrightarrow & (F^\bullet, d_F^\bullet) & \longrightarrow & (G^\bullet, d_G^\bullet) & \longrightarrow & 0 \end{array} \tag{1.0.3}$$

of complexes of objects in $\mathcal{A}$ such that the upper horizontal sequence is exact, it is easy to check that the boundary morphism ∂ above is equal to the traditional boundary morphism obtained by the lower short exact sequence of (1.0.3). (For a short exact sequence in (3), the existence of the commutative diagram (1.0.3) has been proved in, e.g., [NS, (2.7)] as a very special case.)

(6) For a complex $(E^\bullet, d^\bullet)$ of objects in $\mathcal{A}$, the identity $\mathrm{id}\colon E^q \longrightarrow E^q$ $(\forall q \in \mathbb{Z})$ induces an isomorphism $\mathcal{H}^q((E^\bullet, -d^\bullet)) \overset{\sim}{\longrightarrow} \mathcal{H}^q((E^\bullet, d^\bullet))$ $(\forall q \in \mathbb{Z})$ of cohomologies. We sometimes use this convention.

(7) We often denote a complex $(E^\bullet, d^\bullet)$ simply by $(E^\bullet, d)$ or $E^\bullet$ as usual when there is no danger of confusion.

(8) Let $(\mathcal{T}, \mathcal{R})$ be a ringed topos. For two complexes $(E^\bullet, d_E^\bullet)$ and $(F^\bullet, d_F^\bullet)$ of $\mathcal{R}$-modules, the tensor product of $(E^\bullet, d_E^\bullet)$ and $(F^\bullet, d_F^\bullet)$ is defined as follows as usual:

$$(E^\bullet \otimes F^\bullet)^n := \bigoplus_{p+q=n} E^p \otimes_{\mathcal{R}} F^q \quad (n \in \mathbb{Z})$$

with a boundary morphism $d^n|_{(E^p \otimes_{\mathcal{R}} F^q)} := d_E^p \otimes \mathrm{id}_{F^q} + (-1)^p \mathrm{id}_{E^p} \otimes d_F^q$.

(9) ([BBM, p. 4], [Co, p. 10]) Let $(\mathcal{T}, \mathcal{R})$ be a ringed topos. For two complexes $(E^\bullet, d_E^\bullet)$ and $(F^\bullet, d_F^\bullet)$ of $\mathcal{R}$-modules, set

$$Hom_{\mathcal{R}}^n(E^\bullet, F^\bullet) := \prod_q Hom_{\mathcal{R}}(E^q, F^{q+n}).$$

Then $Hom^{\bullet}_{\mathcal{R}}(E^{\bullet},F^{\bullet})$ becomes a complex with the following boundary morphism:

$$(1.0.4)\quad d^n := \prod_q ((-1)^{n+1}d_E^q + d_F^{q+n})\colon Hom^n_{\mathcal{R}}(E^{\bullet},F^{\bullet}) \longrightarrow Hom^{n+1}_{\mathcal{R}}(E^{\bullet},F^{\bullet}).$$

(Note that this boundary morphism is different from that in [Ha1, p. 64].) For a morphism $f\colon(\mathcal{T},\mathcal{R})\longrightarrow(\mathcal{T}',\mathcal{R}')$ of ringed topoi and for a bounded above complex $E^{\bullet}$ of $\mathcal{R}'$-modules (resp. a bounded below complex $F^{\bullet}$ of $\mathcal{R}$-modules), there exists a canonical isomorphism

$$(1.0.5)\qquad \mathrm{RHom}_{\mathcal{R}}(Lf^*(E^{\bullet}),F^{\bullet}) \xrightarrow{=} \mathrm{RHom}_{\mathcal{R}'}(E^{\bullet},Rf_*(F^{\bullet}))$$

in $\mathrm{D}^+(\Gamma(\mathcal{T}',\mathcal{R}'))$ by the proof of [B, V Proposition 3.3.1].

(10) ([Co, (1.3.15)]) Let A be a not necessarily commutative ring. Let M and N be left A-modules. Let $\{(P^{\bullet},d_P^{\bullet})\}_{\bullet\leq 0}$ (resp. $\{(I^{\bullet},d_I^{\bullet})\}_{\bullet\geq 0}$) be a projective (resp. injective) resolution of M (resp. N). Then we have two natural isomorphisms

$$(1.0.6)\qquad \mathrm{Hom}^{\bullet}_A(P^{\bullet},N) \xrightarrow{\sim} \mathrm{Hom}^{\bullet}_A(P^{\bullet},I^{\bullet}) \xleftarrow{\sim} \mathrm{Hom}^{\bullet}_A(M,I^{\bullet})$$

in the derived category $\mathrm{D}(A)$ of complexes of left A-modules. Following ([Co, (1.3.15)]), we fix an isomorphism

$$(1.0.7)\qquad \mathrm{Hom}_A(P^{\bullet},N) \xrightarrow{\sim} \mathrm{Hom}^{\bullet}_A(P^{\bullet},N)$$

of complexes of A-modules as follows:

$$\mathrm{Hom}_A(P^{-n},N)\ni f\longmapsto(-1)^{n(n+1)/2}f\in\mathrm{Hom}^n_A(P^{\bullet},N)=\mathrm{Hom}_A(P^{-n},N)\ (n\in\mathbb{N}).$$

Part I. Signs in weight spectral sequences.

2. Weight spectral sequences.

In this section we recall four weight spectral sequences of a proper SNCL variety over a log point: the l-adic weight spectral sequence, the p-adic weight spectral sequence by the use of log de Rham-Witt complexes and two ∞-adic weight spectral sequences in [FN]. We have to pay careful attention to the identification of the E_1-terms of the weight spectral sequences with classical cohomologies. In the l-adic case, two identifications are possible by cycle classes of smooth divisors and by the log Kummer sequence (see (5.16) below for the two identifications); in the p-adic case, an identification is possible by the use of the Poincaré residue isomorphism. In the ∞-adic case, three identifications are possible by the methods above. These three methods naturally give different descriptions of the boundary morphisms between the E_1-terms of the weight spectral sequences. Because we would like to give the same description of the boundary morphisms and because we use the Poincaré residue isomorphism as the most basic tool as in [D2] and [Nakk3], we twist the other identifications by signs in the l-adic and the ∞-adic cases.

(A) l-adic case

Let κ be a field of characteristic $p \geq 0$ and let $s := (\operatorname{Spec} \kappa, \mathcal{M}_s)$ be a log point with standard chart $\mathbb{N} \ni 1 \longmapsto 0 \in \kappa$. Let $l \neq p$ be a prime number. Let X/s be a proper SNCL variety (see [Nakk1, §2] for the definition; in this paper, we do not assume that $\overset{\circ}{X}$ is of pure dimension, geometrically connected, nor that the irreducible components of $\overset{\circ}{X}$ are geometrically irreducible.). Set $\overline{X} := X \otimes_{\kappa} \kappa_{\rm sep}$. Let $\overset{\circ}{X}{}^{(r)}$ (resp. $\overset{\circ}{\overline{X}}{}^{(r)}$) ($r \in \mathbb{Z}_{\geq 1}$) be the disjoint union of all r-fold intersections of the different irreducible components of $\overset{\circ}{X}$ (resp. $\overline{X}$). Then $\overline{X}$ is a SNCL variety over $\overline{s}$. Set also $X_{\overline{s}} := \varprojlim_{m \in \mathbb{N}} \varprojlim (\overline{X} \otimes_{\mathbb{Z}[\mathbb{N}]} \mathbb{Z}[\mathbb{N}^{1/l^m}])$ by abuse of notation, and

$$H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Z}_l) := \varprojlim_{n \in \mathbb{N}} H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Z}/l^n) \quad (h \in \mathbb{Z})$$

and

$$H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l) := H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \quad (h \in \mathbb{Z}).$$

See [Nak1] for the definition of the log etale cohomology.

Let $\overline{X}{}^{\log}_{\rm et}$ be the log etale site of $\overline{X}$ and $\widetilde{\overline{X}}{}^{\log}_{\rm et}$ the log etale topos of $\overline{X}$ ([Nak1, (2.2)]). Let $\varepsilon = \varepsilon_{\rm et} \colon \widetilde{\overline{X}}{}^{\log}_{\rm et} \longrightarrow \widetilde{\overset{\circ}{\overline{X}}}_{\rm et}$ be the forgetting log morphism of topoi. Fix a total order on the irreducible components of $\overset{\circ}{\overline{X}}$. Then, by the proof of [Nak3, (1.8.3)], we have an isomorphism $R^r\varepsilon_*(\mathbb{Z}/l^n) \overset{\sim}{\longrightarrow} (\mathbb{Z}/l^n)(-r)_{\overset{\circ}{\overline{X}}{}^{(r)}}$. Here, in the target of this isomorphism, we omit the notation of the direct image of a natural morphism $\overset{\circ}{\overline{X}}{}^{(r)} \longrightarrow \overset{\circ}{\overline{X}}$ for simplicity of notation. Let us recall this isomorphism.

Let m be a positive integer which is prime to p. Let $\mathcal{M}_{\overline{X},\log}$ be a sheaf of monoids on $\overline{X}{}^{\log}_{\rm et}$ which is associated to the presheaf $U \longmapsto \Gamma(U, \mathcal{M}_U)$ ($U \in \overline{X}{}^{\log}_{\rm et}$) ([KN, p. 169]). Let $\mathcal{M}_{\overline{X}}$ be the log structure of $\overline{X}$ in $\widetilde{\overset{\circ}{\overline{X}}}_{\rm et}$. Then the log Kummer sequence

$$(2.0.1; m) \qquad 0 \longrightarrow (\mathbb{Z}/m)(1) \longrightarrow \mathcal{M}^{\rm gp}_{\overline{X},\log} \overset{m}{\longrightarrow} \mathcal{M}^{\rm gp}_{\overline{X},\log} \longrightarrow 0$$

in $\widetilde{\overline{X}}{}^{\log}_{\text{et}}$ ([KN, (2.3)]) gives an isomorphism

$$(2.0.2; m) \qquad (\mathcal{M}^{\text{gp}}_{\overline{X}}/\mathcal{O}^*_{\overline{X}}) \otimes_{\mathbb{Z}} (\mathbb{Z}/m)(-1) \xrightarrow{\sim} R^1\varepsilon_*(\mathbb{Z}/m).$$

Furthermore, the cup product $R^1\varepsilon_*(\mathbb{Z}/m)^{\otimes r} \longrightarrow R^r\varepsilon_*(\mathbb{Z}/m)$ induces an isomorphism $\bigwedge^r R^1\varepsilon_*(\mathbb{Z}/m) \xrightarrow{\sim} R^r\varepsilon_*(\mathbb{Z}/m)$. The isomorphism $R^r\varepsilon_*(\mathbb{Z}/m) \xrightarrow{\sim} (\mathbb{Z}/m)(-r)_{\overset{\circ}{\overline{X}}{}^{(r)}}$ in the proof of [Nak3, (1.8.3)] is the inverse isomorphism of the following composite isomorphism

$$(2.0.3; m) \quad (\mathbb{Z}/m)(-r)_{\overset{\circ}{\overline{X}}{}^{(r)}} \xrightarrow{\sim} \bigwedge^r\{(\mathcal{M}^{\text{gp}}_{\overline{X}}/\mathcal{O}^*_{\overline{X}}) \otimes_{\mathbb{Z}} (\mathbb{Z}/m)(-1)\} \xrightarrow{\sim} \bigwedge^r R^1\varepsilon_*(\mathbb{Z}/m) \xrightarrow{\sim} R^r\varepsilon_*(\mathbb{Z}/m).$$

By [Nak3, (1.4)] we have the following spectral sequence

$$(2.0.4; l) \quad E_{1,l}^{-k,h+k} = \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}_{\text{et}}(\overset{\circ}{\overline{X}}, R^{2j+k+1}\varepsilon_*(\mathbb{Z}/l^n)(j+1)) \Longrightarrow H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Z}/l^n)$$

for a positive integer n. Using the identification $(2.0.3; l^n)$ and taking the projective limit with respect to n, we obtain the following l-adic weight spectral sequence of X/s:

$$(2.0.5; l) \quad E_{1,l}^{-k,h+k} = \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}_{\text{et}}(\overset{\circ}{\overline{X}}{}^{(2j+k+1)}, \mathbb{Z}_l)(-j-k) \Longrightarrow H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Z}_l).$$

However we do not use this spectral sequence in this paper. Instead of using $(2.0.3; l^n)$, we use the following isomorphism

$$(2.0.6; l) \qquad (\mathbb{Z}/l^n)(-2j-k-1)_{\overset{\circ}{\overline{X}}{}^{(2j+k+1)}} \overset{(-1)^{j+k}}{\xrightarrow{\sim}} (\mathbb{Z}/l^n)(-2j-k-1)_{\overset{\circ}{\overline{X}}{}^{(2j+k+1)}} \overset{(2.0.3;l^n)}{\xrightarrow{\sim}} R^{2j+k+1}\varepsilon_*(\mathbb{Z}/l^n)$$

for the sheaf $R^{2j+k+1}\varepsilon_*(\mathbb{Z}/l^n)$ in the direct factor of the E_1-term of $(2.0.4; l)$ and we use the following l-adic weight spectral sequence of X/s by the use of the identification $(2.0.6; l)$:

$$(2.0.7; l) \quad E_{1,l}^{-k,h+k} = \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}_{\text{et}}(\overset{\circ}{\overline{X}}{}^{(2j+k+1)}, \mathbb{Z}_l)(-j-k) \Longrightarrow H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Z}_l).$$

In (5.16.5) below, the reader shall know the reason for making the twist by the sign $(-1)^{j+k}$ by remembering the introduction in this section. Tensorizing (2.0.7; l) with $\mathbb{Q}_l$, we obtain the following l-adic weight spectral sequence of X/s:

$$(2.0.8;l)\quad E_{1,l}^{-k,h+k} = \bigoplus_{j\geq \max\{-k,0\}} H_{\rm et}^{h-2j-k}(\overset{\circ}{X}{}^{(2j+k+1)}, \mathbb{Q}_l)(-j-k) \Longrightarrow H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l).$$

The weight spectral sequence (2.0.8; l) is isomorphic to (2.0.5; l)$\otimes_{\mathbb{Z}_l}\mathbb{Q}_l$. By [Nak3, (2.1)], (2.0.8; l) degenerates at E_2.

Let T be a basis of $\mathbb{Z}_l(1)$ and $\check{T} \in \mathbb{Z}_l(-1)$ the dual basis of T. On $H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Z}_l)$, there exists an operator $\upsilon_l := (T - \mathrm{id}) \otimes \check{T}\colon H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Z}_l) \longrightarrow$ $\longrightarrow H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Z}_l)(-1)$ ([Nak3, (2.4) (1)], cf. [Il2, p. 39]), which we call the *l-adic quasi-monodromy operator*. By using the l-adic Steenbrink complex of X/s (§5 below) and by interpreting υ_l by an endomorphism of it (cf. the p-adic and ∞-adic cases below), we see that υ_l is nilpotent. Hence, on $H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l)$, the l-adic monodromy operator

$$N_l := \log T \otimes \check{T}\colon H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l) \longrightarrow H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l)(-1)$$

is well-defined.

K. Kato has conjectured the following, which we call the *l-adic monodromy weight conjecture* ([Nak3, (2.4) (2)]) (this is a generalization of a well-known conjecture):

(2.0.9; l): If $\overset{\circ}{X}$ is projective over κ, then the monodromy filtration on $H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l)$ and the weight filtration on $H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l)$ coincide. That is, the morphism

$$N_l^r\colon \mathrm{gr}^P_{h+r} H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l) \longrightarrow \mathrm{gr}^P_{h-r} H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l)(-r) \quad (r \in \mathbb{Z}_{\geq 0})$$

is an isomorphism. Here P is the weight filtration associated to the spectral sequence (2.0.8; l) such that $\mathrm{gr}^P_k H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l) = E_{\infty,l}^{h-k,k}$.

Nakayama has reduced (2.0.9;l) for any base field κ to (2.0.9;l) for a finite field ([loc. cit., (2.4) (4)]).

(B) p-adic case

Assume that κ is a perfect field of characteristic $p > 0$. Let W be the Witt ring of κ and K_0 the fraction field of W. Let $s = (\mathrm{Spec}\,\kappa, \mathcal{M}_s)$ be a log point and let X/s be a proper SNCL variety. Let $W\Lambda^\bullet_X$ be the "reverse" log de Rham-Witt complex of X defined in [Hy] and [HK] and denoted by $W\omega^\bullet_X$ in [loc. cit.]. Then, in [Nakk3, (7.19)], I have completed the proof of [HK, (4.19)] which claims, as a special case, that there exists a canonical morphism

$$(2.0.1;p) \qquad H^h_{\text{log-crys}}(X/W) \longrightarrow H^h(X, WA^\bullet_X)$$

and that it is an isomorphism. See [Nakk3, (7.19)] for details.

Let WA^{ij}_X $(i,j \in \mathbb{N})$ be a $W(\mathcal{O}_X)$-module defined in [Mo, 3.8] with the filtration $P = \{P_k\}_{k\in\mathbb{Z}}$ which is the projective limit of the filtration in [Mo, 3.21]; to take the projective limit of the filtration is a nontrivial operation; I have completed it in [Nakk3, §8]. By following the proof of [RZ, (1.7)], our boundary morphisms of a p-adic double Steenbrink complex $WA^{\bullet\bullet}_X$ are as follows:

$$(2.0.2;p) \qquad \begin{array}{ccc} WA^{i,j+1}_X & & \\ {\scriptstyle (-1)^i\theta\wedge}\big\uparrow & & \\ WA^{ij}_X & \xrightarrow{(-1)^{j+1}d} & WA^{i+1,j}_X. \end{array}$$

Let $WA^\bullet_X$ be the p-adic single Steenbrink complex associated to $WA^{\bullet\bullet}_X$. Let $W\widetilde{\Lambda}^i_X$ $(i \in \mathbb{N})$ be a $W(\mathcal{O}_X)$-module defined in [Hy, (1.2)] and [Mo, 2.3], and denoted by $W\widetilde{\omega}^i_X$ in [loc. cit.]. Let $\theta \in \Gamma(X, W\widetilde{\Lambda}^1_X)$ be a global section defined in [Hy, (1.2.2)], [Mo, 3.4 (3)] and [Nakk3, (8.1.3)]. Then, by [Mo, Lemme 3.15.1], the morphism $\theta\wedge\colon W\widetilde{\Lambda}^\bullet_X \longrightarrow WA^\bullet_X$ of complexes induces a quasi-isomorphism

$$\theta\wedge\colon W\Lambda^\bullet_X \xrightarrow{\sim} WA^\bullet_X.$$

(For the proof of [Mo, Lemme 3.15.1], we need the second isomorphism in [Mo, 1.3.3]; in [Nakk3, (6.28) (7)] we have given the precise proof for the second isomorphism in [loc. cit.].) Hence we have a composite isomorphism

$$(2.0.3;p) \qquad H^h_{\text{log-crys}}(X/W) \xrightarrow{\sim} H^h(X, W\Lambda^\bullet_X) \xrightarrow[\sim]{\theta\wedge} H^h(X, WA^\bullet_X).$$

Let us also recall the p-adic weight spectral sequence of X/s in [Nakk3, (9.11.1)], which is a correction of the p-adic weight spectral sequence in [Mo, 3.23] (The construction of the weight spectral sequence in [Mo, 3.23] is incomplete. See [Nakk3, §6∼11] for details (cf. (2.2) (1) below).).

Consider the following exact sequence

$$(2.0.4;p) \qquad 0 \longrightarrow \mathrm{gr}^P_{k-1}WA^\bullet_X \longrightarrow (P_k/P_{k-2})WA^\bullet_X \longrightarrow \mathrm{gr}^P_k WA^\bullet_X \longrightarrow 0.$$

Fix a total order on the irreducible components of $\overset{\circ}{X}$. Then we have

$$(2.0.5;p) \qquad \mathrm{gr}^P_k WA^\bullet_X \xrightarrow[\sim]{\mathrm{Res}} \bigoplus_{j\geq\max\{-k,0\}} (W\Omega^\bullet_{\overset{\circ}{X}{}^{(2j+k+1)}}(-j-k), (-1)^{j+1}d)\{-2j-k\}$$

by [Nakk3, (9.9)]. Hence we have

$$(2.0.6;p)\ H^h(X,\mathrm{gr}_k^PWA_X^\bullet)=\bigoplus_{j\geq\max\{-k,0\}}H^{h-2j-k}(X,(W\Omega^\bullet_{\overset{\circ}{X}{}^{(2j+k+1)}},(-1)^{j+1}d))(-j-k)$$
$$=\bigoplus_{j\geq\max\{-k,0\}}H^{h-2j-k}(X,W\Omega^\bullet_{\overset{\circ}{X}{}^{(2j+k+1)}})(-j-k)$$
$$=\bigoplus_{j\geq\max\{-k,0\}}H^{h-2j-k}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k+1)}/W)(-j-k).$$

Here we have obtained the middle equality by using the Convention (6). Using the identification $(2.0.6;p)$, we have the following p-adic weight spectral sequence of X/s:

$$(2.0.7;p)\ \ E_{1,p}^{-k,h+k}=\bigoplus_{j\geq\max\{-k,0\}}H^{h-2j-k}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k+1)}/W)(-j-k)\Longrightarrow H^h_{\text{log-crys}}(X/W).$$

Tensorizing $(2.0.7;p)$ with K_0, we obtain the following p-adic weight spectral sequence of X/s:

$$(2.0.8;p)\ \ E_{1,p}^{-k,h+k}=\bigoplus_{j\geq\max\{-k,0\}}H^{h-2j-k}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k+1)}/W)(-j-k)\otimes_W K_0\Longrightarrow$$
$$H^h_{\text{log-crys}}(X/W)\otimes_W K_0.$$

Mokrane has practically conjectured that $(2.0.8;p)$ degenerates at E_2 ([Mo, 3.24]). If κ is a finite field, one can immediately prove this conjecture by using the purity of the weight [CL, (1.2)] or [Nakk3, (2.2) (4)] as in [Mo, 3.32]. In [Nakk3, (3.6)] we have generalized this fact: $(2.0.8;p)$ degenerates at E_2 for any perfect field κ of characteristic $p>0$.

By following [Nakk3, (11.3.6)] (cf. [HK, (3.6)]), let

$$N_{p,n}\colon H^h_{\text{log-crys}}(X/W_n)\longrightarrow H^h_{\text{log-crys}}(X/W_n)(-1)$$

be the boundary morphism obtained from the following triangle

$$R\theta_*(C_{X/(W_n,W_n(\mathcal{M}_s))})(-1)[-1]\overset{d\log t\wedge}{\longrightarrow}R\theta_*(W_n\otimes_{W_n\langle t\rangle}C_{X/W_n})\longrightarrow$$
$$R\theta_*(C_{X/(W_n,W_n(\mathcal{M}_s))})\overset{+1}{\longrightarrow}$$

(see the notation in [HK, (3.6)]; the θ above is not the θ in this paper). Here we have used the Convention (4) and (5). Let

$$N_p\colon H^h_{\text{log-crys}}(X/W)\longrightarrow H^h_{\text{log-crys}}(X/W)(-1)$$

be the p-adic monodromy operator obtained from $\{N_{p,n}\}_{n\in\mathbb{Z}_{>0}}$ (cf.

[HK, (3.6)]). Let $\upsilon_p\colon WA^\bullet_X \longrightarrow WA^\bullet_X(-1)$ be a morphism of complexes defined by a family of morphisms $(-1)^{i+j+1}\mathrm{proj.}\colon WA^{ij}_X \longrightarrow WA^{i-1,j+1}_X$ $(i,j \in \mathbb{N})$ ([Nakk3, §11], cf. [St1, (4.22)], [RZ, (1.6.2)]); the morphism υ in [Mo, 3.13] is *not* a morphism of complexes if $\dim \overset{\circ}{X} \geq 2$; see [Nakk3, (11.9)] for more details). Denote also by υ_p the induced morphism

$$H^h(X, WA^\bullet_X) \longrightarrow H^h(X, WA^\bullet_X)(-1) \tag{2.0.8.1; p}$$

of cohomologies. Then, under the identification (2.0.3; p), $N_p = \upsilon_p$ ([Nakk3, (11.4) (2), (11.10)]). In [Nakk3, (11.7)] we have proved the following: let k be a positive integer. Under the identification (2.0.6; p), the induced isomorphism

$$\upsilon^k_p\colon \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}(X, W\Omega^\bullet_{\overset{\circ}{X}{}^{(2j+k+1)}})(-j-k) \overset{\sim}{\longrightarrow} \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}(X, W\Omega^\bullet_{\overset{\circ}{X}{}^{(2j+k+1)}})(-j-k) \tag{2.0.8.2; p}$$

by the isomorphism $\upsilon^k_p\colon \mathrm{gr}^P_k WA^\bullet_X \overset{\sim}{\longrightarrow} \mathrm{gr}^P_{-k} WA^\bullet_X(-k)$ is the identity if k is even and $(-1)^{h+1}$ if k is odd. In the l-adic case and the ∞-adic case stated later, one can prove the obvious analogues.

Let G be the sum with signs of Gysin morphisms defined in [Mo, 4.10] and ρ the sum with signs of induced morphisms by closed immersions defined in [Mo, 4.12]. Then the boundary morphism between the E_1-terms of (2.0.7; p) is as follows ([Nakk3, (10.1)]):

$$\sum_{j\geq \max\{-k,0\}} \{(-1)^j G + (-1)^{j+k}\rho\}\colon \tag{2.0.8.3; p}$$

$$E^{-k,h+k}_{1,p} = \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k+1)}/W)(-j-k)$$

$$\longrightarrow E^{-k+1,h+k}_{1,p} = \bigoplus_{j\geq \max\{-k+1,0\}} H^{h-2j-k+2}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k)}/W)(-j-k+1).$$

Here we omit the shift $\{*\}$ in [loc. cit.] for the morphisms G and ρ. Note that [Mo, 4.13] is wrong and the proof of it is incomplete; see [Nakk3, (10.2) (4), (5)] for more details.

The following conjecture is the p-adic analogue of (2.0.9;l) (cf. [Mo, 3.27]):

(2.0.9;p): If $\overset{\circ}{X}$ is projective over κ, then the monodromy filtration on $H^h_{\text{log-crys}}(X/W) \otimes_W K_0$ and the weight filtration on $H^h_{\text{log-crys}}(X/W) \otimes_W K_0$ coincide.

(C) ∞-adic case

Let $s := ((\operatorname{Spec} \mathbb{C})_{\rm an}, \mathbb{N} \oplus \mathbb{C}^*)$ be a log point and let X/s be a proper SNCL analytic variety. Though it has been thought that Steenbrink has first constructed the weight spectral sequence of X/s in [St2], we follow the formulation in [FN] which stems from that in [RZ] and [SZ] (cf. [KwN]) because some constructions in [St2] obviously depend on the choice of local charts of X (e.g., $L^1_{\underline{D}}$ in [St2, (4.5)]); Steenbrink's construction alone is incomplete because we cannot discuss, for example, the functoriality of his weight spectral sequence; the $\mathbb{Z}$-structure (resp. $\mathbb{Q}$-structure) of the Steenbrink complex may (resp. might) depend on the choice of local charts of X; in [FN], Fujisawa and Nakayama have shown that Steenbrink's construction over $\mathbb{Q}$ does not depend on the choice of the local charts by constructing an isomorphism from his complex over $\mathbb{Q}$ to their complex.

Let us first construct a spectral sequence over $\mathbb{Z}$ which is closely related to the weight spectral sequence over $\mathbb{Q}$ in [FN]. We change some signs in some morphisms of complexes in [SZ] and [FN] to obtain the same description of the boundary morphism of the ∞-adic weight spectral sequences $(2.0.7;\infty)$ and (2.1.10) below as that of the l-adic weight spectral sequence $(2.0.5;l)$; see (5.5) and (5.10) below.

Let $\varepsilon = \varepsilon_{\rm top}\colon X^{\log} \longrightarrow \overset{\circ}{X}$ be the real blow up of X ([KN, (1.2)], (cf. [KwN, p. 404–405]) (we do not use the notation τ in [KN] for the real blow up because we have to use the standard notation τ for the canonical filtration later (see, e.g., (4.5.2) below)): as a set,

$$X^{\log} = \{(x,h) | x \in X, h \in \operatorname{Hom}_{\rm gp}(\mathcal{M}^{\rm gp}_{X,x}, \mathbb{S}^1) \text{ such that } h(f) = \frac{f(x)}{|f(x)|} (f \in \mathcal{O}^*_{X,x})\},$$

where $\mathbb{S}^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. Then there is a natural map $X^{\log} \longrightarrow \mathbb{S}^1$ of topological spaces. Let $\mathbb{R} \ni x \longmapsto \exp(2\pi\sqrt{-1}x) \in \mathbb{S}^1$ be the universal cover of $\mathbb{S}^1$. Take the fiber product $X_\infty := X^{\log} \times_{\mathbb{S}^1} \mathbb{R}$ ([Us], cf. [KwN, pp. 404–405]). Let $\pi\colon X_\infty \longrightarrow X^{\log}$ be the projection. Let $I^\bullet_{\mathbb{Z}}$ be an injective resolution of $\mathbb{Z}_{X^{\log}}$. Set $J^\bullet_{\mathbb{Z}} := (\varepsilon\pi)_*\pi^{-1}(I^\bullet_{\mathbb{Z}})$ (cf. [FN, §3]). Note that $\pi^{-1}(I^\bullet_{\mathbb{Z}})$ is an injective resolution of $\mathbb{Z}_{X_\infty}$ by the local property of the injectivity ([KS, (2.4.10)]). Let $T\colon J^\bullet_{\mathbb{Z}} \overset{\sim}{\longrightarrow} J^\bullet_{\mathbb{Z}}$ be the induced automorphism by the covering transformation $\mathbb{R} \ni x \longmapsto x+1 \in \mathbb{R}$ over $\mathbb{S}^1$. We consider the following morphism

$$\delta := -(2\pi\sqrt{-1})^{-1}(T - \mathrm{id})\colon J^\bullet_{\mathbb{Z}} \longrightarrow J^\bullet_{\mathbb{Z}}(-1)$$

of complexes. Let $\mathrm{MF}(\delta)$ be the mapping fiber of δ: $\mathrm{MF}(\delta) := := J^\bullet_{\mathbb{Z}} \oplus J^\bullet_{\mathbb{Z}}(-1)[-1]$; $\mathrm{MF}(\delta)^q := J^q_{\mathbb{Z}} \oplus J^{q-1}_{\mathbb{Z}}(-1)$, $d(x,y) = (dx, -dy + \delta(x))$.

Let

$$\text{(2.0.1;}\infty\text{)}\qquad \theta\colon \mathrm{MF}(\delta) \longrightarrow \mathrm{MF}(\delta)(1)[1]$$

be a morphism of complexes on $\overset{\circ}{X}$ defined by

$$(x, y) \longmapsto (0, -x) \quad ((x, y) \in \mathrm{MF}(\delta)^q)$$

(in [SZ, (5.12)] and [FN, (1.2.2)], the morphism $(x, y) \longmapsto (0, x)$ was considered). Define a double complex $A_{\mathbb{Z}}^{\bullet\bullet}$ as follows: the component $A_{\mathbb{Z}}^{ij}$ ($i \in \mathbb{Z}, j \in \mathbb{N}$) is, by definition, $\mathrm{MF}(\delta)^{i+j+1}(j+1)/(\tau_j \mathrm{MF}(\delta))^{i+j+1}(j+1)$, where $\tau_\bullet$ means the canonical filtration on the complex $\mathrm{MF}(\delta)$. Following the proof of [RZ, (1.7)], consider the following boundary morphisms of $A_{\mathbb{Z}}^{\bullet\bullet}$:

$$\text{(2.0.2;}\infty\text{)}\qquad \begin{array}{ccc} A_{\mathbb{Z}}^{i,j+1} & & \\ {\scriptstyle (-1)^i\theta\wedge}\uparrow & & \\ A_{\mathbb{Z}}^{ij} & \xrightarrow{(-1)^{j+1}d} & A_{\mathbb{Z}}^{i+1,j}. \end{array}$$

Let $A_{\mathbb{Z}}^{\bullet}$ be the single complex of $A_{\mathbb{Z}}^{\bullet\bullet}$. Consider a composite morphism

$$\mu_{\mathbb{Z}}\colon J_{\mathbb{Z}}^{\bullet} \ni x \longmapsto (0, -x) \in \mathrm{MF}(\delta)(1)[1]/\tau_0 \mathrm{MF}(\delta)(1)[1] \overset{\subset}{\longrightarrow} A_{\mathbb{Z}}^{\bullet}.$$

of complexes. By the proof of [FN, (3.17)], the morphism $\mathcal{H}^h(\delta)\colon \mathcal{H}^h(J_{\mathbb{Z}}^{\bullet}) \longrightarrow \mathcal{H}^h(J_{\mathbb{Z}}^{\bullet})(-1)$ ($h \in \mathbb{Z}$) of cohomologies is the zero. Hence, by the argument of [SZ, (5.13)], $\mu_{\mathbb{Z}}$ is a quasi-isomorphism. Therefore $H^h(X, A_{\mathbb{Z}}^{\bullet}) = H^h(X, J_{\mathbb{Z}}^{\bullet}) = H^h(X_\infty, \mathbb{Z})$. Let $\{P_k\}_{k\in\mathbb{Z}}$ be the following filtration on $A_{\mathbb{Z}}^{\bullet}$ defined by

$$\text{(2.0.3;}\infty\text{)}\quad P_k A_{\mathbb{Z}}^{ij} := (\{(\tau_{2j+k+1} + \tau_j)\mathrm{MF}(\delta)\}^{i+j+1}(j+1))/(\tau_j \mathrm{MF}(\delta))^{i+j+1}(j+1).$$

Then we have the following spectral sequence

$$\text{(2.0.4;}\infty\text{)}\quad E_{1,\infty}^{-k,h+k} = \bigoplus_{j \geq \max\{-k,0\}} H^{h-2j-k}(\overset{\circ}{X}, \mathcal{H}^{2j+k+1}(\mathrm{MF}(\delta)(j+1))) \Longrightarrow H^h(X_\infty, \mathbb{Z}).$$

By [FN, (3.2)] the following sequence

$$0 \longrightarrow \varepsilon_*(I_{\mathbb{Z}}^{\bullet}) \longrightarrow J_{\mathbb{Z}}^{\bullet} \overset{T-\mathrm{id}}{\longrightarrow} J_{\mathbb{Z}}^{\bullet} \longrightarrow 0$$

is exact. In particular, the morphism $\delta\colon J_{\mathbb{Z}}^{\bullet} \longrightarrow J_{\mathbb{Z}}^{\bullet}(-1)$ is surjective. Hence the morphism $\mathrm{Ker}\,\delta \ni x \longmapsto (x, 0) \in \mathrm{MF}(\delta)$ is a quasi-isomorphism (the Convention (3)). Fix a total order on the irreducible components of $\overset{\circ}{X}$. Then we have a canonical isomorphism

$$\text{(2.0.4.1;}\infty\text{)}\quad R^r\varepsilon_*(\mathbb{Z}_{X^{\log}}) \overset{\sim}{\longleftarrow} \bigwedge^r (\mathcal{M}_X^{\mathrm{gp}}/\mathcal{O}_X^*)(-r) \overset{\sim}{\longrightarrow} \mathbb{Z}(-r)_{\overset{\circ}{X}{}^{(r)}} \quad (r \in \mathbb{Z}_{>0})$$

by [KN, (1.5)] and [Nak3, (1.8.1)] (cf. the proof of [Nak3, (1.8.3)]). Putting these together, we have the following:

$$(2.0.5;\infty)\quad \mathcal{H}^r(\mathrm{MF}(\delta)) = \mathcal{H}^r(\mathrm{Ker}\,\delta) = \mathcal{H}^r(\varepsilon_*(I^\bullet_{\mathbb{Z}})) = R^r\varepsilon_*(\mathbb{Z}_{X^{\log}}) = \mathbb{Z}(-r)_{\overset{\circ}{X}{}^{(r)}}.$$

As in the l-adic case, we make the following identification:

$$(2.0.6;\infty)\quad \mathcal{H}^{2j+k+1}(\mathrm{MF}(\delta)) \overset{(2.0.5;\infty)}{=} \mathbb{Z}(-2j-k-1)_{\overset{\circ}{X}{}^{(2j+k+1)}} \overset{(-1)^{j+k}}{\underset{\sim}{\longrightarrow}} \mathbb{Z}(-2j-k-1)_{\overset{\circ}{X}{}^{(2j+k+1)}}.$$

Using (2.0.4; ∞) and the identification (2.0.6; ∞), we obtain the following spectral sequence

$$(2.0.7;\infty)\quad E_{1,\infty}^{-k,h+k} = \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}(\overset{\circ}{X}{}^{(2j+k+1)}, \mathbb{Z})(-j-k) \Longrightarrow H^h(X_\infty, \mathbb{Z}).$$

Tensorizing (2.0.7; ∞) with $\mathbb{Q}$, we obtain the following spectral sequence

$$(2.0.8;\infty)\quad E_{1,\infty}^{-k,h+k} = \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}(\overset{\circ}{X}{}^{(2j+k+1)}, \mathbb{Q})(-j-k) \Longrightarrow H^h(X_\infty, \mathbb{Q}).$$

DEFINITION 2.1. (1) We call the cohomology $H^h(X_\infty, \mathbb{Z})$ (resp. $H^h(X_\infty, \mathbb{Q})$) the *log Betti cohomology* of X/s with coefficient $\mathbb{Z}$ (resp. $\mathbb{Q}$).

(2) We call the spectral sequences (2.0.7; ∞) and (2.0.8; ∞) the *weight spectral sequences* of $H^h(X_\infty, \mathbb{Z})$ and $H^h(X_\infty, \mathbb{Q})$, respectively.

(3) We call the filtrations on $H^h(X_\infty, \mathbb{Z})$ and $H^h(X_\infty, \mathbb{Q})$ induced by the weight spectral sequences (2.0.7; ∞) and (2.0.8; ∞), respectively, the *weight filtrations*.

If X is algebraic, (2.0.7; ∞)$\otimes_{\mathbb{Z}}\mathbb{Z}_l$ is canonically isomorphic to (2.0.7; l) by the proof of [FN, (7.1)]. In (5.10.1) below, we shall describe the boundary morphism between the E_1-terms of (2.0.7; ∞) by the sum with signs of Gysin morphisms and the induced morphisms of closed immersions.

By the proof of [RZ, (1.7)], the morphism $\delta\colon A^\bullet_{\mathbb{Z}} \longrightarrow A^\bullet_{\mathbb{Z}}(-1)$ is homotopic to a morphism $\nu_\infty\colon A^\bullet_{\mathbb{Z}} \longrightarrow A^\bullet_{\mathbb{Z}}(-1)$ induced by a family of morphisms $(-1)^{i+j}\mathrm{proj} : A^{ij}_{\mathbb{Z}} \longrightarrow A^{i-1,j+1}_{\mathbb{Z}}(-1)$ ($i\in\mathbb{Z}, j\in\mathbb{N}$). (See (3.6.5) below why the sign $(-1)^{i+j+1}$ does not appear.) Hence $\delta\colon A^\bullet_{\mathbb{Z}} \longrightarrow A^\bullet_{\mathbb{Z}}(-1)$ is nilpotent. Therefore the morphism

$$(2.1.1)\qquad N_\infty := -(2\pi\sqrt{-1})^{-1}\log T\colon A^\bullet_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{Q} \longrightarrow A^\bullet_{\mathbb{Z}}(-1)\otimes_{\mathbb{Z}}\mathbb{Q}$$

is well-defined. Let $E^{**}_{r,\infty}$ ($r\in\mathbb{Z}_{\geq 1}$) be the E_r-term of the spectral sequence (2.0.8; ∞). Since $(-1)^{i+j}\mathrm{proj}(P_kA^{ij}_{\mathbb{Z}}) \subset P_{k-2}A^{i-1,j+1}_{\mathbb{Z}}(-1)$, the morphism $\nu^k_\infty\colon E^{-k,h+k}_{r,\infty} \longrightarrow E^{k,h-k}_{r,\infty}(-k)$ is equal to $N^k_\infty\colon E^{-k,h+k}_{r,\infty} \longrightarrow E^{k,h-k}_{r,\infty}(-k)$.

Let us recall another more traditional Steenbrink complex defined in [FN, §3] (cf. [SZ, §5]) and another more traditional weight spectral sequence.

Set $I^{\bullet}_{\mathbb{Q}} := I^{\bullet}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $I^{\bullet}_{\mathbb{Q}}$ is an injective resolution of $\mathbb{Q}_{X^{\log}}$. Set $J^{\bullet}_{\mathbb{Q}} := (\varepsilon\pi)_*\pi^{-1}(I^{\bullet}_{\mathbb{Q}})$. Let $B(J^{\bullet}_{\mathbb{Q}}) := \bigcup_{e=1}^{\infty} \mathrm{Ker}(T - \mathrm{id})^e \otimes_{\mathbb{Z}} \mathbb{Q}$ be a subcomplex of $J^{\bullet}_{\mathbb{Q}}$. Then Fujisawa and Nakayama have proved that the inclusion $B(J^{\bullet}_{\mathbb{Q}}) \overset{\subset}{\longrightarrow} J^{\bullet}_{\mathbb{Q}}$ is a quasi-isomorphism ([FN, (3.5)], cf. [SZ, (5.9)]); thus we have $H^h(X_\infty, \mathbb{Q}) = H^h(X, B(J^{\bullet}_{\mathbb{Q}}))$ and the well-defined morphism

(2.1.2) $$N_\infty := -(2\pi\sqrt{-1})^{-1} \log T \colon B(J^{\bullet}_{\mathbb{Q}}) \longrightarrow B(J^{\bullet}_{\mathbb{Q}})(-1)$$

induces a morphism

(2.1.3) $$N_\infty \colon H^h(X_\infty, \mathbb{Q}) = H^h(X, B(J^{\bullet}_{\mathbb{Q}})) \longrightarrow H^h(X, B(J^{\bullet}_{\mathbb{Q}}))(-1) = \\ = H^h(X_\infty, \mathbb{Q})(-1),$$

which we called the *monodromy operator*. Let $\mathrm{MF}_{\mathbb{Q}}(N_\infty)$ be the mapping fiber of the morphism $N_\infty \colon B(J^{\bullet}_{\mathbb{Q}}) \longrightarrow B(J^{\bullet}_{\mathbb{Q}})(-1)$. Let

(2.1.4) $$\theta \colon \mathrm{MF}_{\mathbb{Q}}(N_\infty) \ni (x, y) \longmapsto (0, -x) \in \mathrm{MF}_{\mathbb{Q}}(N_\infty)(1)[1]$$

be a morphism of complexes on $\overset{\circ}{X}$. Set

(2.1.5) $$A^{ij}_{\mathbb{Q}}(N_\infty) := \mathrm{MF}_{\mathbb{Q}}(N_\infty)^{i+j+1}(j+1)/(\tau_j \mathrm{MF}_{\mathbb{Q}}(N_\infty))^{i+j+1}(j+1) \\ (i \in \mathbb{Z}, j \in \mathbb{N})$$

and consider the following boundary morphisms:

(2.1.6) $$\begin{array}{ccc} A^{i,j+1}_{\mathbb{Q}}(N_\infty) & & \\ {\scriptstyle (-1)^i\theta\wedge}\Big\uparrow & & \\ A^{ij}_{\mathbb{Q}}(N_\infty) & \xrightarrow{(-1)^{j+1}d} & A^{i+1,j}_{\mathbb{Q}}(N_\infty). \end{array}$$

Then $A^{\bullet\bullet}_{\mathbb{Q}}(N_\infty)$ is a double complex. Let $A^{\bullet}_{\mathbb{Q}}(N_\infty)$ be the associated single complex to $A^{\bullet\bullet}_{\mathbb{Q}}(N_\infty)$ (cf. [SZ, (5.12)], [FN, (3.6), (3.16)]). By [FN, (1.14), (3.17)] (cf. [SZ, (5.13)]), the following composite morphism

(2.1.7) $$\mu_{\mathbb{Q}} \colon B(J^{\bullet}_{\mathbb{Q}}) \ni x \longmapsto (0, -x) \in \\ \in \mathrm{MF}_{\mathbb{Q}}(N_\infty)(1)[1]/\tau_0 \mathrm{MF}_{\mathbb{Q}}(N_\infty)(1)[1] \overset{\subset}{\longrightarrow} A^{\bullet}_{\mathbb{Q}}(N_\infty)$$

is a quasi-isomorphism as in the integral version above. On the other hand, the morphism $\delta \colon J^{\bullet}_{\mathbb{Z}} \longrightarrow J^{\bullet}_{\mathbb{Z}}(-1)$ induces surjective morphisms

$\delta'_{\mathbb{Q}}\colon J^{\bullet}_{\mathbb{Q}}\longrightarrow J^{\bullet}_{\mathbb{Q}}(-1)$ and $\delta_{\mathbb{Q}}\colon B(J^{\bullet}_{\mathbb{Q}})\longrightarrow B(J^{\bullet}_{\mathbb{Q}})(-1)$. By using $\delta_{\mathbb{Q}}$, we have a double complex $A^{\bullet\bullet}_{\mathbb{Q}}(\delta_{\mathbb{Q}})$ and the associated single complex $A^{\bullet}_{\mathbb{Q}}(\delta_{\mathbb{Q}})$ to $A^{\bullet\bullet}_{\mathbb{Q}}(\delta_{\mathbb{Q}})$ (cf. [FN, (6.2)]). Note that $\mathrm{Ker}(\delta'_{\mathbb{Q}}\colon J^{\bullet}_{\mathbb{Q}}\longrightarrow J^{\bullet}_{\mathbb{Q}}(-1)) = \mathrm{Ker}(N_{\infty}\colon B(J^{\bullet}_{\mathbb{Q}})\longrightarrow B(J^{\bullet}_{\mathbb{Q}})(-1))$ and that the morphism $N_{\infty}\colon B(J^{\bullet}_{\mathbb{Q}})\longrightarrow B(J^{\bullet}_{\mathbb{Q}})(-1)$ is surjective. Hence we have $\mathcal{H}^r(\mathrm{MF}_{\mathbb{Q}}(N_{\infty})) = R^r\varepsilon_*(\mathbb{Q}_{X^{\log}})$ $(r\in\mathbb{Z})$. As in (2.0.3; ∞), set

(2.1.8) $$P_kA^{ij}_{\mathbb{Q}}(N_{\infty}) := (\{(\tau_{2j+k+1}+\tau_j)\mathrm{MF}_{\mathbb{Q}}(N_{\infty})\}^{i+j+1}(j+1))/(\tau_j\mathrm{MF}_{\mathbb{Q}}(N_{\infty}))^{i+j+1}(j+1).$$

Then, using the following identification

(2.1.9) $$\mathcal{H}^{2j+k+1}(\mathrm{MF}_{\mathbb{Q}}(N_{\infty}))=\mathcal{H}^{2j+k+1}(\mathrm{MF}(\delta'_{\mathbb{Q}}))\overset{(2.0.5;\infty)\otimes\mathbb{Q}}{=}\mathbb{Q}(-2j-k-1)_{\overset{\circ}{X}{}^{(2j+k+1)}} \overset{(-1)^{j+k}}{\underset{\sim}{\longrightarrow}} \mathbb{Q}(-2j-k-1)_{\overset{\circ}{X}{}^{(2j+k+1)}},$$

we have the following weight spectral sequence

(2.1.10) $$E_{1,\infty}^{-k,h+k} = \bigoplus_{j\geq\max\{-k,0\}} H^{h-2j-k}(\overset{\circ}{X}{}^{(2j+k+1)},\mathbb{Q})(-j-k)\Longrightarrow H^h(X_{\infty},\mathbb{Q}).$$

In (5.10.2) below, we shall describe the boundary morphism between the E_1-terms of (2.1.10) by the sum with signs of Gysin morphisms and the induced morphisms of closed immersions.

Fujisawa and Nakayama have proved that the weight spectral sequence (2.0.8; ∞) is isomorphic to the weight spectral sequence (2.1.10) ([FN, (6.5)]).

Following the method of [SZ], Fujisawa and Nakayama have proved in [FN] that, if the irreducible components of $\overset{\circ}{X}$ are compact and Kähler or the analytifications of proper smooth schemes over $\mathbb{C}$, $H^h(X_{\infty},\mathbb{Q})$ has a natural mixed $\mathbb{Q}$-Hodge structure (and in particular, $H^h(X_{\infty},\mathbb{Z})$ has a mixed $\mathbb{Z}$-Hodge structure) and that the spectral sequence (2.1.10) with signs which are different from our signs is a spectral sequence of mixed $\mathbb{Q}$-Hodge structures. Because we have to recall some fundamental objects in [SZ] and [FN] for the proof, we shall also show this fact in §3 below with the change of signs in [SZ] and [FN].

The spectral sequence (2.1.10) degenerates at E_2 by the yoga of weights in mixed Hodge theory if the irreducible components of $\overset{\circ}{X}$ are compact and Kähler or the analytifications of proper smooth schemes over $\mathbb{C}$. As a corollary, (2.0.8; ∞) also degenerates at E_2 under the same assumption because (2.0.8; ∞) is isomorphic to (2.1.10) ([FN, (6.5)]).

By the argument due to M. Saito [SaM, §4] (cf. [St2, p. 117]), if $\overset{\circ}{X}$ is projective, the monodromy filtration with respect to N_∞ and the weight filtration on $H^h(X_\infty, \mathbb{Q})$ coincide: the morphism $N_\infty : E_{2,\infty}^{-k,h+k} \longrightarrow E_{2,\infty}^{k,h+k}(-k)$ is an isomorphism. Here $E_{2,\infty}^{**}$ is the E_2-term of the spectral sequence (2.1.10).

REMARK 2.2. (1) There are many non-minor mistakes and many unproven facts in the theory of log de Rham-Witt complexes and p-adic Steenbrink complexes in published papers. In [Nakk3, §6∼11], we have corrected all mistakes and checked all unproven facts in [Hy], [HK] and [Mo] which are needed in this paper except counter-examples (6.8) (2) and (4) below.

(2) In [FN], Fujisawa and Nakayama have proved that their method and Kawamata-Namikawa's method give the same cohomological mixed $\mathbb{Q}$-Hodge complex on $\overset{\circ}{X}$ up to canonical isomorphisms.

(3) (cf. [RZ, (1.7)], [SZ, (5.14)]) Let $f : E^\bullet \longrightarrow E^\bullet$ be an endomorphism of a complex of objects of an abelian category. Let $\mathrm{MF}(f) = s((E^\bullet, d_E) \longrightarrow (E^\bullet, -d_E))$ be the mapping fiber of f. Let d be the boundary morphism of $\mathrm{MF}(f)$ and let $\theta : \mathrm{MF}(f) \ni (x, y) \longmapsto (0, x) \in \mathrm{MF}(f)[1]$ be a morphism of complexes. Consider a double complex $G^{\bullet\bullet}$ defined by $G^{ij} = \mathrm{MF}(f)^{i+j+1}/(\tau_j \mathrm{MF}(f))^{i+j+1}$ with the following boundary morphism

$$
\begin{array}{ccc}
G^{i,j+1} & & \\
\uparrow{\scriptstyle\theta} & & \\
G^{ij} & \xrightarrow{d} & G^{i+1,j}.
\end{array}
$$

Let $\varepsilon : \mathbb{Z} \times \mathbb{Z} \longrightarrow \{\pm 1\}$ be a map. Let $H : G^{ij} \ni (x, y) \longmapsto (\varepsilon(i,j)y, 0) \in G^{i-1,j}$ be a morphism. Then, for $(x, y) \in G^{ij}$, we have the following formulas:

$$
\begin{aligned}
(\theta H + H\theta)(x, y) &= \theta(\varepsilon(i,j)y, 0) + H(0, x) \\
&= (\varepsilon(i,j+1)x, \varepsilon(i,j)y)
\end{aligned}
$$

and

$$
\begin{aligned}
(dH + Hd)(x, y) &= d(\varepsilon(i,j)y, 0) + H(dx, -dy + f(x)) \\
&= \varepsilon(i,j)(dy, f(y)) + \varepsilon(i+1,j)(-dy + f(x), 0) \\
&= ((\varepsilon(i,j) - \varepsilon(i+1,j))dy + \varepsilon(i+1,j)f(x), \varepsilon(i,j)f(y)).
\end{aligned}
$$

Hence $(\theta H + H\theta)(x, y) = \pm(x, y)$ and $(dH + Hd)(x, y) = \pm(f(x), f(y))$ and the term before dy vanishes for any $(x, y) \in G^{ij}$ and for all i, j if and only if $\varepsilon(i,j+1) = \varepsilon(i,j) = \varepsilon(i+1,j)$ for all i, j. Let $G^\bullet$ be the single complex of

$G^{\bullet\bullet}$. If $\varepsilon(i,j)=1$ (resp. $\varepsilon(i,j)=-1$) for all i,j, then the morphism $G^{\bullet}\ni(x,y)\longmapsto(f(x),f(y))\in G^{\bullet}$ is homotopic to the morphism $v\colon G^{\bullet}\longrightarrow G^{\bullet}$ defined by $-$proj.: $G^{ij}\longrightarrow G^{i-1,j+1}$. Thus I think that the statement [SZ, (5.14)] and the proof of [loc. cit.] is not right in signs: the sign $(-1)^{p+q+1}$ in [loc. cit.] must not appear.

3. Steenbrink mixed $\mathbb{Q}$-Hodge complex.

Let $s:=((\mathrm{Spec}\,\mathbb{C})_{\mathrm{an}},\mathbb{N}\oplus\mathbb{C}^*)$ be a log point. Let X be a proper SNCL analytic variety over s. Fix a total order on the irreducible components of $\overset{\circ}{X}$ as in §2. In this section, as a corollary of (3.8) below, we show that (2.1.10) is a spectral sequence of mixed Hodge structures if the irreducible components of $\overset{\circ}{X}$ are compact and Kähler or the analytifications of proper smooth schemes over $\mathbb{C}$. The proof of (3.8) is essentially the same as that in [FN]: in [loc. cit.] Fujisawa and Nakayama have already obtained (3.8) with signs which are different from those in §2 and this section. We add only (3.2) below to [FN]. We shall use (3.2) for an argument about a sign ((4.6) below) which gives an influence to almost all results in §4 and §5: only to give the uniform descriptions of the boundary morphisms of the E_1-terms of the p-adic, the l-adic and the ∞-adic weight spectral sequences ((2.0.8.3; p), (5.5.1), (5.10.1), (5.10.2) below), we have changed and change signs in [SZ] and [FN]. We also add (3.5) below to [St2]; (3.5) is an easy corollary of (3.2) and (3.4), and a complement for [St2, (4.6), (4.7)], though our statement in (3.5) is different from [St2, (4.6), (4.7)] whose proofs seem mistaken (see (3.6) (2) below).

First let us recall some complexes in [SZ] and [FN].

By abuse of notation, we sometimes omit the symbol $\circ$ in the notation $\overset{\circ}{X}$ below. Let $I^{\bullet}_{\mathbb{Q}}$ (resp. $I^{\bullet}_{\mathbb{C}}$) be an injective resolution of $\mathbb{Q}_{X^{\log}}$ (resp. $\mathbb{C}_{X^{\log}}$). Let us recall natural morphisms $\pi\colon X_{\infty}\longrightarrow X^{\log}$ and $\varepsilon=\varepsilon_{\mathrm{top}}\colon X^{\log}\longrightarrow\overset{\circ}{X}$ in §2. Set $J^{\bullet}_{\mathbb{Q}}:=(\varepsilon\pi)_*\pi^{-1}(I^{\bullet}_{\mathbb{Q}})$ and $J^{\bullet}_{\mathbb{C}}:=(\varepsilon\pi)_*\pi^{-1}(I^{\bullet}_{\mathbb{C}})$. Let $B(J^{\bullet}_{\mathbb{Q}}):=\bigcup_{e=1}^{\infty}\mathrm{Ker}(T-\mathrm{id})^e$ be a subcomplex of $J^{\bullet}_{\mathbb{Q}}$ and $B(J^{\bullet}_{\mathbb{C}})$ a similar subcomplex of $J^{\bullet}_{\mathbb{C}}$.

In this section we denote $A^{\bullet}_{\mathbb{Q}}(N_{\infty})$ in §2 by $A^{\bullet}(J^{*}_{\mathbb{Q}})$. Let $\mathrm{MF}_{\mathbb{C}}(N_{\infty})=$ $=\mathrm{MF}_{\mathbb{C}}(N_{\infty})^{\bullet}$ be the mapping fiber of the morphism

$$N_{\infty}=-(2\pi\sqrt{-1})^{-1}\log T\colon B(J^{\bullet}_{\mathbb{C}})\longrightarrow B(J^{\bullet}_{\mathbb{C}})(-1).$$

As in (2.1.5), by using the canonical filtration on $\mathrm{MF}_{\mathbb{C}}(N_{\infty})$, we have two complexes $A^{\bullet\bullet}(J^{*}_{\mathbb{C}})$ and $A^{\bullet}(J^{*}_{\mathbb{C}})$. As in (2.1.8), $A^{\bullet\bullet}(J^{*}_{\mathbb{C}})$ and $A^{\bullet}(J^{*}_{\mathbb{C}})$ have filtrations P's.

By [FN, (1.14), (3.17)], a composite morphism

(3.0.1) $B(J^{\bullet}_{\mathbb{C}})\ni x\longmapsto(0,-x)\in \mathrm{MF}_{\mathbb{C}}(N_{\infty})(1)[1]/\tau_0\mathrm{MF}_{\mathbb{C}}(N_{\infty})(1)[1]\overset{\subset}{\longrightarrow} A^{\bullet}(J^{*}_{\mathbb{C}})$

is a quasi-isomorphism. By [FN, (3.5)], the natural inclusion $B(J^{\bullet}_{\mathbb{C}})\overset{\subset}{\longrightarrow} J^{\bullet}_{\mathbb{C}}$ is a quasi-isomorphism. Hence, using the quasi-isomorphism (3.0.1), we have an isomorphism

(3.0.2) $$\mu_{\mathbb{C}}\colon J^{\bullet}_{\mathbb{C}}\overset{\sim}{\longrightarrow} A^{\bullet}(J^{*}_{\mathbb{C}})$$

in the derived category $\mathrm{D}^{+}(\mathbb{C}_X)$. Using the quasi-isomorphism (2.1.7), we have an analogous isomorphism

(3.0.3) $$J^{\bullet}_{\mathbb{Q}}\overset{\sim}{\longrightarrow} A^{\bullet}(J^{*}_{\mathbb{Q}})$$

in $\mathrm{D}^{+}(\mathbb{Q}_X)$, which we denote by the same symbol $\mu_{\mathbb{Q}}$. We also have a natural morphism $J^{\bullet}_{\mathbb{Q}}\longrightarrow J^{\bullet}_{\mathbb{C}}$ of complexes. This morphism induces a morphism $A^{\bullet}(J^{*}_{\mathbb{Q}})\longrightarrow A^{\bullet}(J^{*}_{\mathbb{C}})$ of complexes and we have the following commutative diagram

(3.0.4) $$\begin{array}{ccc} J^{\bullet}_{\mathbb{Q}} & \xrightarrow[\sim]{\mu_{\mathbb{Q}}} & A^{\bullet}(J^{*}_{\mathbb{Q}}) \\ \big\downarrow & & \big\downarrow \\ J^{\bullet}_{\mathbb{C}} & \xrightarrow[\sim]{\mu_{\mathbb{C}}} & A^{\bullet}(J^{*}_{\mathbb{C}}). \end{array}$$

Let $\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}}$ be the log de Rham complex of the log analytic space $X/((\mathrm{Spec}\,\mathbb{C})_{\mathrm{an}},\mathbb{C}^{*})$. Let t be a global section of $\mathcal{M}_s$ whose image in $\Gamma(s,\mathcal{M}_s/\mathcal{O}^{*}_s)=\mathbb{N}$ is the generator. Set $u:=(2\pi\sqrt{-1})^{-1}\log t$. Let

$$\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}}:=(\cdots\longrightarrow\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{i}_{X/\mathbb{C}}\longrightarrow\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{i+1}_{X/\mathbb{C}}\longrightarrow\cdots)$$

be a complex with the following boundary morphism:

(3.0.5) $$d\Big(\sum_{j=0}^{r}u^{[j]}\omega_j\Big):=(2\pi\sqrt{-1})^{-1}\sum_{j=1}^{r}(u^{[j-1]}d\log t\wedge\omega_j)+\sum_{j=0}^{r}u^{[j]}d\omega_j$$
$$(\omega_j\in\widetilde{\Lambda}^{i}_{X/\mathbb{C}},\ r\in\mathbb{N}),$$

where $u^{[j]}:=(j!)^{-1}u^{j}$ $(j\in\mathbb{N})$ is a divided power of u. The complex $\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}}$ is the linearization of $\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}}$ with respect to the variable u (cf. [BO1, 6.11 Lemma]); because we have to consider the first term on the right hand side of (3.0.5), we use the notation $\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}}$ and do not use the notation $\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}}[u]$ in [SZ] and [FN]. By [KN, (3.8)] and [FN, (3.3)] (cf. the proof of [FN, (3.5)]), we have the following commutative diagram:

$$(3.0.6)\qquad \begin{array}{ccccc} J^{\bullet}_{\mathbb{C}} & \overset{\supset}{\longleftarrow} & B(J^{\bullet}_{\mathbb{C}}) & \overset{\lambda}{\longleftarrow} & R(\epsilon\pi)_*\pi^{-1}(\widetilde{\Lambda}^{\bullet,\log}_{X/\mathbb{C}}) = \mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}} \\ \uparrow & & \uparrow & & \uparrow \iota \\ \mathbb{C}_X & = & \mathbb{C}_X & \longrightarrow & \widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}}. \end{array}$$

Here $\widetilde{\Lambda}^{\bullet,\log}_{X/\mathbb{C}} = \mathcal{O}^{\log}_X \otimes_{\varepsilon^{-1}(\mathcal{O}_X)} \varepsilon^{-1}(\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}})$ (recall the definition of the sheaf $\mathcal{O}^{\log}_X$ in [KN, (3.2)]) and the right vertical morphism ι is a natural inclusion $\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}} \overset{\subset}{\longrightarrow} \mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}}$ and the upper horizontal morphism λ is a natural quasi-isomorphism. By abuse of notation, we also denote by λ the composite quasi-isomorphism $\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}} \longrightarrow J^{\bullet}_{\mathbb{C}}$. Let $\mathrm{MF}(N_{\tilde{\Lambda}})$ be the mapping fiber of the following morphism

$$(3.0.7)\quad N_{\tilde{\Lambda}} := -(2\pi\sqrt{-1})^{-1}d/du\colon \mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}} \longrightarrow \mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}}(-1)$$

of complexes. Let θ be a morphism $\mathrm{MF}(N_{\tilde{\Lambda}}) \ni (x,y) \longmapsto (0,-x) \in \mathrm{MF}(N_{\tilde{\Lambda}})(1)[1]$ of complexes on $\overset{\circ}{X}$. By using the canonical filtration on $\mathrm{MF}(N_{\tilde{\Lambda}})$, we have a double complex $A^{\bullet\bullet}(\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{*}_{X/\mathbb{C}})$ and a single complex $A^{\bullet}(\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{*}_{X/\mathbb{C}}) := := s(A^{\bullet\bullet}(\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{*}_{X/\mathbb{C}}))$. By using the canonical filtration on $\mathrm{MF}(N_{\tilde{\Lambda}})$ again, we have the filtrations P's on $A^{\bullet\bullet}(\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{*}_{X/\mathbb{C}})$ and $A^{\bullet}(\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{*}_{X/\mathbb{C}})$ (cf. (2.1.8)). In [FN, (3.7)] Fujisawa and Nakayama have essentially proved that the quasi-isomorphism λ induces the following filtered quasi-isomorphism

$$(3.0.8)\qquad (A^{\bullet}(J^{*}_{\mathbb{C}}),P) \longleftarrow (A^{\bullet}(\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{*}_{X/\mathbb{C}}),P).$$

Let $\mu_{\tilde{\Lambda}}$ be the following composite morphism

$$(3.0.9)\quad \begin{aligned} \mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}} \ni x \longmapsto (0,-x) &\in \mathrm{MF}(N_{\tilde{\Lambda}})(1)[1]/\tau_0\mathrm{MF}(N_{\tilde{\Lambda}})(1)[1] \\ &\overset{\subset}{\longrightarrow} A^{\bullet}(\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{*}_{X/\mathbb{C}}) \end{aligned}$$

of complexes. Then we have the following commutative diagram

$$(3.0.10)\qquad \begin{array}{ccc} A^{\bullet}(J^{*}_{\mathbb{C}}) & \overset{\sim}{\longleftarrow} & A^{\bullet}(\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{*}_{X/\mathbb{C}}) \\ {\scriptstyle \mu_{\mathbb{C}}}\uparrow & & \uparrow {\scriptstyle \mu_{\tilde{\Lambda}}} \\ J^{\bullet}_{\mathbb{C}} & \underset{\sim}{\overset{\lambda}{\longleftarrow}} & \mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}}. \end{array}$$

Next, consider the following morphism

$$(3.0.11)\quad \begin{aligned} &\phi^{i}\colon \mathrm{MF}(N_{\tilde{\Lambda}})^{i} = \\ &= ((\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}}) \oplus (\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}})(-1)[-1])^{i} \longrightarrow \widetilde{\Lambda}^{i}_{X/\mathbb{C}} \quad (i\in\mathbb{N}) \end{aligned}$$

of abelian sheaves on X defined by the following formula

$$\phi^i(\omega,\eta)=\omega_0+d\log t\wedge\eta_0, \tag{3.0.12}$$

where $\omega=\sum_{j=0}^{r}u^{[j]}\omega_j$ and $\eta=\sum_{j=0}^{r'}u^{[j]}\eta_j$ ($\omega_j\in\widetilde{\Lambda}^i_{X/\mathbb{C}}$, $\eta_j\in\widetilde{\Lambda}^{i-1}_{X/\mathbb{C}}$, $r,r'\in\mathbb{N}$). Because the formula $\phi(x,y):=x_0-d\log t\wedge y_0$ in [SZ, p. 530] does not give a morphism of complexes and because we claim that the formula (3.0.12) is a right formula as in [FN, (3.9.1)], we give the following proof without omission.

PROPOSITION 3.1. *The family $\phi:=\{\phi^i\}_{i\in\mathbb{N}}$ gives a morphism of complexes.*

PROOF. Indeed, we have the following formula:

$$\begin{aligned}
\phi^{i+1}d(\omega,\eta)&=\phi^{i+1}(d\omega,-d\eta+N_{\widetilde{\Lambda}}(\omega))\\
&=\phi^{i+1}\Big((2\pi\sqrt{-1})^{-1}\sum_{j=1}^{r}u^{[j-1]}d\log t\wedge\omega_j+\sum_{j=0}^{r}u^{[j]}d\omega_j,\\
&\quad-(2\pi\sqrt{-1})^{-1}\sum_{j=1}^{r'}u^{[j-1]}d\log t\wedge\eta_j-\sum_{j=0}^{r'}u^{[j]}d\eta_j-(2\pi\sqrt{-1})^{-1}\sum_{j=1}^{r}u^{[j-1]}\omega_j\Big)\\
&=(2\pi\sqrt{-1})^{-1}d\log t\wedge\omega_1+d\omega_0\\
&\quad+d\log t\wedge\{-(2\pi\sqrt{-1})^{-1}d\log t\wedge\eta_1-d\eta_0-(2\pi\sqrt{-1})^{-1}\omega_1\}\\
&=d\omega_0-d\log t\wedge d\eta_0=d\phi^i(\omega,\eta).
\end{aligned}$$

□

Since $N_{\widetilde{\Lambda}}$ is surjective, the natural inclusion morphism $\widetilde{\iota}\colon\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}}\ni$ $\ni\omega\longmapsto(\omega,0)\in\mathrm{MF}(N_{\widetilde{\Lambda}})$ is a quasi-isomorphism (Convention (3)). Since $\phi\circ\widetilde{\iota}$ is the identity, ϕ is a quasi-isomorphism (cf. [SZ, p. 531], [FN, (3.10)]). Consider the following composite filtered morphism

$$(\mathrm{MF}(N_{\widetilde{\Lambda}}),\tau)\xrightarrow{\ \phi\ }(\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}},\tau)\xrightarrow{\ \subset\ }(\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}},P). \tag{3.1.1}$$

Here $P=\{P_k\}_{k\in\mathbb{Z}}$ is the weight filtration on $\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}}$ defined in [St2, p. 113]; however there is a mistype in [loc. cit.]: we have to replace "$W_m\Omega^p_{\underline{D}}=$ image of $\omega^m_D\otimes\Omega^{p-m}_{\underline{D}}$ in $\Omega^p_{\underline{D}}$" in [loc. cit.] by "$W_m\Omega^p_{\underline{D}}=$ the image of $\omega^{p-m}_D\otimes\Omega^m_{\underline{D}}$ in $\Omega^p_{\underline{D}}$".

LEMMA 3.2. *Let*

$$\mathrm{gr}^{\tau}_k\varepsilon_*(I^{\bullet}_{\mathbb{C}})\longrightarrow\mathrm{gr}^P_k\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}}\quad(k\in\mathbb{Z}_{\geq 0}) \tag{3.2.1}$$

be a morphism in $\mathrm{D}^+(\mathbb{C}_X)$ *obtained by the following natural diagram*

(3.2.2)
$$(\varepsilon_*(I^\bullet_{\mathbb{C}}),\tau) \overset{\subset}{\longrightarrow} (\mathrm{MF}_{\mathbb{C}}(N_\infty),\tau) \overset{\mathrm{MF}(\lambda)}{\underset{\sim}{\longleftarrow}} (\mathrm{MF}(N_{\tilde{\Lambda}}),\tau) \overset{\phi}{\underset{\sim}{\longrightarrow}} (\widetilde{\Lambda}^\bullet_{X/\mathbb{C}},\tau) \overset{\subset}{\longrightarrow} (\widetilde{\Lambda}^\bullet_{X/\mathbb{C}},P).$$

If k is a positive integer, then the following composite morphism

(3.2.3)
$$\mathbb{C}_{\overset{\circ}{X}{}^{(k)}}(-k) \overset{(2.0.4.1;\infty)\otimes_{\mathbb{Z}}\mathbb{C}}{\underset{\sim}{\longrightarrow}} R^k\varepsilon_*(\mathbb{C}_{X^{\log}}) \overset{\sim}{\longleftarrow} \mathrm{gr}^\tau_k\varepsilon_*(I^\bullet_{\mathbb{C}})\{k\} \overset{(3.2.1)}{\longrightarrow} \mathrm{gr}^P_k\widetilde{\Lambda}^\bullet_{X/\mathbb{C}}\{k\} \overset{\mathrm{Res}}{\underset{\sim}{\longrightarrow}} \Omega^\bullet_{\overset{\circ}{X}{}^{(k)}/\mathbb{C}}$$

in $\mathrm{D}^+(\mathbb{C}_X)$ *is the following composite morphism*

(3.2.4)
$$\mathbb{C}_{\overset{\circ}{X}{}^{(k)}}(-k) = \mathbb{C}_{\overset{\circ}{X}{}^{(k)}} \overset{(-1)^k\times}{\longrightarrow} \mathbb{C}_{\overset{\circ}{X}{}^{(k)}} \overset{\subset}{\longrightarrow} \Omega^\bullet_{\overset{\circ}{X}{}^{(k)}/\mathbb{C}}.$$

PROOF. Recall the sheaf $\mathcal{L}_{X^{\log}}$ of the logarithms of local sections of $\varepsilon^{-1}(\mathcal{M}^{\mathrm{gp}}_X)$ ([KN, (1.4)]): $\mathcal{L}_{X^{\log}}$ is the fiber product of the following natural diagram

$$\begin{array}{ccc} & & \epsilon^{-1}(\mathcal{M}^{\mathrm{gp}}_X) \\ & & \downarrow \\ \mathrm{Cont}_{X^{\log}}(\ ,\sqrt{-1}\mathbb{R}) & \overset{\exp}{\longrightarrow} & \mathrm{Cont}_{X^{\log}}(\ ,\mathbb{S}^1), \end{array}$$

where $\mathrm{Cont}_{X^{\log}}(\ ,T)$ is the sheaf of continuous functions on $X^{\log}$ with values in T for a topological space T. Then we have an exact sequence

(3.2.5)
$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathcal{L}_{X^{\log}} \overset{\exp}{\longrightarrow} \varepsilon^{-1}(\mathcal{M}^{\mathrm{gp}}_X) \longrightarrow 0$$

on $X^{\log}$. By [KN, (1.5)], the boundary morphism of $R\varepsilon_*((3.2.5))$ and the cup product induce an isomorphism

(3.2.6)
$$\mathbb{Z}_{\overset{\circ}{X}{}^{(r)}} = \bigwedge^r(\mathcal{M}^{\mathrm{gp}}_X/\mathcal{O}^*_X) \overset{\sim}{\longrightarrow} R^r\varepsilon_*(\mathbb{Z}(r)) \quad (r\in\mathbb{Z}_{\geq 1}).$$

First we prove (3.2) for the case $k=1$.

Let $\{X_m\}_m$ be the irreducible components of $\overset{\circ}{X}$. Because the problem is local on X, we may assume that there exists a section $t_m \in \Gamma(X,\mathcal{M}_X)$ $(\forall m)$ whose image $\bar{t}_m \in \Gamma(X,\mathcal{O}_X)$ defines the closed subscheme X_m of $\overset{\circ}{X}$. Let $\mathcal{U} := \{U_i\}_i$ be an open covering of $X^{\log}$ such that each U_i is so small that there exist sections $T_{m,i} \in \Gamma(U_i,\mathcal{L}_{X^{\log}})$ such that $\exp(T_{m,i}) = \varepsilon^{-1}(t_m)|_{U_i}$. Set $U_{ij} := U_i\cap U_j$ and $U_{ijk} := U_i\cap U_j\cap U_k$. Then the image of $(0,\dots,0,\underset{m}{1},0,\dots,0) \in \mathbb{Z}_{\overset{\circ}{X}{}^{(1)}}$ by the morphism (3.2.6) in $R^1\varepsilon_*(\mathbb{Z}(1))$ for the case $r=1$ is represented by a Čech cocycle $\{T_{m,j}-T_{m,i}\}_{ij}\in\prod_{ij}\Gamma(U_{ij},\mathbb{Z}(1))$. On the other hand, consider the following Čech double complex

$$(3.2.7)\qquad \begin{array}{ccccccc} \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\ {\scriptstyle d}\uparrow & & {\scriptstyle -d}\uparrow & & {\scriptstyle d}\uparrow & & \\ \prod_i \widetilde{\Lambda}^{1,\log}_X(U_i) & \xrightarrow{\partial} & \prod_{i,j}\widetilde{\Lambda}^{1,\log}_X(U_{ij}) & \xrightarrow{\partial} & \prod_{i,j,k}\widetilde{\Lambda}^{1,\log}_X(U_{ijk}) & \xrightarrow{\partial} & \cdots \\ {\scriptstyle d}\uparrow & & {\scriptstyle -d}\uparrow & & {\scriptstyle d}\uparrow & & \\ \prod_i \widetilde{\Lambda}^{0,\log}_X(U_i) & \xrightarrow{\partial} & \prod_{i,j}\widetilde{\Lambda}^{0,\log}_X(U_{ij}) & \xrightarrow{\partial} & \prod_{i,j,k}\widetilde{\Lambda}^{0,\log}_X(U_{ijk}) & \xrightarrow{\partial} & \cdots . \end{array}$$

Then $(d+\partial)\{T_{m,i}\}=\{dT_{m,i}\}+\{T_{m,j}-T_{m,i}\}_{ij}$. Hence $\{T_{m,j}-T_{m,i}\}_{ij}=-\{dT_{m,i}\}$ in $\check{H}^1(\mathcal{U},\widetilde{\Lambda}^{\bullet,\log}_{X/\mathbb{C}})$. By the definition of $dT_{m,i}$ (the last formula in [KN, p. 174]), $dT_{m,i}=d\log\varepsilon^{-1}(t_m)|_{U_i}$. Hence $\{T_{m,j}-T_{m,i}\}_{ij}=-\{d\log\varepsilon^{-1}(t_m)|_{U_i}\}$ in $\check{H}^1(\mathcal{U},\widetilde{\Lambda}^{\bullet,\log}_{X/\mathbb{C}})$. Therefore, by the definition of the morphism (3.2.3), (3.2.3) is equal to the minus natural inclusion morphism $\mathbb{C}_{\overset{\circ}{X}{}^{(1)}}\longrightarrow\Omega^{\bullet}_{\overset{\circ}{X}{}^{(1)}/\mathbb{C}}$.

Next consider the general case $k\geq 1$. Note that the induced morphism

$$\lambda\colon R^k(\varepsilon\pi)_*\pi^{-1}(\widetilde{\Lambda}^{\bullet,\log}_{X/\mathbb{C}})\longrightarrow\mathcal{H}^k(B(J^{\bullet}_{\mathbb{C}}))=\mathcal{H}^k(J^{\bullet}_{\mathbb{C}})$$

is compatible with cup products because $\widetilde{\Lambda}^{\bullet,\log}_{X/\mathbb{C}}$ is a resolution of $\mathbb{C}_{X^{\log}}$ by the logarithmic Poincaré lemma ([KN, (3.8)]). The equality $\mathcal{H}^k(R(\varepsilon\pi)_*\pi^{-1}(\widetilde{\Lambda}^{\bullet,\log}_{X/\mathbb{C}}))=\mathcal{H}^k(\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}})$ is also compatible with cup products because this equality is obtained by the equality $R(\varepsilon\pi)_*\pi^{-1}(\widetilde{\Lambda}^{q,\log}_{X/\mathbb{C}})=\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{q}_{X/\mathbb{C}}$ for each $q\in\mathbb{N}$ ([FN, (3.3)]). Hence we obtain (3.2) for the general case. □

Corollary 3.3. *The morphism* (3.2.1) *for a positive integer k is an isomorphism.*

Proof. By (3.2), the Poincaré lemma for $\overset{\circ}{X}{}^{(k)}$ $(k\geq 1)$ shows that (3.2.1) is an isomorphism. □

Lemma 3.4. *The diagram* (3.2.2) *induces the following isomorphism*

$$(3.4.1)\qquad \mathrm{gr}^{\tau}_0\varepsilon_*(I^{\bullet}_{\mathbb{C}})\xrightarrow{\sim}\mathrm{gr}^{P}_0\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}}.$$

Proof. The source of (3.4.1) is equal to $R^0\varepsilon_*(\mathbb{C}_{X^{\log}})\xleftarrow{\sim}\mathbb{C}_{\overset{\circ}{X}}$ by the proper base change theorem. Because the problem is local on X, we may assume that X is a SNCD on a smooth analytic space Y over $(\mathrm{Spec}\,\mathbb{C})_{\mathrm{an}}$ with canonical log structure. Because a regular local ring is a UFD, we have a surjective morphism $\Omega^1_{Y/\mathbb{C}} \longrightarrow \mathrm{Ker}(\mathrm{Res}\colon \Omega^1_{Y/\mathbb{C}}(\log X) \otimes_{\mathcal{O}_Y} \mathcal{O}_X \longrightarrow \mathcal{O}_{X^{(1)}})$. Hence the target of (3.4.1) is equal to $\Omega^\bullet_{Y/\mathbb{C}}/\Omega^\bullet_{Y/\mathbb{C}}(-\log X)$ (cf. [Mo, (3.1)]). By [DI2, (4.2.2) (c)], the sequence

$$0 \longrightarrow \Omega^\bullet_{Y/\mathbb{C}}(-\log X) \longrightarrow \Omega^\bullet_{Y/\mathbb{C}} \longrightarrow \Omega^\bullet_{\overset{\circ}{X}{}^{(1)}/\mathbb{C}} \longrightarrow \cdots$$

of complexes is exact. Therefore the complex $\Omega^\bullet_{Y/\mathbb{C}}/\Omega^\bullet_{Y/\mathbb{C}}(-\log X)$ is quasi-isomorphic to the single complex of the following double complex

$$(3.4.2) \qquad (\Omega^\bullet_{\overset{\circ}{X}{}^{(1)}/\mathbb{C}}, -d) \longrightarrow (\Omega^\bullet_{\overset{\circ}{X}{}^{(2)}/\mathbb{C}}, d) \longrightarrow (\Omega^\bullet_{\overset{\circ}{X}{}^{(3)}/\mathbb{C}}, -d) \longrightarrow \cdots.$$

By the classical Poincaré lemma for $\overset{\circ}{X}{}^{(k)}$ ($k \in \mathbb{Z}_{\geq 1}$), the single complex is quasi-isomorphic to the following complex

$$(3.4.3) \qquad \mathbb{C}_{\overset{\circ}{X}{}^{(1)}} \longrightarrow \mathbb{C}_{\overset{\circ}{X}{}^{(2)}} \longrightarrow \cdots.$$

Obviously the complex (3.4.3) is isomorphic to $\mathbb{C}_{\overset{\circ}{X}}$. Therefore we obtain (3.4). □

Corollary 3.5. (1) *The diagram* (3.2.2) *induces the following filtered isomorphism*

$$(3.5.1) \qquad (\varepsilon_*(I^\bullet_{\mathbb{C}}), \tau) \overset{\sim}{\longrightarrow} (\widetilde{\Lambda}^\bullet_{X/\mathbb{C}}, P)$$

in $\mathrm{D}^+\mathrm{F}(\mathbb{C}_X)$.

(2) *The natural morphism*

$$(3.5.2) \qquad (\widetilde{\Lambda}^\bullet_{X/\mathbb{C}}, \tau) \longrightarrow (\widetilde{\Lambda}^\bullet_{X/\mathbb{C}}, P)$$

is a filtered quasi-isomorphism.

Proof. (1): (1) is the conjunction of (3.3) and (3.4).

(2): Because the morphisms $(\varepsilon_*(I^\bullet_{\mathbb{C}}), \tau) \overset{\subset}{\longrightarrow} (\mathrm{MF}_{\mathbb{C}}(N_\infty), \tau)$, $\mathrm{MF}(\lambda)$ and ϕ are filtered quasi-isomorphisms, (2) immediately follows from (1). □

Remark 3.6. (1) (cf. [St2, p. 108]) C. Nakayama has kindly pointed out to me that one can easily obtain that the filtered morphism $(\widetilde{\Lambda}^\bullet_{X/\mathbb{C}}, \tau) \longrightarrow (\widetilde{\Lambda}^\bullet_{X/\mathbb{C}}, P)$ is a filtered isomorphism by a well-known method. Indeed we have the following spectral sequence

$$(3.6.1) \qquad E_1^{-k,h+k} = \mathcal{H}^h(\mathrm{gr}^P_k \widetilde{\Lambda}^\bullet_{X/\mathbb{C}}) \Longrightarrow \mathcal{H}^h(\widetilde{\Lambda}^\bullet_{X/\mathbb{C}}).$$

(The spectral sequence

$$E_1^{-m,k} = \mathcal{H}^k(\mathrm{Gr}_m^W \Omega_X^{\bullet}(\log D)) \Longrightarrow \mathcal{H}^k(\Omega_X^{\bullet}(\log D))$$

in [St2, p. 108] is mistaken; the right one is the following:

$$E_1^{-m,k+m} = \mathcal{H}^k(\mathrm{Gr}_m^W \Omega_X^{\bullet}(\log D)) \Longrightarrow \mathcal{H}^k(\Omega_X^{\bullet}(\log D)).)$$

For $k \geq 1$, the Poincaré residue isomorphism gives an isomorphism $\mathrm{gr}_k^P \widetilde{\Lambda}_{X/\mathbb{C}}^{\bullet} \xrightarrow{\sim} \Omega^{\bullet}_{\overset{\circ}{X}{}^{(k)}/\mathbb{C}}\{-k\}$ ([St2, p. 113]). Since the natural morphism $\mathbb{C}_{X^{(k)}} \longrightarrow \Omega^{\bullet}_{\overset{\circ}{X}{}^{(k)}/\mathbb{C}}$ is a quasi-isomorphism by the Poincaré lemma for $\overset{\circ}{X}{}^{(k)}$, we have $E_1^{-k,h+k} = 0$ for $h \neq k$ and $E_1^{-k,2k} = \mathbb{C}_{\overset{\circ}{X}{}^{(k)}} = \mathcal{H}^k(\mathrm{gr}_k^P \widetilde{\Lambda}_{X/\mathbb{C}}^{\bullet})$. If $k = 0$, then the natural morphism $\mathbb{C}_{\overset{\circ}{X}} \longrightarrow \mathrm{gr}_0^P \widetilde{\Lambda}_{X/\mathbb{C}}^{\bullet}$ is a quasi-isomorphism by the proof of (3.4.1). Hence $E_1^{0h} = 0$ if $h \neq 0$ and $E_1^{00} = \mathbb{C}_{\overset{\circ}{X}} = \mathcal{H}^0(\mathrm{gr}_0^P \widetilde{\Lambda}_{X/\mathbb{C}}^{\bullet})$. Therefore the spectral sequence (3.6.1) degenerates at E_1. Consequently we have $\mathcal{H}^m(\mathrm{gr}_k^{\tau} \widetilde{\Lambda}_{X/\mathbb{C}}^{\bullet}) = \mathcal{H}^m(\mathrm{gr}_k^P \widetilde{\Lambda}_{X/\mathbb{C}}^{\bullet})$ for all $m, k \in \mathbb{Z}$. This implies that the morphism (3.5.2) is a filtered isomorphism.

(2) If the proof of [St2, (4.7)] is right, (3.5) (2) also follows from [St2, (4.6)] and [St2, (4.7)]. However, even if X is the special fiber of an analytic semistable family $\mathcal{X}$ with canonical log structure over a unit disk, there exists only a relation a priori between $\widetilde{\Lambda}_{X/\mathbb{C}}^{\bullet}$ and $\Omega_{\mathcal{X}/\mathbb{C}}^{\bullet}(\log X)$: there exists a natural morphism

$$(3.6.2)\quad \Omega_{\mathcal{X}/\mathbb{C}}^{\bullet}(\log X) \longrightarrow (\iota_*(\mathcal{O}_X) \xrightarrow{d} \iota_*(\Omega_{\mathcal{X}/\mathbb{C}}^1(\log X) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_X) \xrightarrow{d} \cdots) = \iota_* \widetilde{\Lambda}_{X/\mathbb{C}}^{\bullet}$$

of complexes on $\mathcal{X}$, where ι is the closed immersion $X \overset{\subset}{\longrightarrow} \mathcal{X}$. (Note that the differential operator $d \colon \Omega_{\mathcal{X}/\mathbb{C}}^i(\log X) \longrightarrow \Omega_{\mathcal{X}/\mathbb{C}}^{i+1}(\log X)$ $(i \in \mathbb{N})$ is not $\mathcal{O}_{\mathcal{X}}$-linear.) Hence I do not understand the reduction to the local case in [St2, (4.7)] obtaining a filtered quasi-isomorphism to $(\widetilde{\Lambda}_{X/\mathbb{C}}^{\bullet}, P)$ by a filtered quasi-isomorphism to $(\Omega_{\mathcal{X}/\mathbb{C}}^{\bullet}(\log X), P)$. Moreover, since $L_{\underline{D}}^1$ in [loc. cit.] depends heavily on the choice of local charts of X as mentioned in §2, the sheaf $W_m K_{\underline{D}}^q$ $(q \geq 1$, $m \in \mathbb{N})$ (in particular, $K_{\underline{D}}^q$) in [loc. cit.] is not (shown to be) well-defined.

Let $A_{X/\mathbb{C}}^{\bullet}$ be the single complex associated to a double complex $A_{X/\mathbb{C}}^{\bullet\bullet}$ defined by the following (cf. [St2, (5.3)]): $A_{X/\mathbb{C}}^{ij} := \widetilde{\Lambda}_{X/\mathbb{C}}^{i+j+1} / P_j \widetilde{\Lambda}_{X/\mathbb{C}}^{i+j+1}$ $(i,j \in \mathbb{N})$ with boundary morphisms

$$(3.6.3)\qquad \begin{array}{ccc} A_{X/\mathbb{C}}^{i,j+1} & & \\ {\scriptstyle (-1)^i d\log t\wedge}\big\uparrow & & \\ A_{X/\mathbb{C}}^{ij} & \xrightarrow{(-1)^{j+1} d} & A_{X/\mathbb{C}}^{i+1,j}. \end{array}$$

Note that the morphism $d\log t\wedge$ is independent of the choice of t. By using the weight filtration P on $\widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}}$, we have filtrations P's on $A^{\bullet\bullet}_{X/\mathbb{C}}$ and $A^{\bullet}_{X/\mathbb{C}}$:

$$P_kA^{ij}_{X/\mathbb{C}} := (P_{2j+k+1}+P_j)\widetilde{\Lambda}^{i+j+1}_{X/\mathbb{C}}/P_j\widetilde{\Lambda}^{i+j+1}_{X/\mathbb{C}} \quad (k\in\mathbb{Z}). \tag{3.6.4}$$

Since $\phi\circ\theta = -(d\log t\wedge)\circ\phi$,

$$\varphi^{ij} := (-1)^{j+1}\phi\colon A^{ij}(\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{*}_{X/\mathbb{C}}) \longrightarrow A^{ij}_{X/\mathbb{C}} \tag{3.6.5}$$

gives a morphism

$$\varphi^{\bullet\bullet}\colon (A^{\bullet\bullet}(\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{*}_{X/\mathbb{C}}),P) \longrightarrow (A^{\bullet\bullet}_{X/\mathbb{C}},P) \tag{3.6.6}$$

of filtered double complexes. Hence we have a morphism

$$\varphi := s(\varphi^{\bullet\bullet})\colon (A^{\bullet}(\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{*}_{X/\mathbb{C}}),P) \longrightarrow (A^{\bullet}_{X/\mathbb{C}},P) \tag{3.6.7}$$

of filtered complexes. We also have the following commutative diagram

$$\begin{array}{ccc} A^{\bullet}(\mathbb{C}[u]\otimes_{\mathbb{C}}\widetilde{\Lambda}^{*}_{X/\mathbb{C}}) & \xrightarrow{\varphi} & A^{\bullet}_{X/\mathbb{C}} \\ {\scriptstyle \mu_{\widetilde{\Lambda}}\circ\iota}\uparrow & & \uparrow{\scriptstyle d\log t\wedge} \\ \widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}} & \xrightarrow{\text{proj.}} & \Lambda^{\bullet}_{X/\mathbb{C}}. \end{array} \tag{3.6.8}$$

By the filtered quasi-isomorphism (3.0.8) and a morphism (3.6.7), we have the following morphism

$$\psi\colon (A^{\bullet}(J^{*}_{\mathbb{Q}})\otimes_{\mathbb{Q}}\mathbb{C},P) \longrightarrow (A^{\bullet}_{X/\mathbb{C}},P) \tag{3.6.9}$$

in $\mathrm{D}^{+}\mathrm{F}(\mathbb{C}_X)$.

LEMMA 3.7. *The morphism* ψ *in* (3.6.9) *is an isomorphism in* $\mathrm{D}^{+}\mathrm{F}(\mathbb{C}_X)$.

PROOF. (3.7) immediately follows from (3.3). □

By [St2, (5.5)] (cf. [Mo, Proposition 3.15]), the morphism $d\log t\wedge\colon \Lambda^{\bullet}_{X/\mathbb{C}} \longrightarrow \widetilde{\Lambda}^{\bullet}_{X/\mathbb{C}}[1]$ induces the following quasi-isomorphism

$$d\log t\wedge\colon \Lambda^{\bullet}_{X/\mathbb{C}} \longrightarrow A^{\bullet}_{X/\mathbb{C}}. \tag{3.7.1}$$

The complexes $A^{\bullet}_{X/\mathbb{C}}$ and $\Lambda^{\bullet}_{X/\mathbb{C}}$ have Hodge filtrations $\mathrm{Fil}_{\mathrm{H}}$ defined by the following stupid filtrations:

$$\mathrm{Fil}^{i}_{\mathrm{H}}(A^{\bullet}_{X/\mathbb{C}}) := A^{\bullet\geq i,\bullet}_{X/\mathbb{C}}, \quad \mathrm{Fil}^{i}_{\mathrm{H}}(\Lambda^{\bullet}_{X/\mathbb{C}}) := \Lambda^{\bullet\geq i}_{X/\mathbb{C}} \quad (i\in\mathbb{Z}). \tag{3.7.2}$$

Hence we obtain the following which has been obtained in [FN, (3.12)] with signs which are different from ours:

THEOREM 3.8. *The pair* $((A^\bullet(J^*_{\mathbb{Q}}), P), (A^\bullet_{X/\mathbb{C}}, P, \mathrm{Fil}_H))$ *is a cohomological mixed* $\mathbb{Q}$*-Hodge complex if the irreducible components of* $\overset{\circ}{X}$ *are compact and Kähler or the analytifications of proper smooth schemes over* $\mathbb{C}$.

DEFINITION 3.9. Assume that the irreducible components of $\overset{\circ}{X}$ are compact and Kähler or the analytifications of proper smooth schemes over $\mathbb{C}$. Then we call $((A^\bullet(J^*_{\mathbb{Q}}), P), (A^\bullet_{X/\mathbb{C}}, P, \mathrm{Fil}_H))$ the *Steenbrink cohomological mixed* $\mathbb{Q}$*-Hodge complex* of X/s.

REMARK 3.10. (cf. [KwN, p. 406]) Since the diagram $B(J^\bullet_{\mathbb{Q}}) \overset{\subset}{\longrightarrow} J^\bullet_{\mathbb{Q}} \longleftarrow J^\bullet_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ induces an isomorphism $B(J^\bullet_{\mathbb{Q}}) \overset{\sim}{\longrightarrow} J^\bullet_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ in $\mathrm{D}^+(\mathbb{Q}_X)$, the triple

$$(J^\bullet_{\mathbb{Z}}, (A^\bullet(J^*_{\mathbb{Q}}), P), (A^\bullet_{X/\mathbb{C}}, P, \mathrm{Fil}_H))$$

is a cohomological mixed $\mathbb{Z}$-Hodge complex. We also call the triple *Steenbrink cohomological mixed* $\mathbb{Z}$*-Hodge complex* of X/s.

Combining (3.0.6), (3.0,10) and (3.6.8), we have the following commutative diagram, which will be used in §10 below:

$$\begin{array}{ccc}
A^\bullet(J^*_{\mathbb{C}}) & \overset{\sim}{\longleftarrow} & A^\bullet(\mathbb{C}[u] \otimes_{\mathbb{C}} \widetilde{\Lambda}^*_{X/\mathbb{C}}) \\
{\scriptstyle \mu_{\mathbb{C}}}\uparrow \simeq & & {\scriptstyle \varphi}\downarrow \simeq \\
J^\bullet_{\mathbb{C}} & & A^\bullet_{X/\mathbb{C}} \\
\uparrow & & {\scriptstyle d\log t\wedge}\uparrow \simeq \\
\mathbb{C}_X & \longrightarrow & \Lambda^\bullet_{X/\mathbb{C}}.
\end{array} \tag{3.10.1}$$

Therefore we have the following commutative diagram for $h \in \mathbb{Z}$:

$$\begin{array}{ccc}
H^h(X, A^\bullet(J^*_{\mathbb{C}})) & \overset{=}{\longleftarrow} & H^h(X, A^\bullet(\mathbb{C}[u] \otimes_{\mathbb{C}} \widetilde{\Lambda}^*_{X/\mathbb{C}})) \\
{\scriptstyle =}\uparrow & & {\scriptstyle \varphi}\downarrow \simeq \\
H^h(X_\infty, \mathbb{C}) & & H^h(X, A^\bullet_{X/\mathbb{C}}) \\
\uparrow & & {\scriptstyle d\log t\wedge}\uparrow \simeq \\
H^h(X, \mathbb{C}) & \longrightarrow & H^h(X, \Lambda^\bullet_{X/\mathbb{C}}).
\end{array} \tag{3.10.1; H}$$

4. l-adic and ∞-adic Gysin morphisms of smooth divisors.

In this section, paying attention to signs, we study the l-adic and the ∞-adic Gysin morphisms of smooth divisors on an irreducible component of a SNCL variety over a log point. This study is necessary for the descriptions in §5 below of the boundary morphisms between the E_1-terms of (2.0.7; l), (2.0.7; ∞) and (2.1.10).

First we consider the case where the base field is $\mathbb{C}$ and where the ambient space is smooth over $\mathbb{C}$.

Let Y be a smooth analytic space over $\mathbb{C}$. Let D be a smooth divisor on Y. Consider the following exact sequence

$$(4.0.1) \qquad 0 \longrightarrow \Omega^{\bullet}_{Y/\mathbb{C}} \longrightarrow \Omega^{\bullet}_{Y/\mathbb{C}}(\log D) \overset{\mathrm{Res}}{\longrightarrow} \Omega^{\bullet}_{D/\mathbb{C}}\{-1\} \longrightarrow 0.$$

Here the Poincaré residue morphism is locally defined by $\omega \wedge \wedge\, d\log f \longmapsto \omega|_D$, where $f = 0$ ($f \in \Gamma(Y, \mathcal{O}_Y)$) is a local equation of D. Let $\mathbb{C}_Y = \Omega^{\bullet}_{Y/\mathbb{C}}$ and $\mathbb{C}_D = \Omega^{\bullet}_{D/\mathbb{C}}$ be the natural identifications in the derived categories $\mathrm{D}^+(\mathbb{C}_Y)$ and $\mathrm{D}^+(\mathbb{C}_D)$. By these identifications we have the following triangle

$$(4.0.2) \qquad \mathbb{C}_Y \longrightarrow \Omega^{\bullet}_{Y/\mathbb{C}}(\log D) \longrightarrow \mathbb{C}_D\{-1\} \overset{+1}{\longrightarrow} .$$

Let

$$(4.0.3) \qquad d \colon \mathbb{C}_D\{-1\} \longrightarrow \mathbb{C}_Y[1]$$

be the boundary morphism of the triangle (4.0.2) obtained by the Convention (4).

LEMMA 4.1. *Let $G \colon \mathbb{C}_D\{-1\} \longrightarrow \mathbb{C}_Y[1]$ be the Gysin morphism of D on Y. Then $d = -G$.*

PROOF. Let $\iota \colon D \overset{\subset}{\longrightarrow} Y$ be the natural closed immersion. Because ι_* is exact, we have the following formula

$$\begin{aligned}
(4.1.1)\ H^0(\mathrm{RHom}_{\mathbb{C}_Y}(\iota_*(\mathbb{C}_D)\{-1\}, \mathbb{C}_Y[1])) &= H^0(\mathrm{RHom}_{\mathbb{C}_D}(L\iota_*(\mathbb{C}_D), \mathbb{C}_Y\{1\}[1])) \\
&\simeq H^0(\mathrm{RHom}_{\mathbb{C}_D}(\mathbb{C}_D, R\iota^!\mathbb{C}_Y\{1\}[1])) \\
&= H^0(D, R\iota^!\mathbb{C}_Y\{1\}[1]) \\
&= H^2(Y, R\iota^!\mathbb{C}_Y) = H^2(D, R^2\iota^!\mathbb{C}_Y) \\
&= H^2(D, \mathbb{C}_D[-2]) = H^0(D, \mathbb{C}_D).
\end{aligned}$$

Here we obtain the isomorphism in (4.1.1) by (1.0.5) under the Convention (1.0.4) and we have used the Convention (6) in the third equality in (4.1.1).

Since the problem is local, we may assume that there exists a locally finite open covering $\{Y_i\}_i$ of Y such that $D \cap Y_i$ is defined by an analytic equation $t_i = 0$ ($t_i \in \Gamma(Y_i, \mathcal{O}_{Y_i})$). Set $Y_{ij} := Y_i \cap Y_j$ and $Y_{ijk} := Y_i \cap Y_j \cap Y_k$. We follow the convention on the signs of torsors in [SGA $4\frac{1}{2}$, Cycle 1.1]. Then the section $1 \in H^0(D, \mathcal{O}_D)$ defines the class $[\{s_{ij}\}] \in \check{H}^1_D(Y, \Omega^1_{Y/\mathbb{C}})$ of a Čech 1-cocycle defined by $s_{ij} := d\log t_j - d\log t_i$ by the exact sequence (4.0.1) and by the convention in [loc. cit.]. Here we omit the restriction $|_{Y_{ij}}$ for the sections $d\log t_j$ and $d\log t_i$. Set $t_{ij} := t_i/t_j \in \Gamma(Y_{ij}, \mathcal{O}^*_{Y_{ij}})$. Retaking an open covering $\{Y_i\}_i$ of Y, we may assume that a branch $\log(t_{ij}) \in \Gamma(Y_{ij}, \mathcal{O}_{Y_{ij}})$ of t_{ij} is defined.

On the other hand, since $t_j^{-1}/t_i^{-1} = t_{ij}$, the line bundle $\mathcal{O}_Y(D)$ defines a 1-cocycle $\{t_{ij}\} \in \check{H}^1_D(Y, \mathcal{O}^*_Y)$. Set $U := Y \setminus D$ and let $j\colon U \overset{\subset}{\longrightarrow} Y$ be the natural open immersion. For an abelian sheaf E on Y, set $K^U(E) :=$ $:= \mathrm{Ker}(E \longrightarrow j_*(E|_U))$. The 1-cocycle $\{t_{ij}\}$ and the exponential sequence

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathcal{O}_Y \overset{\exp}{\longrightarrow} \mathcal{O}^*_Y \longrightarrow 0$$

gives a 2-cocycle $\{u_{ijk}\}$ defined by $u_{ijk} = \log t_{jk} - \log t_{ik} + \log t_{ij} \in$ $\in K^{U\cap Y_{ijk}}(\mathbb{Z}(1)_{Y_{ijk}})$. Consider the following Čech double complex

$$(4.1.2)\qquad \begin{array}{ccccccc}
\cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\
{\scriptstyle d}\uparrow & & {\scriptstyle -d}\uparrow & & {\scriptstyle d}\uparrow & & \\
\prod_i K^{U\cap Y_i}(\Omega^2_Y)(Y_i) & \overset{\delta}{\longrightarrow} & \prod_{i,j} K^{U\cap Y_{ij}}(\Omega^2_Y)(Y_{ij}) & \overset{\delta}{\longrightarrow} & \prod_{i,j,k} K^{U\cap Y_{ijk}}(\Omega^2_Y)(Y_{ijk}) & \overset{\delta}{\longrightarrow} & \cdots \\
{\scriptstyle d}\uparrow & & {\scriptstyle -d}\uparrow & & {\scriptstyle d}\uparrow & & \\
\prod_i K^{U\cap Y_i}(\Omega^1_Y)(Y_i) & \overset{\delta}{\longrightarrow} & \prod_{i,j} K^{U\cap Y_{ij}}(\Omega^1_Y)(Y_{ij}) & \overset{\delta}{\longrightarrow} & \prod_{i,j,k} K^{U\cap Y_{ijk}}(\Omega^1_Y)(Y_{ijk}) & \overset{\delta}{\longrightarrow} & \cdots \\
{\scriptstyle d}\uparrow & & {\scriptstyle -d}\uparrow & & {\scriptstyle d}\uparrow & & \\
\prod_i K^{U\cap Y_i}(\Omega^0_Y)(Y_i) & \overset{\delta}{\longrightarrow} & \prod_{i,j} K^{U\cap Y_{ij}}(\Omega^0_Y)(Y_{ij}) & \overset{\delta}{\longrightarrow} & \prod_{i,j,k} K^{U\cap Y_{ijk}}(\Omega^0_Y)(Y_{ijk}) & \overset{\delta}{\longrightarrow} & \cdots .
\end{array}$$

Here the horizontal morphisms δ's are the usual boundary morphisms of the Čech complex. Then $\{u_{ijk}\} + \{s_{ij}\} = (\delta - d)(\{\log t_{ij}\})$. Hence $[\{s_{ij}\}] = -[\{u_{ijk}\}]$ in $\check{H}^2_D(Y, \Omega^\bullet_{Y/\mathbb{C}})$. Because the image of $[\{u_{ijk}\}]$ in $H^2_D(Y, \Omega^\bullet_{Y/\mathbb{C}}) = H^2_D(Y, \mathbb{C}) = H^2_D(Y, \mathbb{Z}(1)) \otimes_{\mathbb{Z}} \mathbb{C}$ is the cycle class $c(D)$ of D, the image of $[\{s_{ij}\}]$ in $H^2_D(Y, \mathbb{C})$ is equal to $-c(D)$. □

REMARK 4.2. The similar calculation at the end of [Gro, II §3] is mistaken in signs if we follow the convention on the signs of torsors in [SGA $4\frac{1}{2}$, Cycle 1.1].

Let the notations be as in [loc. cit.]. Because J is represented by a cocycle $\{t_{(j)}/t_{(i)}\}_{i<j}$, $c_1(J)$ is represented by a cocycle $\{d\log(t_{(j)}/t_{(i)})\}_{i<j}$ (not

$\{d\log(t_{(i)}/t_{(j)})\}_{i<j}$ in [loc. cit.]). On the other hand, the image of the boundary morphism of the Čech 0-cocycle $\{d\log t_{(i)}\}$ is equal to $d\log t_{(j)} - d\log t_{(i)} = d\log(t_{(j)}/t_{(i)})$. Hence

$$f_*(1) = c_1(J) \tag{4.2.1}$$

in $H^1(X, W_n\Omega^1_{X,\log})$ if one uses [Gro, II Proposition 3.5.6]. Obviously the formula (4.2.1) is not a desirable formula.

Next we consider the case of a SNCD on a smooth analytic space over $\mathbb{C}$.

Most of the following arguments except arguments on signs are included in [Mo, §4].

Let Y be a smooth analytic space over $\mathbb{C}$. Let D be a SNCD on $Y/\mathbb{C}$. Let $D = \bigcup_{i\in I} D_i$ be a union of smooth divisors. Fix a total order on I. Let k be a positive integer. Set $I_k := \{(i_0,\ldots,i_{k-1}) \mid i_0 < \cdots < i_{k-1}\ (i_j \in I)\}$ and $\underline{i} := (i_0,\ldots,i_{k-1})$. For an integer $0 \leq j \leq k-1$, set $\underline{i}_j := (i_0,\ldots,\hat{i}_j,\ldots,i_{k-1})$. Here $\hat{i}_j$ means to omit i_j. Set $D_{\underline{i}} := D_{i_0}\cap\cdots\cap D_{i_{k-1}}$ and $D_{\underline{i}_j} := D_{i_0}\cap\cdots\cap\hat{D}_{i_j}\cap\cdots\cap D_{i_{k-1}}$ for $k\geq 2$ and $D_{\underline{i}_0} := Y$. Set also $D^{(k)} := \coprod_{\underline{i}\in I_k} D_{\underline{i}}$ for $k \geq 1$ and $D^{(0)} := Y$. As in [Mo, p. 323], we have the following commutative diagram

$$\begin{array}{ccccc}
0 \longrightarrow & \mathrm{gr}^P_{k-1}\Omega^\bullet_{Y/\mathbb{C}}(\log D) & \longrightarrow & (P_k/P_{k-2})\Omega^\bullet_{Y/\mathbb{C}}(\log D) \\
& \downarrow{\scriptstyle \mathrm{Res}_{\underline{i}_j}} & & \downarrow{\scriptstyle \mathrm{Res}^{\underline{i}}_{\underline{i}_j}} \\
0 \longrightarrow & \Omega^\bullet_{D_{\underline{i}_j}/\mathbb{C}}\{-(k-1)\} & \longrightarrow & \Omega^\bullet_{D_{\underline{i}_j}/\mathbb{C}}(\log D_{\underline{i}})\{-(k-1)\}
\end{array}$$

$$\begin{array}{ccc}
\longrightarrow & \mathrm{gr}^P_k\Omega^\bullet_{Y/\mathbb{C}}(\log D) & \longrightarrow 0 \\
& \downarrow{\scriptstyle \mathrm{Res}_{\underline{i}}} & \\
\xrightarrow{(-1)^j\mathrm{Res}_{(\underline{i},\underline{i}_j)}} & \Omega^\bullet_{D_{\underline{i}}/\mathbb{C}}\{-k\} & \longrightarrow 0.
\end{array} \tag{4.2.2}$$

Here $\mathrm{Res}_{\underline{i}_j}$, $\mathrm{Res}_{\underline{i}}$ and $\mathrm{Res}_{(\underline{i},\underline{i}_j)}$ are usual Poincaré residue morphisms with respect to $D_{\underline{i}_j}$, $D_{\underline{i}}$ and $D_{\underline{i}}$, respectively. The morphism $\mathrm{Res}^{\underline{i}}_{\underline{i}_j}$ is locally defined by a morphism $\omega d\log x_{i_0}\wedge\cdots\wedge d\log x_{i_{k-1}} \longmapsto (-1)^j\omega d\log x_{i_j}$, where $x_{i_m} = 0$ $(x_{i_m}\in\mathcal{O}_Y)$ is a local equation of D_{i_m} $(0\leq m\leq k-1)$. Note that the formula $\mathrm{Rés}^I_{I_q}(\omega) = \alpha\wedge dx_{i_q}/x_{i_q}|_{D_{I_q}}$ in [Mo, p. 323, l. -9] have to be replaced by $\mathrm{Rés}^I_{I_q}(\omega) = (-1)^{q-1}\alpha\wedge dx_{i_q}/x_{i_q}|_{D_{I_q}}$.

The boundary morphism

$$\mathbb{C}_{D_{\underline{i}}}\{-k\} \longrightarrow \mathbb{C}_{D_{\underline{i}_j}}\{-(k-1)\}[1] \tag{4.2.3}$$

by the lower exact sequence of (4.2.2), by the Poincaré lemma and by the use of the Convention (4) is equal to $-((-1)^j G^{\underline{i}_j}_{\underline{i}})$ by (4.1), where $G^{\underline{i}_j}_{\underline{i}} \colon \mathbb{C}_{D_{\underline{i}}}\{-1\} \longrightarrow \mathbb{C}_{D_{\underline{i}_j}}[1]$ is the Gysin morphism of the closed immersion $D_{\underline{i}} \xrightarrow{\subset} D_{\underline{i}_j}$. Set $G := \sum_{\underline{i}\in I_k} \sum_{j=0}^{k-1} (-1)^j G^{\underline{i}_j}_{\underline{i}}$. Then we have the following:

PROPOSITION 4.3 (cf. [Mo, Proposition 4.4]). *Let k be a positive integer. Let*

$$d\colon \mathrm{gr}^P_k \Omega^\bullet_{Y/\mathbb{C}}(\log D) \longrightarrow \mathrm{gr}^P_{k-1}\Omega^\bullet_{Y/\mathbb{C}}(\log D)[1]$$

be the boundary morphism of the upper exact sequence of (4.2.2) *by the use of the Convention* (4)*. Then the following diagram*

$$\begin{array}{ccc}
\mathrm{gr}^P_k \Omega^\bullet_{Y/\mathbb{C}}(\log D) & \xrightarrow{\ d\ } & \mathrm{gr}^P_{k-1}\Omega^\bullet_{Y/\mathbb{C}}(\log D)[1] \\
{\scriptstyle \mathrm{Res}}\downarrow \simeq & & {\scriptstyle \mathrm{Res}}\downarrow \simeq \\
\Omega^\bullet_{Y^{(k)}/\mathbb{C}}\{-k\} & & \Omega^\bullet_{Y^{(k-1)}/\mathbb{C}}\{-(k-1)\}[1] \\
\| & & \| \\
\mathbb{C}_{Y^{(k)}}\{-k\} & \xrightarrow{\ -G\ } & \mathbb{C}_{Y^{(k-1)}}\{-(k-1)\}[1]
\end{array}$$

is commutative.

REMARK 4.4. If one considers the short exact sequence in [Mo, Proposition 4.4] as a triangle and if one considers d_1 in [loc. cit.] as the boundary morphism in [Ha1] of a triangle, our commutative diagram in (4.3) is the same as the ∞-adic analytic analogue of the commutative diagram in [Mo, Proposition 4.4]. (Note that the boundary morphism of a triangle in [Ha1] induces the minus traditional boundary morphism on cohomologies as remarked in the Convention (5).) However I suspect that the traditional morphism has been considered in [Mo, Proposition 4.4] and hence that [loc. cit.] is mistaken in a sign.

Next we consider the case of a SNCL analytic variety over $\mathbb{C}$.

Let $s := ((\mathrm{Spec}\,\mathbb{C})_{\mathrm{an}}, \mathbb{N} \oplus \mathbb{C}^*)$ be a log point. Let X be a proper SNCL analytic variety over s. Let $\overset{\circ}{X} := \bigcup_{i\in I} X_i$ be the union of the irreducible components of $\overset{\circ}{X}$. Fix a total order on I. Let $k \geq 2$ be an integer. Let I_k, $\underline{i}$, $\underline{i}_j$, $X_{\underline{i}}$ and $X_{\underline{i}_j}$ be analogous objects for X and I to those for D and I in the SNCD case above. As in [Mo, p. 326] (cf. (4.2.2)), we have the following commutative diagram

$$\begin{array}{ccccc} 0 \longrightarrow & \mathrm{gr}^P_{k-1}\widetilde{\Lambda}^\bullet_{X/\mathbb{C}} & \longrightarrow & (P_k/P_{k-2})\widetilde{\Lambda}^\bullet_{X/\mathbb{C}} \\ & \mathrm{Res}_{\underline{i}_j}\Big\downarrow\simeq & & \mathrm{Res}^{\underline{i}}_{\underline{i}_j}\Big\downarrow\simeq \\ 0 \longrightarrow & \Omega^\bullet_{X_{\underline{i}_j}/\mathbb{C}}\{-(k-1)\} & \longrightarrow & \Omega^\bullet_{X_{\underline{i}_j}/\mathbb{C}}(\log X_{\underline{i}})\{-(k-1)\} \end{array}$$

(4.4.1)

$$\begin{array}{ccc} \longrightarrow & \mathrm{gr}^P_k\widetilde{\Lambda}^\bullet_{X/\mathbb{C}} & \longrightarrow 0 \\ & \mathrm{Res}_{\underline{i}}\Big\downarrow\simeq & \\ \xrightarrow{(-1)^j\mathrm{Res}_{(\underline{i},\,\underline{i}_j)}} & \Omega^\bullet_{X_{\underline{i}}/\mathbb{C}}\{-k\} & \longrightarrow 0. \end{array}$$

Hence we have the following:

PROPOSITION 4.5. *Let $k \geq 2$ be an integer. Let*

$$d\colon \mathrm{gr}^P_k\widetilde{\Lambda}^\bullet_{X/\mathbb{C}} \longrightarrow \mathrm{gr}^P_{k-1}\widetilde{\Lambda}^\bullet_{X/\mathbb{C}}[1]$$

be the boundary morphism of the upper exact sequence of (4.4.1) *by the use of the Convention* (4). *Let $G^{\underline{i}_j}_{\underline{i}}\colon \mathbb{C}_{X_{\underline{i}}}\{-1\} \longrightarrow \mathbb{C}_{X_{\underline{i}_j}}[1]$ be the Gysin morphism of the closed immersion $X_{\underline{i}} \xrightarrow{\subset} X_{\underline{i}_j}$. Set $G := \sum_{\underline{i}\in I_k}\sum_{j=0}^{k-1}(-1)^j G^{\underline{i}_j}_{\underline{i}}$. Then the following diagram*

$$\begin{array}{ccc} \mathrm{gr}^P_k\widetilde{\Lambda}^\bullet_{X/\mathbb{C}} & \xrightarrow{d} & \mathrm{gr}^P_{k-1}\widetilde{\Lambda}^\bullet_{X/\mathbb{C}}[1] \\ \mathrm{Res}\Big\downarrow\simeq & & \mathrm{Res}\Big\downarrow\simeq \\ \Omega^\bullet_{X^{(k)}/\mathbb{C}}\{-k\} & & \Omega^\bullet_{X^{(k-1)}/\mathbb{C}}\{-(k-1)\}[1] \\ \Big\| & & \Big\| \\ \mathbb{C}_{\overset{\circ}{X}{}^{(k)}}\{-k\} & \xrightarrow{-G} & \mathbb{C}_{\overset{\circ}{X}{}^{(k-1)}}\{-(k-1)\}[1] \end{array} \tag{4.5.1}$$

is commutative.

Let $\varepsilon_{\mathrm{top}}\colon X^{\log} \longrightarrow \overset{\circ}{X}$ be the real blow up of X. Then we have the following triangle:

$$\mathrm{gr}^\tau_{k-1}R\varepsilon_{\mathrm{top}*}(\mathbb{Z}) \longrightarrow (\tau_k/\tau_{k-2})R\varepsilon_{\mathrm{top}*}(\mathbb{Z}) \longrightarrow \mathrm{gr}^\tau_k R\varepsilon_{\mathrm{top}*}(\mathbb{Z}) \xrightarrow{+1} \tag{4.5.2}$$

By (4.5.2), (2.0.4.1; ∞) and the Convention (4), we have the following boundary morphism

$$d\colon \mathbb{Z}(-k)_{\overset{\circ}{X}{}^{(k)}}\{-k\} \longrightarrow \mathbb{Z}(-(k-1))_{\overset{\circ}{X}{}^{(k-1)}}\{-(k-1)\}[1] \tag{4.5.3}$$

Corollary 4.6. *The boundary morphism* (4.5.3) *is equal to G.*

Proof. Let $\underline{i}=(i_0,\dots,i_{k-1})$ be an element of I_k. Let $j\leq k-1$ be a nonnegative integer. Let $\iota\colon X_{\underline{i}} \overset{\subset}{\longrightarrow} X_{\underline{i}_j}$ be the natural closed immersion. We would like to prove that the morphism (4.5.3) induces a morphism

$$(4.6.1)\qquad (-1)^j G_{\underline{i}}^{\underline{i}_j}\colon \iota_*(\mathbb{Z}_{X_{\underline{i}}})(-k)\{-k\} \longrightarrow \mathbb{Z}_{X_{\underline{i}_j}}(-(k-1))\{-(k-1)\}[1]$$

Because ι_* is exact, we have the following formula as in (4.1.1):

$$(4.6.2)\ H^0(\mathrm{RHom}_{\mathbb{Z}_{X_{\underline{i}_j}}}(\iota_*(\mathbb{Z}_{X_{\underline{i}}})(-k)\{-1\},\mathbb{Z}_{X_{\underline{i}_j}}(-(k-1))[1]))=H^0(D,R^2\iota^!\mathbb{Z}_{X_{\underline{i}_j}}(1)) =H^0(X_{\underline{i}},\mathbb{Z}_{X_{\underline{i}}}).$$

Hence it suffices to prove that the morphism (4.5.3) $\otimes_{\mathbb{Z}}\mathbb{C}$

$$(4.6.3)\qquad \mathbb{C}(-k)_{\overset{\circ}{X}{}^{(k)}}\{-k\} \overset{d}{\longrightarrow} \mathbb{C}(-(k-1))_{\overset{\circ}{X}{}^{(k-1)}}\{-(k-1)\}[1]$$

is equal to G. Because the natural morphism $R\varepsilon_{\mathrm{top}*}(\mathbb{C}_{X^{\log}})\longrightarrow \widetilde{\Lambda}^\bullet_{X/\mathbb{C}}$ induces an isomorphism $\mathrm{gr}^\tau_r R\varepsilon_{\mathrm{top}*}(\mathbb{C}_{X^{\log}}) \overset{\sim}{\longrightarrow} \mathrm{gr}^P_r\widetilde{\Lambda}^\bullet_{X/\mathbb{C}}$ ($r\in\mathbb{Z}_{>0}$) by (3.2.1), we have the following commutative diagram by (3.2.3) and (4.5.1):

$$(4.6.4)\qquad \begin{array}{ccc} \mathbb{C}(-k)_{\overset{\circ}{X}{}^{(k)}}\{-k\} & \overset{d}{\longrightarrow} & \mathbb{C}(-(k-1))_{\overset{\circ}{X}{}^{(k-1)}}\{-(k-1)\}[1] \\ {\scriptstyle(-1)^k\times}\downarrow & & \downarrow{\scriptstyle(-1)^{k-1}\times} \\ \mathbb{C}(-k)_{\overset{\circ}{X}{}^{(k)}}\{-k\} & \overset{-G}{\longrightarrow} & \mathbb{C}(-(k-1))_{\overset{\circ}{X}{}^{(k-1)}}\{-(k-1)\}[1]. \end{array}$$

The commutative diagram (4.6.4) shows (4.6). □

Next we recall a well-known method ([SGA 4-3, XI 4]) quickly.

Let T be a topological space. Let T_{cl} be a site defined by the following:

(4.6.5) An object of T_{cl} is a local isomorphism $U\longrightarrow T$ of topological spaces.

(4.6.6) A morphism in T_{cl} is a morphism of topological spaces over T.

(4.6.7) A family $\{U_\lambda\longrightarrow U\}_\lambda$ of morphisms in T_{cl} is called a covering family if the union of the images of U_λ's is U; the covering families define a Grothendieck pretopology and hence a Grothendieck topology on the category T_{cl}.

Let X be an fs log analytic space over $\mathbb{C}$ in the sense of [KN, §1]. Let $X^{\log}_{\mathrm{cl}}$ be the site defined above for the topological space $X^{\log}$. Let $\widetilde{X}^{\log}$ and $\overset{\circ}{\widetilde{X}}$ be the topoi defined by the classical topologies of $X^{\log}$ and $\overset{\circ}{X}$, respectively. Then we have natural morphisms $\varepsilon_{\mathrm{cl}}\colon \widetilde{X}^{\log}_{\mathrm{cl}} \longrightarrow \widetilde{\overset{\circ}{X}}_{\mathrm{cl}}$, $\varepsilon_{\mathrm{top}}\colon \widetilde{X}^{\log} \longrightarrow \widetilde{\overset{\circ}{X}}$, $\mu^{\log}\colon \widetilde{X}^{\log}_{\mathrm{cl}} \longrightarrow \widetilde{X}^{\log}$ and $\mu\colon \widetilde{\overset{\circ}{X}}_{\mathrm{cl}} \longrightarrow \widetilde{\overset{\circ}{X}}$ of topoi fitting into the following commutative diagram

$$
\begin{array}{ccc}
\widetilde{X}^{\log}_{\mathrm{cl}} & \xrightarrow{\mu^{\log}} & \widetilde{X}^{\log} \\
{\scriptstyle \epsilon_{\mathrm{cl}}}\downarrow & & \downarrow{\scriptstyle \epsilon_{\mathrm{top}}} \\
\widetilde{\overset{\circ}{X}}_{\mathrm{cl}} & \xrightarrow{\mu} & \widetilde{\overset{\circ}{X}}.
\end{array}
\tag{4.6.8}
$$

The morphism $\mu^{\log}_{*}$ (resp. μ_{*}) gives an equivalence of categories. Henceforth, in this section, we do not consider $\widetilde{X}^{\log}$ and $\widetilde{\overset{\circ}{X}}$ except the first part of the proof of (4.9) below. Set $\mathcal{M}_{X_{\mathrm{cl}}} := \mu^{-1}(\mathcal{M}_X)$ and $\mathcal{O}_{X_{\mathrm{cl}}} := \mu^{-1}(\mathcal{O}_X)$.

We obtain a topos $\widetilde{X}^{\log}_{\mathrm{et}}$ ([IKN, §2]) which is the analogue of the log etale topos of an fs log scheme ([Nak1, (2.2)]). (In [loc. cit.] this is called the ket topos of X and denoted by $\widetilde{X}^{\mathrm{ket}}$.) By using the local description of a Kummer log etale morphism of fs log analytic spaces over $\mathbb{C}$ ([IKN, (2.3)]) and using [KN, (1.3) (3)] and [KN, (1.2.1.1)], for a Kummer etale morphism $f\colon U \longrightarrow V$ of fs log analytic spaces over $\mathbb{C}$, the associated morphism $f^{\log}\colon U^{\log} \longrightarrow V^{\log}$ is a local isomorphism of topological spaces by the same proof as that of [KN, (2.2)]. Hence we have a natural morphism $\beta_X\colon \widetilde{X}^{\log}_{\mathrm{cl}} \longrightarrow \widetilde{X}^{\log}_{\mathrm{et}}$ of topoi. We also have a natural morphism $\varepsilon_{\mathrm{an}}\colon \widetilde{X}^{\log}_{\mathrm{et}} \longrightarrow \widetilde{\overset{\circ}{X}}_{\mathrm{et}}$ of topoi fitting into the following commutative diagram

$$
\begin{array}{ccc}
\widetilde{X}^{\log}_{\mathrm{cl}} & \xrightarrow{\beta_X} & \widetilde{X}^{\log}_{\mathrm{et}} \\
{\scriptstyle \epsilon_{\mathrm{cl}}}\downarrow & & \downarrow{\scriptstyle \epsilon_{\mathrm{an}}} \\
\widetilde{\overset{\circ}{X}}_{\mathrm{cl}} & \xrightarrow{\beta_{\overset{\circ}{X}}} & \widetilde{\overset{\circ}{X}}_{\mathrm{et}}.
\end{array}
\tag{4.6.9}
$$

The morphism $\beta_{\overset{\circ}{X}*}\colon \widetilde{\overset{\circ}{X}}_{\mathrm{cl}} \longrightarrow \widetilde{\overset{\circ}{X}}_{\mathrm{et}}$ gives an equivalence of categories. Henceforth we identify $\widetilde{\overset{\circ}{X}}_{\mathrm{cl}}$ with $\widetilde{\overset{\circ}{X}}_{\mathrm{et}}$ by $\beta_{\overset{\circ}{X}*}$ and denote β_X only by β. Using this identification, we have a formula $\varepsilon_{\mathrm{cl}} = \varepsilon_{\mathrm{an}} \circ \beta$ by (4.6.9).

Let $\mathcal{M}_{X,\log}$ be a sheaf of monoids in $\widetilde{X}^{\log}_{\mathrm{et}}$ which is associated to the presheaf $U \longmapsto \Gamma(U, \mathcal{M}_U)$ ($U \in X^{\log}_{\mathrm{et}}$). Then we have a natural morphism $\varepsilon^{-1}_{\mathrm{an}}(\mathcal{M}_{X_{\mathrm{cl}}}) \longrightarrow \mathcal{M}_{X,\log}$, which induces a morphism

$$
\varepsilon^{-1}_{\mathrm{cl}}(\mathcal{M}_{X_{\mathrm{cl}}}) \longrightarrow \beta^{-1}(\mathcal{M}_{X,\log}). \tag{4.6.10}
$$

LEMMA 4.7 [Analytic log Kummer sequence]. *Let m be a positive integer. Then the following sequence*

$$0 \longrightarrow (\mathbb{Z}/m)(1) \longrightarrow \mathcal{M}^{\mathrm{gp}}_{X,\log} \overset{m\times}{\longrightarrow} \mathcal{M}^{\mathrm{gp}}_{X,\log} \longrightarrow 0 \tag{4.7.1}$$

is exact in $\widetilde{X}^{\log}_{\mathrm{et}}$.

PROOF. The obvious analytic analogue of the proof of [KN, (2.3)] works. □

Let E be an m-torsion abelian sheaf in $\widetilde{\overset{\circ}{X}}_{\mathrm{cl}}$. Using (4.7.1) and (4.6.10), we have a canonical morphism

$$\bigwedge^{k}(\mathcal{M}^{\mathrm{gp}}_{X_{\mathrm{cl}}}/\mathcal{O}^{*}_{X_{\mathrm{cl}}}) \otimes_{\mathbb{Z}} (\mathbb{Z}/m)(-k) \otimes_{\mathbb{Z}} E \longrightarrow R^{k}\varepsilon_{\mathrm{cl}*}(\varepsilon_{\mathrm{cl}}^{-1}(E)) \quad (k \in \mathbb{Z}_{\geq 0}). \tag{4.7.2}$$

By the same proof as that of [KN, (1.5)], we see that (4.7.2) is an isomorphism.

By [KN, (1.5)], for an abelian sheaf E on $\widetilde{\overset{\circ}{X}_{\mathrm{cl}}}$, we have a canonical isomorphism

$$\bigwedge^{k}(\mathcal{M}^{\mathrm{gp}}_{X_{\mathrm{cl}}}/\mathcal{O}^{*}_{X_{\mathrm{cl}}})(-k) \otimes_{\mathbb{Z}} E \overset{\sim}{\longrightarrow} R^{k}\varepsilon_{\mathrm{cl}*}(\varepsilon_{\mathrm{cl}}^{-1}(E)) \quad (k \in \mathbb{Z}_{\geq 0}). \tag{4.7.3}$$

Next we consider the algebraic case. Let X be an fs log scheme over $\mathbb{C}$ whose underlying scheme $\overset{\circ}{X}$ is locally of finite type over $\mathbb{C}$. Let $\varepsilon_{\mathrm{et}}\colon \widetilde{X}^{\log}_{\mathrm{et}} \longrightarrow \widetilde{\overset{\circ}{X}}_{\mathrm{et}}$ be the forgetting log morphism. Then Kato and Nakayama have proved that the log Kummer sequence (2.0.1; m) gives the following canonical isomorphism ([KN, (2.4)]):

$$\bigwedge^{k}(\mathcal{M}^{\mathrm{gp}}_{X}/\mathcal{O}^{*}_{X})(-k) \otimes_{\mathbb{Z}} E \overset{\sim}{\longrightarrow} R^{k}\varepsilon_{\mathrm{et}*}(\varepsilon_{\mathrm{et}}^{-1}(E)) \quad (k \in \mathbb{Z}_{\geq 0}). \tag{4.7.4}$$

Let $\eta^{\log}\colon (\widetilde{X}^{\log}_{\mathrm{an}})_{\mathrm{cl}} \longrightarrow \widetilde{X}^{\log}_{\mathrm{et}}$ ([KN, (2.1), (2.2)]) and $\eta\colon (\widetilde{\overset{\circ}{X}_{\mathrm{an}}})_{\mathrm{cl}} \longrightarrow \widetilde{\overset{\circ}{X}}_{\mathrm{et}}$ be the natural morphisms of topoi.

Because a functor $U \longmapsto U_{\mathrm{an}}$ ($U \in X^{\log}_{\mathrm{et}}$) defines a continuous functor $X^{\log}_{\mathrm{et}} \longrightarrow (X_{\mathrm{an}})^{\log}_{\mathrm{et}}$, we have a morphism

$$\eta_{\mathrm{et}}\colon \widetilde{(X_{\mathrm{an}})}^{\log}_{\mathrm{et}} \longrightarrow \widetilde{X}^{\log}_{\mathrm{et}} \tag{4.7.5}$$

of topoi. Then we have the following commutative diagram of topoi (cf. [KN, p. 171]):

(4.7.6)
$$\begin{array}{ccccc}
(\widetilde{X}^{\log}_{\rm an})_{\rm cl} & & & & \\
 & \overset{\beta}{\searrow} & & \overset{\eta^{\log}}{\searrow\!\!\!\longrightarrow} & \\
 & & \widetilde{(X_{\rm an})}{}^{\log}_{\rm et} & \xrightarrow[\eta_{\rm et}]{} & \widetilde{X}^{\log}_{\rm et} \\
 & {\scriptstyle \epsilon_{\rm cl}}\searrow & \downarrow{\scriptstyle \epsilon_{\rm an}} & & \downarrow{\scriptstyle \epsilon_{\rm et}} \\
 & & (\widetilde{\overset{\circ}{X}}_{\rm an})_{\rm cl} & \xrightarrow[\eta]{} & \widetilde{\overset{\circ}{X}}_{\rm et}.
\end{array}$$

Let K be an abelian sheaf in $\widetilde{X}^{\log}_{\rm et}$. Then we have the following base change morphism

(4.7.7)
$$\eta^{-1}R\varepsilon_{{\rm et}*}(K) \longrightarrow R\varepsilon_{{\rm cl}*}(\eta^{\log,-1}(K)).$$

In particular, for a nonnegative integer k, we have the following morphism

(4.7.8)
$$\eta^{-1}R^k\varepsilon_{{\rm et}*}(K) \longrightarrow R^k\varepsilon_{{\rm cl}*}(\eta^{\log,-1}(K)).$$

Hence we have a canonical morphism

(4.7.9)
$$R^k\varepsilon_{{\rm et}*}(K) \longrightarrow R\eta_*(R^k\varepsilon_{{\rm cl}*}(\eta^{\log,-1}(K))).$$

THEOREM 4.8. *Let $\eta^*(\mathcal{M}_X) \in (\widetilde{\overset{\circ}{X}_{\rm an}})_{\rm cl}$ be the associated log structure to the composite morphism $\eta^{-1}(\mathcal{M}_X) \longrightarrow \eta^{-1}(\mathcal{O}_X) \longrightarrow \mathcal{O}_{(X_{\rm an})_{\rm cl}}$. Let m be a positive integer and let E be an m-torsion abelian sheaf in $\widetilde{\overset{\circ}{X}}_{\rm et}$. Let k be a nonnegative integer. Then the following diagram is commutative*:

(4.8.1)
$$\begin{array}{ccc}
R\eta_*(\bigwedge^k(\eta^*(\mathcal{M}^{\rm gp}_X)/\mathcal{O}^*_{(X_{\rm an})_{\rm cl}})(-k)\otimes_{\mathbb{Z}}\eta^{-1}(E)) & \xrightarrow[\sim]{R\eta_*((4.7.3))} & R\eta_*(R^k\epsilon_{{\rm cl}*}\epsilon^{-1}_{\rm cl}\eta^{-1}(E)) \\
{\scriptstyle R\eta_*({\rm id}\otimes\exp(m^{-1}\times)^{\otimes k}\otimes{\rm id})}\downarrow\simeq & & \| \\
R\eta_*(\bigwedge^k(\eta^*(\mathcal{M}^{\rm gp}_X)/\mathcal{O}^*_{(X_{\rm an})_{\rm cl}})(-k)\otimes_{\mathbb{Z}}\eta^{-1}(E)) & \xrightarrow[\sim]{R\eta_*((4.7.2))} & R\eta_*(R^k\epsilon_{{\rm cl}*}\epsilon^{-1}_{\rm cl}\eta^{-1}(E)) \\
\uparrow & & \uparrow{\scriptstyle (4.7.9)} \\
\bigwedge^k(\mathcal{M}^{\rm gp}_X/\mathcal{O}^*_X)(-k)\otimes_{\mathbb{Z}}E & \xrightarrow[\sim]{(4.7.4)} & R^k\epsilon_{{\rm et}*}(\epsilon^{-1}_{\rm et}(E)),
\end{array}$$

where the lower left vertical morphism is induced by the adjunction morphism ${\rm id} \longrightarrow R\eta_*\eta^{-1}$. *Furthermore, if E is constructible, then the lower left vertical morphism is an isomorphism.*

PROOF. First we prove the commutativity of the upper square of the diagram (4.8.1). Let Y be an fs log analytic space over $\mathbb{C}$. Let F be an abelian sheaf in $\widetilde{\overset{\circ}{Y}}_{\mathrm{cl}}$. To prove the commutativity, it suffices to prove the commutativity of the following diagram

$$\begin{array}{ccc} \bigwedge^k(\mathcal{M}^{\mathrm{gp}}_{Y_{\mathrm{cl}}}/\mathcal{O}^*_{Y_{\mathrm{cl}}})(-k)\otimes_{\mathbb{Z}} F & \xrightarrow[\sim]{(4.7.3)} & R^k\epsilon_{\mathrm{cl}*}(\epsilon_{\mathrm{cl}}^{-1}(F)) \\ {\scriptstyle \mathrm{id}\otimes\exp(m^{-1}\times)^{\otimes k}\otimes\mathrm{proj}}\downarrow & & \downarrow \\ \bigwedge^k(\mathcal{M}^{\mathrm{gp}}_{Y_{\mathrm{cl}}}/\mathcal{O}^*_{Y_{\mathrm{cl}}})\otimes_{\mathbb{Z}}(\mathbb{Z}/m)(-k)\otimes_{\mathbb{Z}}(F/mF) & \xrightarrow[\sim]{(4.7.2)} & R^k\epsilon_{\mathrm{cl}*}(\epsilon_{\mathrm{cl}}^{-1}(F/mF)). \end{array} \tag{4.8.2}$$

We have a natural morphism

$$\beta^{-1}(\mathcal{M}^{\mathrm{gp}}_{Y,\log})\longrightarrow \mathrm{Cont}_{Y^{\log}_{\mathrm{cl}}}(\ ,\mathbb{S}^1)$$

of abelian sheaves in $\widetilde{Y}^{\log}$ induced by the natural morphism

$$\Gamma(V,\mathcal{M}^{\mathrm{gp}}_{Y,\log})\longrightarrow \mathrm{Cont}_{Y^{\log}_{\mathrm{cl}}}(V^{\log},\mathbb{S}^1)\quad (V\in Y^{\log}_{\mathrm{et}})$$

of presheaves on $Y^{\log}_{\mathrm{cl}}$. Set

$$\mathcal{L}^{\dagger}_{Y^{\log}_{\mathrm{cl}}}:=\mathrm{Cont}_{Y^{\log}_{\mathrm{cl}}}(\ ,\sqrt{-1}\mathbb{R})\times_{\exp,\mathrm{Cont}_{Y^{\log}_{\mathrm{cl}}}(\ ,\mathbb{S}^1)}\beta^{-1}(\mathcal{M}^{\mathrm{gp}}_{Y,\log}). \tag{4.8.3}$$

Then we have an exponential sequence

$$0\longrightarrow \mathbb{Z}(1)\longrightarrow \mathcal{L}^{\dagger}_{Y^{\log}_{\mathrm{cl}}}\overset{\exp}{\longrightarrow}\beta^{-1}(\mathcal{M}^{\mathrm{gp}}_{Y,\log})\longrightarrow 0. \tag{4.8.4}$$

We claim that, for a positive integer m, the multiplication morphism

$$m\times\colon \mathcal{L}^{\dagger}_{Y^{\log}_{\mathrm{cl}}}\longrightarrow \mathcal{L}^{\dagger}_{Y^{\log}_{\mathrm{cl}}} \tag{4.8.5}$$

is an isomorphism. (Note that an analogous claim in the proof of [Il4, (5.9)] that the sheaf of logarithms $\mathcal{L}_{X^{\log}_{\mathrm{an}}}$ in [KN, (1.4)] is uniquely m-divisible ($m\geq 2$) is mistaken because $\varepsilon_{\mathrm{cl}}^{-1}(\mathcal{M}^{\mathrm{gp}}_{(X_{\mathrm{an}})_{\mathrm{cl}}})$ is not m-divisible for the case $X=(\mathrm{Spec}\,\mathbb{C},\mathbb{N}\oplus\mathbb{C}^*)$.) First we show the injectivity of (4.8.5). Let (a,s) ($a\in\sqrt{-1}\mathbb{R}, s\in\beta^{-1}(\mathcal{M}^{\mathrm{gp}}_{Y,\log})$) be a local section of $\mathcal{L}^{\dagger}_{Y^{\log}_{\mathrm{cl}}}$ such that $m(a,s)=0$. Then $a=0$ and $s\in(\mathbb{Z}/m)(1)$. Because the natural composite morphism

$$\mathbb{C}^*\overset{\subset}{\longrightarrow}\beta^{-1}(\mathcal{M}^{\mathrm{gp}}_{Y,\log})\longrightarrow \mathrm{Cont}_{Y^{\log}_{\mathrm{cl}}}(\ ,\mathbb{S}^1) \tag{4.8.6}$$

of abelian sheaves in $\widetilde{Y}^{\log}_{\mathrm{cl}}$ is induced by the map $c\longmapsto c/|c|$ ($c\in\mathbb{C}^*$), we see that $s=1$. Hence the morphism (4.8.5) is injective. Next we show the surjectivity of (4.8.5). For an object U of $Y^{\log}_{\mathrm{cl}}$, let (a,u)

$(a \in \sqrt{-1}\mathbb{R}, u \in \Gamma(U, \beta^{-1}(\mathcal{M}^{\rm gp}_{Y,\log})))$ be a section of $\Gamma(U, \mathcal{L}^{\dagger}_{Y^{\log}_{\rm cl}})$. We may assume that $u \in \Gamma(U, \beta^{-1}(\mathcal{M}_{Y,\log}))$. Since $\mathcal{M}_{Y,\log}$ is m-divisible by (4.7) and since the functor β^{-1} is right-exact, there exists a section v_λ of $\Gamma(U_\lambda, \beta^{-1}(\mathcal{M}_{Y,\log}))$ for some covering $(U_\lambda \longrightarrow U)_\lambda$ of U in $Y^{\log}_{\rm cl}$ such that $v^m_\lambda = u|_{U_\lambda}$. Let w_λ be the image of v_λ in ${\rm Cont}_{Y^{\log}_{\rm cl}}(U_\lambda, \mathbb{S}^1)$. Then $\zeta_\lambda := w_\lambda \exp(-m^{-1}a)$ is an m-th root of unity. It is easy to check that $(m^{-1}a, v_\lambda\zeta_\lambda^{-1})$ is indeed an element of $\Gamma(U_\lambda, \mathcal{L}^{\dagger}_{Y^{\log}})$ and that $m(m^{-1}a, v_\lambda\zeta_\lambda^{-1}) = (a, u)|_{U_\lambda}$. Hence the morphism (4.8.5) is surjective. Consider the following well-defined morphism

$$\exp(m^{-1}\times): \mathcal{L}^{\dagger}_{Y^{\log}_{\rm cl}} \longrightarrow \beta^{-1}(\mathcal{M}^{\rm gp}_{Y,\log}). \tag{4.8.7}$$

Because $(2\pi\sqrt{-1}n/m, \exp(2\pi\sqrt{-1}n/m))$ $(n \in \mathbb{Z})$ is a section of $\mathcal{L}^{\dagger}_{Y^{\log}}$, we have a formula

$$m^{-1}(2\pi\sqrt{-1}n, 1) = (2\pi\sqrt{-1}n/m, \exp(2\pi\sqrt{-1}n/m)).$$

Hence we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}(1) & \longrightarrow & \mathcal{L}_{Y^{\log}_{\rm cl}} & \xrightarrow{\exp} & \epsilon_{\rm cl}^{-1}(\mathcal{M}^{\rm gp}_{Y_{\rm cl}}) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle (4.6.10)^{\rm gp}} & & \\
0 & \longrightarrow & \mathbb{Z}(1) & \longrightarrow & \mathcal{L}^{\dagger}_{Y^{\log}_{\rm cl}} & \xrightarrow{\exp} & \beta^{-1}(\mathcal{M}^{\rm gp}_{Y,\log}) & \longrightarrow & 0 \\
 & & {\scriptstyle \exp(m^{-1}\times)}\downarrow & & {\scriptstyle \exp(m^{-1}\times)}\downarrow & & \| & & \\
0 & \longrightarrow & (\mathbb{Z}/m)(1) & \longrightarrow & \beta^{-1}(\mathcal{M}^{\rm gp}_{Y,\log}) & \xrightarrow{m\times} & \beta^{-1}(\mathcal{M}^{\rm gp}_{Y,\log}) & \longrightarrow & 0
\end{array} \tag{4.8.8}$$

of exact sequences. Now the commutativity of the diagram (4.8.2) follows from the commutative diagram (4.8.8) and from the definitions of the isomorphisms (4.7.3) and (4.7.2).

As to the commutativity of the lower square diagram of (4.8.1), it suffices to prove that the following diagram is commutative:

$$\begin{array}{ccc}
\bigwedge^k(\eta^*(\mathcal{M}^{\rm gp}_X)/\mathcal{O}^*_{(X_{\rm an})_{\rm cl}})(-k) \otimes_{\mathbb{Z}} \eta^{-1}(E)) & \xrightarrow[\sim]{(4.7.2)} & R^k\epsilon_{{\rm cl}*}(\epsilon_{\rm cl}^{-1}\eta^{-1}(E)) \\
\| & & \uparrow{\scriptstyle (4.7.9)} \\
\eta^{-1}(\bigwedge^k(\mathcal{M}^{\rm gp}_X/\mathcal{O}^*_X)(-k) \otimes_{\mathbb{Z}} E) & \xrightarrow[\sim]{\eta^{-1}((4.7.4))} & \eta^{-1}R^k\epsilon_{{\rm et}*}(\epsilon_{\rm et}^{-1}(E)).
\end{array} \tag{4.8.9}$$

Let

$$\eta_{\rm et}^{-1}(\mathcal{M}_{X,\log}) \longrightarrow \mathcal{M}_{X_{\rm an},\log} \tag{4.8.10}$$

be a natural morphism. Then we have the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathbb{Z}/m)(1) & \longrightarrow & \mathcal{M}^{\rm gp}_{X_{\rm an},\log} & \xrightarrow{m\times} & \mathcal{M}^{\rm gp}_{X_{\rm an},\log} & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle (4.8.10)^{\rm gp}} & & \uparrow{\scriptstyle (4.8.10)^{\rm gp}} & & \\
0 & \longrightarrow & (\mathbb{Z}/m)(1) & \longrightarrow & \eta_{\rm et}^{-1}(\mathcal{M}^{\rm gp}_{X,\log}) & \xrightarrow{m\times} & \eta_{\rm et}^{-1}(\mathcal{M}^{\rm gp}_{X,\log}) & \longrightarrow & 0
\end{array}
\tag{4.8.11}
$$

of exact sequences.

Using (4.8.11), we have the following commutative diagram of triangles

$$
\begin{array}{ccccccc}
R\epsilon_{\rm cl*}((\mathbb{Z}/m)(1)) & \longrightarrow & R\epsilon_{\rm cl*}(\beta^{-1}(\mathcal{M}^{\rm gp}_{X_{\rm an},\log})) & \xrightarrow{m\times} & R\epsilon_{\rm cl*}(\beta^{-1}(\mathcal{M}^{\rm gp}_{X_{\rm an},\log})) & \xrightarrow{+1} & \\
\| & & \uparrow & & \uparrow & & \\
R\epsilon_{\rm cl*}((\mathbb{Z}/m)(1)) & \longrightarrow & R\epsilon_{\rm cl*}(\eta^{\log,-1}(\mathcal{M}^{\rm gp}_{X,\log})) & \xrightarrow{m\times} & R\epsilon_{\rm cl*}(\eta^{\log,-1}(\mathcal{M}^{\rm gp}_{X,\log})) & \xrightarrow{+1} & \\
\uparrow & & \uparrow{\scriptstyle (4.7.7)} & & \uparrow{\scriptstyle (4.7.7)} & & \\
\eta^{-1}R\epsilon_{\rm et*}((\mathbb{Z}/m)(1)) & \longrightarrow & \eta^{-1}R\epsilon_{\rm et*}(\mathcal{M}^{\rm gp}_{X,\log}) & \xrightarrow{m\times} & \eta^{-1}R\epsilon_{\rm et*}(\mathcal{M}^{\rm gp}_{X,\log}) & \xrightarrow{+1} & .
\end{array}
\tag{4.8.12}
$$

(Here we have used the Convention (4).) In particular, we have the commutativity of the diagram (4.8.9) for the case $k=1$ and $E=\mathbb{Z}/m$. In the general case, by using the Godement resolution of an abelian sheaf in a topos with enough points and using the definition of the cup product, we obtain the commutativity of the diagram (4.8.9).

Assume now that E is constructible. Since $(\mathcal{M}^{\rm gp}_X/\mathcal{O}^*_X)\otimes_{\mathbb{Z}}\mathbb{Z}/m$ is a constructible torsion abelian sheaf in $\overset{\circ}{\widetilde{X}}_{\rm et}$, the lower left vertical morphism in (4.8.1) is an isomorphism by Artin-Grothendieck's comparison theorem ([SGA 4-3, XVI (4.1)]) as used in the proof of [KN, (2.6)]. □

Let X be a SNCL variety over $s=(\operatorname{Spec}\mathbb{C},\mathbb{N}\oplus\mathbb{C}^*)$. Fix a total order on the irreducible components of $\overset{\circ}{X}$. Let $k\geq 2$ be an integer. Let m be a positive integer. The triangle

$$
\mathrm{gr}^{\tau}_{k-1}R\varepsilon_{\rm et*}(\mathbb{Z}/m)\longrightarrow(\tau_k/\tau_{k-2})R\varepsilon_{\rm et*}(\mathbb{Z}/m)\longrightarrow\mathrm{gr}^{\tau}_kR\varepsilon_{\rm et*}(\mathbb{Z}/m)\xrightarrow{+1}, \tag{4.8.13}
$$

the isomorphism (2.0.3; m) and the Convention (4) give the following boundary morphism

$$
d\colon(\mathbb{Z}/m)(-k)_{\overset{\circ}{X}{}^{(k)}}\{-k\}\longrightarrow(\mathbb{Z}/m)_{\overset{\circ}{X}{}^{(k-1)}}(-(k-1))\{-(k-1)\}[1]. \tag{4.8.14}
$$

We also have the Čech-Gysin morphism

$$
G\colon(\mathbb{Z}/m)_{\overset{\circ}{X}{}^{(k)}}(-k)\{-k\}\longrightarrow(\mathbb{Z}/m)_{\overset{\circ}{X}{}^{(k-1)}}(-(k-1))\{-(k-1)\}[1] \tag{4.8.15}
$$

which is analogous to the Čech-Gysin morphism in the analytic case.

THEOREM 4.9. *The boundary morphism d in* (4.8.14) *is equal to G in* (4.8.15).

PROOF. Let Z be a SNCL analytic variety over $s_{\rm an}$. For a positive integer m, we have the following commutative diagram

$$
\begin{array}{ccccccc}
\mathrm{gr}^{\tau}_{k-1}R\epsilon_{\mathrm{top}*}(\mathbb{Z}) & \longrightarrow & (\tau_k/\tau_{k-2})R\epsilon_{\mathrm{top}*}(\mathbb{Z}) & \longrightarrow & \mathrm{gr}^{\tau}_{k}R\epsilon_{\mathrm{top}*}(\mathbb{Z}) & \xrightarrow{+1} & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{gr}^{\tau}_{k-1}R\epsilon_{\mathrm{top}*}(\mathbb{Z}/m) & \longrightarrow & (\tau_k/\tau_{k-2})R\epsilon_{\mathrm{top}*}(\mathbb{Z}/m) & \longrightarrow & \mathrm{gr}^{\tau}_{k}R\epsilon_{\mathrm{top}*}(\mathbb{Z}/m) & \xrightarrow{+1} & .
\end{array}
\tag{4.9.1}
$$

Fix a total order on the irreducible components of $\overset{\circ}{Z}$. Then we have a canonical isomorphism

$$
\bigwedge^{k}(\mathcal{M}^{\mathrm{gp}}_{Z}/\mathcal{O}^{*}_{Z})\otimes_{\mathbb{Z}}\mathbb{Z}/m \overset{\sim}{\longrightarrow} (\mathbb{Z}/m)_{Z^{(k)}}
\tag{4.9.2}
$$

and a canonical boundary morphism

$$
d\colon (\mathbb{Z}/m)_{Z^{(k)}}(-k)\{-k\} \longrightarrow (\mathbb{Z}/m)_{Z^{(k-1)}}(-(k-1))\{-(k-1)\}[1]
\tag{4.9.3}
$$

by the lower triangle of (4.9.1), by (4.7.3) and by the Convention (4). By (4.6) and (4.9.1), the boundary morphism d in (4.9.3) is also equal to

$$
G\colon (\mathbb{Z}/m)_{Z^{(k)}}(-k)\{-k\} \longrightarrow (\mathbb{Z}/m)_{Z^{(k-1)}}(-(k-1))\{-(k-1)\}[1].
$$

Using the identification $\widetilde{Z}^{\log}_{\mathrm{cl}}$ (resp. $\widetilde{\overset{\circ}{Z}}_{\mathrm{cl}}$) with $\widetilde{Z}^{\log}$ (resp. $\widetilde{\overset{\circ}{Z}}$) by $\mu^{\log}_{*}$ (resp. μ_{*}), we obtain the following commutative diagram

$$
\begin{array}{ccccccc}
\mathrm{gr}^{\tau}_{k-1}R\epsilon_{\mathrm{cl}*}(\mathbb{Z}) & \longrightarrow & (\tau_k/\tau_{k-2})R\epsilon_{\mathrm{cl}*}(\mathbb{Z}) & \longrightarrow & \mathrm{gr}^{\tau}_{k}R\epsilon_{\mathrm{cl}*}(\mathbb{Z}) & \xrightarrow{+1} & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{gr}^{\tau}_{k-1}R\epsilon_{\mathrm{cl}*}(\mathbb{Z}/m) & \longrightarrow & (\tau_k/\tau_{k-2})R\epsilon_{\mathrm{cl}*}(\mathbb{Z}/m) & \longrightarrow & \mathrm{gr}^{\tau}_{k}R\epsilon_{\mathrm{cl}*}(\mathbb{Z}/m) & \xrightarrow{+1} & ,
\end{array}
\tag{4.9.4}
$$

and a similar boundary morphism

$$
d\colon (\mathbb{Z}/m)_{Z^{(k)}_{\mathrm{cl}}}(-k)\{-k\} \longrightarrow (\mathbb{Z}/m)_{Z^{(k-1)}_{\mathrm{cl}}}(-(k-1))\{-(k-1)\}[1]
\tag{4.9.5}
$$

by the lower triangle of (4.9.4) is also equal to the Gysin morphism

$$
G\colon (\mathbb{Z}/m)_{Z^{(k)}_{\mathrm{cl}}}(-k)\{-k\} \longrightarrow (\mathbb{Z}/m)_{Z^{(k-1)}_{\mathrm{cl}}}(-(k-1))\{-(k-1)\}[1].
$$

By applying $R\eta_{*}$ to the lower triangle of (4.9.4) and by setting $K = \mathbb{Z}/m$ in (4.7.7), we have the following commutative diagram

$$
(4.9.6)\quad
\begin{array}{ccc}
R\eta_*(R^{k-1}\epsilon_{\mathrm{cl}*}(\mathbb{Z}/m)\{-(k-1)\}) & \longrightarrow & R\eta_*((\tau_k/\tau_{k-2})R\epsilon_{\mathrm{cl}*}(\mathbb{Z}/m)) \\
\| & & \| \\
R\eta_*(\mathrm{gr}^{\tau}_{k-1}R\epsilon_{\mathrm{cl}*}(\mathbb{Z}/m)) & \longrightarrow & R\eta_*((\tau_k/\tau_{k-2})R\epsilon_{\mathrm{cl}*}(\mathbb{Z}/m)) \\
\uparrow & & \uparrow \\
\mathrm{gr}^{\tau}_{k-1}R\epsilon_{\mathrm{et}*}(\mathbb{Z}/m) & \longrightarrow & (\tau_k/\tau_{k-2})R\epsilon_{\mathrm{et}*}(\mathbb{Z}/m) \\
\| & & \| \\
R^{k-1}\epsilon_{\mathrm{et}*}(\mathbb{Z}/m)\{-(k-1)\} & \longrightarrow & (\tau_k/\tau_{k-2})R\epsilon_{\mathrm{et}*}(\mathbb{Z}/m)
\end{array}
$$

$$
\begin{array}{cccc}
\longrightarrow & R\eta_*(R^{k}\epsilon_{\mathrm{cl}*}(\mathbb{Z}/m)\{-k\}) & \xrightarrow{+1} \\
 & \| & \\
\longrightarrow & R\eta_*(\mathrm{gr}^{\tau}_{k}R\epsilon_{\mathrm{cl}*}(\mathbb{Z}/m)) & \xrightarrow{+1} \\
 & \uparrow & \\
\longrightarrow & \mathrm{gr}^{\tau}_{k}R\epsilon_{\mathrm{et}*}(\mathbb{Z}/m) & \xrightarrow{+1} \\
 & \| & \\
\longrightarrow & R^{k}\epsilon_{\mathrm{et}*}(\mathbb{Z}/m)\{-k\} & \xrightarrow{+1}
\end{array}
$$

of triangles. In fact, the three middle vertical morphisms in (4.9.6) are isomorphisms by (4.8).

Now (4.9) follows from the commutative diagrams (4.9.6) and (4.8.1), from the proved fact that the morphism (4.9.5) is equal to G and from the compatibility of the cycle class of an algebraic smooth divisor on a smooth scheme over $\mathbb{C}$ with that of the associated analytic smooth divisor. □

Let κ be a separably closed field of characteristic $p \geq 0$. Let X be a SNCL variety over s. Fix a total order on the irreducible components of $\overset{\circ}{X}$. Then we have the boundary morphism

$$(4.9.7)\quad d\colon (\mathbb{Z}/m)_{\overset{\circ}{X}{}^{(k)}}(-k)\{-k\} \longrightarrow (\mathbb{Z}/m)_{\overset{\circ}{X}{}^{(k-1)}}(-(k-1))\{-(k-1)\}[1]$$

obtained from the triangle

$$(4.9.8)\quad \mathrm{gr}^{\tau}_{k-1}R\varepsilon_{\mathrm{et}*}(\mathbb{Z}/m) \longrightarrow (\tau_k/\tau_{k-2})R\varepsilon_{\mathrm{et}*}(\mathbb{Z}/m) \longrightarrow \mathrm{gr}^{\tau}_{k}R\varepsilon_{\mathrm{et}*}(\mathbb{Z}/m) \xrightarrow{+1},$$

from the isomorphism (2.0.3; m) and from the Convention (4). We also have the Čech-Gysin morphism

(4.9.9) $\quad G\colon (\mathbb{Z}/m)_{\overset{\circ}{X}{}^{(k)}}(-k)\{-k\} \longrightarrow (\mathbb{Z}/m)_{\overset{\circ}{X}{}^{(k-1)}}(-(k-1))\{-(k-1)\}[1].$

COROLLARY 4.10. *The boundary morphism d in* (4.9.7) *is equal to G in* (4.9.9).

PROOF. First assume that $p=0$. Let κ' be an algebraically closed field contained in κ or containing κ. Then (4.10) for κ is equivalent to (4.10) for κ' by the functoriality of the cycle class of a smooth divisor in a smooth scheme over a field and by the similar calculation to that in (4.1.1). Hence, by the Lefschetz principle, we may assume that $\kappa=\mathbb{C}$. In this case, (4.10) is nothing but (4.9).

Next assume that $p>0$. Because the problem is local, we may assume that there exists a classically etale morphism $\overset{\circ}{f}\colon \overset{\circ}{X} \longrightarrow \overset{\circ}{X}{}' := := \mathrm{Spec}(\kappa[x_0,\dots,x_d]/(x_0\cdots x_r))$ such that the log structure of X is the pull-back of that on $\mathrm{Spec}(\kappa[x_0,\dots,x_d]/(x_0\cdots x_r))$ associated to a morphism

$$\mathbb{N}^{r+1} \ni e_i := (0,\dots,0,\overset{i}{1},0,\dots,0) \longmapsto x_{i-1} \in \kappa[x_0,\dots,x_d]/(x_0\cdots x_r) \quad (1\le i\le r+1).$$

By the functoriality stated in the previous paragraph, it suffices to prove (4.10) for X'.

Let W be a Cohen ring of κ. Let K_0 be the fraction field of W. Let $\mathcal{X}$ be the log scheme whose underlying scheme is $\mathrm{Spec}(W[x_0,\dots,x_d]/(x_0\cdots x_r))$ and whose log structure is associated to a morphism $\mathbb{N}^{r+1}\ni e_i \longmapsto x_{i-1} \in \in W[x_0,\dots,x_d]/(x_0\cdots x_r)$ $(1\le i\le r+1)$. Set $Y:=\mathcal{X}\otimes_W K_0$. Let $\varepsilon_{\mathcal{X}}\colon \widetilde{\mathcal{X}}{}^{\log}_{\mathrm{et}} \longrightarrow \widetilde{\overset{\circ}{\mathcal{X}}}_{\mathrm{et}}$ and $\varepsilon_Y\colon \widetilde{Y}{}^{\log}_{\mathrm{et}} \longrightarrow \widetilde{\overset{\circ}{Y}}_{\mathrm{et}}$ be forgetting log morphisms. Let m be a positive integer prime to p. Then $R^r\varepsilon_{\mathcal{X}*}(\mathbb{Z}/m) = (\mathbb{Z}/m)_{\overset{\circ}{\mathcal{X}}{}^{(r)}}(-r)$ and $R^r\varepsilon_{Y*}(\mathbb{Z}/m) = = (\mathbb{Z}/m)_{\overset{\circ}{Y}{}^{(r)}}(-r)$ by [KN, (2.4)]. To prove (4.10) for X', it suffices to show, by the functoriality of the cycle class of a smooth divisor in a smooth scheme, that the boundary morphism

$$R^{r+1}\varepsilon_{\mathcal{X}*}(\mathbb{Z}/m)\{-r-1\} \longrightarrow R^r\varepsilon_{\mathcal{X}*}(\mathbb{Z}/m)\{-r\}[1]$$

of the following triangle

$$\mathrm{gr}^{\tau}_r R\varepsilon_{\mathcal{X}*}(\mathbb{Z}/m) \longrightarrow (\tau_{r+1}/\tau_{r-1})R\varepsilon_{\mathcal{X}*}(\mathbb{Z}/m) \longrightarrow \mathrm{gr}^{\tau}_{r+1}\varepsilon_{\mathcal{X}*}(\mathbb{Z}/m) \xrightarrow{+1}$$

is given by the Čech-Gysin morphism for the irreducible components of $\overset{\circ}{\mathcal{X}}$. By the similar calculation to that in (4.1.1), it suffices to prove the claim for $Y \otimes_{K_0} \overline{K}_0$. Consequently (4.10) in positive characteristic follows from (4.10) in characteristic 0, which we have already proved. $\square$

5. Boundary morphisms between the E_1-terms of l-adic and ∞-adic weight spectral sequences.

In this section we prove that the three boundary morphisms $d_1^{\bullet\bullet}$ between the E_1-terms of (2.0.7; l), (2.0.7; ∞) and (2.1.10) are described by the sum with signs of Gysin morphisms and the induced morphisms of closed immersions. In the l-adic case, this is a generalization of a correction of [RZ, (2.10)] (cf. (5.8) (1), (2) below).

First we consider the l-adic case.

Let us recall some facts in [Nak3] briefly. Let $\kappa = \kappa_{\mathrm{sep}}$ be a separably closed field. Let X be a (not necessarily proper) SNCL variety over the log point $s = (\mathrm{Spec}\,\kappa, \mathcal{M}_s)$. Recall the topos $\widetilde{X_{\overline{s}}} := \varprojlim_m \widetilde{(X \otimes_{\mathbb{Z}[\mathbb{N}]} \mathbb{Z}[\mathbb{N}^{1/l^m}])^{\log}_{\mathrm{et}}}$ in §2. Let $\pi_X\colon \widetilde{X_{\overline{s}}} \longrightarrow \widetilde{X}^{\log}_{\mathrm{et}}$ be the projection and let $\varepsilon := \varepsilon_X\colon \widetilde{X}^{\log}_{\mathrm{et}} \longrightarrow \overset{\circ}{\widetilde{X}}_{\mathrm{et}}$ be the forgetting log morphism ([Nak1, (1.1.2)]). Fix a generator T of $\mathbb{Z}_l(1)$. Let $I^\bullet$ be an injective resolution of $\mathbb{Z}/l^n$ in $\widetilde{X}^{\log}_{\mathrm{et}}$. Set $K^\bullet := (\varepsilon_X \pi_X)_* \pi_X^{-1}(I^\bullet)$, and let $L^\bullet$ be the mapping fiber of $T-1\colon K^\bullet \longrightarrow K^\bullet$: $L^\bullet := s((K^\bullet, d) \overset{T-1}{\longrightarrow} (K^\bullet, -d))$, where s means the single complex of a double complex (cf. Convention (8)). Then [loc. cit., (1.3.1)] tells us that the natural morphism $\varepsilon_{X*}(I^\bullet) \longrightarrow L^\bullet$ is a quasi-isomorphism. Fix a total order on the irreducible components of $\overset{\circ}{X}$. Then, by the proof of [Nak3, (1.8.3)] using the log Kummer sequence (cf. §2), we have a canonical isomorphism

(5.0.1) $$\mathcal{H}^r(L^\bullet) = R^r\varepsilon_{X*}(\mathbb{Z}/l^n) \overset{\sim}{\longrightarrow} (\mathbb{Z}/l^n)_{\overset{\circ}{X}^{(r)}}(-r) \quad (r \in \mathbb{Z}_{\geq 1}).$$

By following [SaT, (1.6)], let $\theta\colon L^\bullet \longrightarrow L^\bullet(1)[1]$ be the following vertical morphism

(5.0.2) $$\begin{array}{ccc} (K^\bullet(1), -d(1)) & \xrightarrow{-(T-1)} & (K^\bullet(1), d(1)) \\ & & \uparrow{\scriptstyle \mathrm{id}\otimes T} \\ & & (K^\bullet, d) \quad \xrightarrow{T-1} \quad (K^\bullet, -d). \end{array}$$

Let $A^{\bullet\bullet}_{X,l,n}$ be the double complex defined by sheaves $A^{ij}_{X,l,n} := (L^\bullet(j+1)[j+1]/\tau_j L^\bullet(j+1)[j+1])^i$ $(i,j \in \mathbb{Z})$ of $(\mathbb{Z}/l^n)_{\overset{\circ}{X}}$-modules with the following boundary morphisms as in the proof of [RZ, (1.7)]

(5.0.3)
$$\begin{array}{ccc} A^{i,j+1}_{X,l,n} & & \\ {\scriptstyle (-1)^i\theta}\uparrow & & \\ A^{ij}_{X,l,n} & \xrightarrow{(-1)^{j+1}d} & A^{i+1,j}_{X,l,n}. \end{array}$$

Let $A^{\bullet}_{X,l,n}$ be the single complex of $A^{\bullet\bullet}_{X,l,n}$.

To give the explicit description of the boundary morphism between the E_1-terms of (2.0.7; l), we prove the following (cf. [Mo, 4.12] in the p-adic case and [RZ, (2.9)] in the l-adic case for a semistable family):

LEMMA 5.1. *Let the notations be as above. Let $\overset{\circ}{X} := \bigcup_{i \in I} X_i$ be the union of the irreducible components of $\overset{\circ}{X}$. Fix a total order on I. Let I_r ($r \in \mathbb{Z}_{\geq 1}$) be the set defined in §4 for I. Then the following diagram*

(5.1.1; l)
$$\begin{array}{ccc} \mathcal{H}^r(L^\bullet)(r) & \xrightarrow{\theta} & \mathcal{H}^{r+1}(L^\bullet)(r+1) \\ {\scriptstyle (5.0.1)}\downarrow\simeq & & {\scriptstyle (5.0.1)}\downarrow\simeq \\ (\mathbb{Z}/l^n)_{\overset{\circ}{X}{}^{(r)}} & \xrightarrow{\rho} & (\mathbb{Z}/l^n)_{\overset{\circ}{X}{}^{(r+1)}} \end{array}$$

is commutative. Here ρ is defined by the following formula

(5.1.2; l)
$$\rho := \sum_{\underline{i} \in I_r} \sum_{j=0}^{r} (-1)^j (\iota_{\underline{i}}^{\underline{i}_j})^*,$$

where $\iota_{\underline{i}}^{\underline{i}_j} : X_{\underline{i}} \overset{\subset}{\longrightarrow} X_{\underline{i}_j}$ is the natural closed immersion.

PROOF. By [Nak1, (4.6)] the category of $\mathbb{Z}/l^n$-modules on the site $s^{\log}_{\text{et}}$ is equivalent to the category of $\mathbb{Z}/l^n$-modules with $\pi_1(s)$-continuous actions, where $\pi_1(s)$ is the log fundamental group of s. Let G_l be the pro-l-part of $\pi_1(s)$. The group G_l is identified with $\mathbb{Z}_l(1)$ by the following isomorphism

(5.1.3)
$$G_l \ni T \longmapsto (T(m_1^{l^{-i}})/m_1^{l^{-i}})_{i \in \mathbb{N}} \in \mathbb{Z}_l(1),$$

where m_1 is a section of $\mathcal{M}_s$ whose image in $\mathcal{M}_s/\mathcal{O}_s^* \simeq \mathbb{N}$ is the generator. By using the identification (5.1.3) and by abuse of notation, we denote simply by T the element $T(m_1^{l^{-n}})/m_1^{l^{-n}} \in \mathbb{Z}/l^n(1)$. We endow $\mathbb{Z}/l^n(1)$ with the trivial action of G_l as in [RZ, §1].

Step 1. As in [RZ, (1.2)], consider the following extension of $\mathbb{Z}/l^n[G_l]$-

modules:

$$0 \longrightarrow \mathbb{Z}/l^n(1) \overset{\subset}{\longrightarrow} \mathbb{Z}/l^n \underset{1}{\oplus} \mathbb{Z}/l^n(1) \overset{\text{proj}}{\longrightarrow} \mathbb{Z}/l^n \longrightarrow 0, \tag{5.1.4}$$

where the G_l-action on the middle term is given by $T(x,y) = (x, y + x \otimes T)$ $(x \in \mathbb{Z}/l^n, y \in \mathbb{Z}/l^n(1))$. Let $R_{G_l}\colon \mathrm{D}^+(\mathbb{Z}/l^n[G_l]) \ni (C^\bullet, d) \longmapsto s((C^\bullet, d) \overset{T-1}{\longrightarrow} (C^\bullet, -d)) \in \mathrm{D}^+(\mathbb{Z}/l^n)$ be a functor of derived categories ([RZ, (1.1)]). As in [RZ, (1.2)], we have the following triangle by using (5.1.4):

$$\cdots \longrightarrow R_{G_l}(K^\bullet(1)) \longrightarrow R_{G_l}(K^\bullet \underset{1}{\oplus} K^\bullet(1)) \longrightarrow R_{G_l}(K^\bullet) \overset{+1}{\longrightarrow} \cdots. \tag{5.1.5}$$

In particular, we have the boundary morphism

$$L^\bullet \longrightarrow L^\bullet(1)[1] \tag{5.1.6}$$

by using the Convention (4). Let $(x,y) \in K^r \oplus K^{r-1}$ be a local section such that $dx = 0$ and $-dy + (T-1)(x) = 0$. For a local section $((x,0),(y,0)) \in \in \{K^r \underset{1}{\oplus} K^r(1)\} \oplus \{K^{r-1} \underset{1}{\oplus} K^{r-1}(1)\}$, we have the following formula

$$\begin{aligned} d((x,0),(y,0)) &= ((0,0),(T-1)(x,0)) + ((0,0),(-dy,0)) \\ &= ((0,0),((T-1)(x), x \otimes T)) + ((0,0),(-(T-1)(x),0)) \\ &= ((0,0),(0, x \otimes T)). \end{aligned}$$

Hence the induced morphism $\theta\colon \mathcal{H}^r(L^\bullet) \longrightarrow \mathcal{H}^{r+1}(L^\bullet(1))$ is equal to the boundary morphism $\mathcal{H}^r(L^\bullet) \longrightarrow \mathcal{H}^r(L^\bullet(1)[1])$ induced by the morphism (5.1.6).

Step 2. Let $\partial\colon \mathcal{H}^r(L^\bullet) \longrightarrow \mathcal{H}^r(L^\bullet(1)[1])$ be the induced morphism by the boundary morphism (5.1.6). Let 1 be the unit element of $\mathbb{Z}/l^n$. Then, by a general property of the cup product, $\partial(a) = \partial(1 \cup a) = \partial(1) \cup a$, where a is a local section of $\mathcal{H}^r(L^\bullet)$. Here $\partial(1)$ is an element of $H^1(\mathrm{Hom}^\bullet_{\mathbb{Z}/l^n[G_l]}(\mathbb{Z}/l^n, I^\bullet))$, where $I^\bullet$ is an injective resolution of $\mathbb{Z}/l^n(1)$ as a $\mathbb{Z}/l^n[G_l]$-module. By the definition of (5.1.4) and by (5.2) below, the image of 1 by the composite morphism (5.2.1) below is equal to the cocycle

$$G_l \in T \longmapsto T \in \mathbb{Z}/l^n(1) \tag{5.1.7}$$

in $H^1(G_l, \mathbb{Z}/l^n(1))$.

Step 3. We calculate the cocycle (5.1.7) by another exact sequence. Namely, consider the following exact sequence of $\mathbb{Z}[G_l]$-modules:

$$0 \longrightarrow \mathbb{Z}/l^n(1) \longrightarrow \mathcal{M}^{\mathrm{gp}}_{s,\log} \overset{l^n \times}{\longrightarrow} \mathcal{M}^{\mathrm{gp}}_{s,\log} \longrightarrow 0. \tag{5.1.8}$$

Then we have an isomorphism

$$(5.1.9)\qquad (\mathcal{M}_s^{\rm gp}/\mathcal{O}_s^*)\otimes_{\mathbb{Z}}\mathbb{Z}/l^n \overset{\sim}{\longrightarrow} H^1(G_l,\mathbb{Z}/l^n(1)) = R^1\varepsilon_{s*}(\mathbb{Z}/l^n(1))$$

as the boundary morphism by using injective resolutions of the terms in (5.1.8) (see the Convention (5)).

The image of $m_1\otimes 1$ in $H^1(G_l,\mathbb{Z}/l^n(1))$ by the composite isomorphism (5.2.1) below is equal to the following cocycle

$$(5.1.10)\qquad [(T\longmapsto T(m_1^{l^{-n}})/m_1^{l^{-n}})] = [T\longmapsto T]\in H^1(G_l,\mathbb{Z}/l^n(1)).$$

Furthermore there exists a canonical isomorphism

$$(5.1.11)\qquad (\mathcal{M}_s^{\rm gp}/\mathcal{O}_s^*)\otimes_{\mathbb{Z}}\mathbb{Z}/l^n \ni m_1\otimes 1 \overset{\sim}{\longmapsto} 1\in\mathbb{Z}/l^n.$$

Hence we have the following composite isomorphism

$$(5.1.12)\quad \mathbb{Z}/l^n \overset{\sim}{\longrightarrow} (\mathcal{M}_s^{\rm gp}/\mathcal{O}_s^*)\otimes_{\mathbb{Z}}\mathbb{Z}/l^n \overset{\sim}{\longrightarrow} R^1\varepsilon_{s*}(\mathbb{Z}/l^n(1)) = H^1(G_l,\mathbb{Z}/l^n(1)).$$

The image of 1 by the isomorphism (5.1.12) is the cocycle $[T\longmapsto T]$.

Step 4. Let $f\colon X\longrightarrow s$ be the structural morphism. Then we have the following obvious commutative diagram

$$\begin{array}{ccc}
(\mathcal{M}_X^{\rm gp}/\mathcal{O}_X^*)\otimes_{\mathbb{Z}}\mathbb{Z}/l^n & \xrightarrow{=} & R^1\epsilon_{X*}(\mathbb{Z}/l^n(1)) \\
\uparrow & & \uparrow \\
f^{-1}(\mathbb{Z}/l^n) = f^{-1}((\mathcal{M}_s^{\rm gp}/\mathcal{O}_s^*)\otimes_{\mathbb{Z}}\mathbb{Z}/l^n) & \xrightarrow{=} & f^{-1}(R^1\epsilon_{s*}(\mathbb{Z}/l^n(1)))(= f^{-1}(H^1(G_l,\mathbb{Z}/l^n(1)))).
\end{array}$$

Let θ' be the image of $1\in\mathbb{Z}/l^n$ by the composite morphism of the horizontal lower arrow and the right vertical arrow. The cup product $\theta'\wedge\colon R^r\varepsilon_{X*}(\mathbb{Z}/l^n) = \mathcal{H}^r(L^\bullet)\longrightarrow R^{r+1}\varepsilon_{X*}(\mathbb{Z}/l^n(1)) = \mathcal{H}^{r+1}(L^\bullet(1))$ is the boundary morphism obtained from the triangle (5.1.5) by the Step 2 and the Step 3. By the Step 1, this boundary morphism is equal to $\theta\colon\mathcal{H}^r(L^\bullet)\longrightarrow\mathcal{H}^{r+1}(L^\bullet(1))$. Obviously the image of $1\in\mathbb{Z}/l^n$ by the left vertical morphism is $(1,\dots,1)\in(\mathbb{Z}/l^n)_{\overset{\circ}{X}{}^{(1)}} = (\mathcal{M}_X^{\rm gp}/\mathcal{O}_X^*)\otimes_{\mathbb{Z}}\mathbb{Z}/l^n$. Because the isomorphism $(\bigwedge^r(\mathcal{M}_X^{\rm gp}/\mathcal{O}_X^*))\otimes_{\mathbb{Z}}(\mathbb{Z}/l^n)\overset{\sim}{\longrightarrow}R^r\varepsilon_{X*}(\mathbb{Z}/l^n(r))$ is obtained by the isomorphism $(\mathcal{M}_X^{\rm gp}/\mathcal{O}_X^*)\otimes_{\mathbb{Z}}\mathbb{Z}/l^n\overset{\sim}{\longrightarrow}R^1\varepsilon_{X*}(\mathbb{Z}/l^n(1))$ and the cup product $(R^1\varepsilon_{X*}(\mathbb{Z}/l^n)(1))^{\otimes r}\longrightarrow R^r\varepsilon_{X*}(\mathbb{Z}/l^n(r))$ (cf. [KN, (2.4)], the proof of [Nak3, (1.8.3)]), we obtain (5.1.1; l). □

The following lemma is only roughly well-known; I give the proof because the sign is considerably delicate in the Hom-complex and because I cannot find the following quite delicate calculation in the references.

LEMMA 5.2. *Let G be a group and let R be a commutative ring with trivial G-action. Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence of left $R[G]$-modules. Let c be an element of C^G. Let $b \in B$ be a lift of c. Let $(I^\bullet, d_I)$, $(J^\bullet, d_J)$ and $(K^\bullet, d_K)$ be injective resolutions of A, B and C, respectively, as left $R[G]$-modules fitting into the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (I^\bullet, d_I) & \xrightarrow{\alpha^\bullet} & (J^\bullet, d_J) & \xrightarrow{\beta^\bullet} & (K^\bullet, d_K) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0.
\end{array}$$

Assume that the upper horizontal sequence is exact. Let $(R^\bullet, d_R)$ be the standard homogeneous projective resolution of R. Then the image of $c \in C^G$ by the following composite morphism

$$\begin{aligned}
(5.2.1) \qquad C^G \xrightarrow{\partial} & H^1(\mathrm{Hom}_{R[G]}(R, I^\bullet)) = H^1(\mathrm{Hom}^\bullet_{R[G]}(R, I^\bullet)) \\
\xrightarrow{\sim} & H^1(\mathrm{Hom}^\bullet_{R[G]}(R^\bullet, I^\bullet)) \xleftarrow{\sim} H^1(\mathrm{Hom}^\bullet_{R[G]}(R^\bullet, A)) \\
\overset{(1.0.7)}{\xleftarrow{\sim}} & H^1(\mathrm{Hom}_{R[G]}(R^\bullet, A)) = H^1(G, A)
\end{aligned}$$

is a 1-cocycle $G \ni \sigma \longmapsto \alpha^{-1}(\sigma(b) - b) \in A$.

PROOF. Let $\iota_I : A \longrightarrow I^0$, $\iota_J : B \longrightarrow J^0$ and $\iota_K : C \longrightarrow K^0$ be the injective morphisms. Let b' be an element of $(J^0)^G$ such that $\beta^0(b') = \iota_K(c)$ in $(K^0)^G$. Then there exists an element $a^1 \in (I^1)^G$ such that $\alpha^1(a^1) = d_J(b') \in (J^1)^G$. By the definition of $\partial(c)$, $\partial(c)$ in $H^1(\mathrm{Hom}_{R[G]}(R, I^\bullet))$ has a representative $1 \longmapsto a^1$. Since $\beta^0(b' - \iota_J(b)) = 0$, there exists an element $a^0 \in I^0$ such that $\alpha^0(a^0) = b' - \iota_J(b)$. Since

$$\alpha^1(d_I(a^0)) = d_J(\alpha^0(a^0)) = d_J(b' - \iota_J(b)) = d_J(b') = \alpha^1(a^1)$$

and since α^1 is injective, $d_I(a^0) = a^1$. By the following diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{R[G]}(R, I^1) & \longrightarrow & \mathrm{Hom}_{R[G]}(R^0, I^1) & \xrightarrow{d_R^*} & \mathrm{Hom}_{R[G]}(R^{-1}, I^1) \\
 & & \uparrow & & \uparrow \\
 & & \mathrm{Hom}_{R[G]}(R^0, I^0) & \xrightarrow{-d_R^*} & \mathrm{Hom}_{R[G]}(R^{-1}, I^0) \\
 & & & & \uparrow \\
 & & & & \mathrm{Hom}_{R[G]}(R^{-1}, A),
\end{array}$$

the image of $a^0 \in I^0 = \mathrm{Hom}^0_{R[G]}(R^\bullet, I^\bullet)$ by the boundary morphism $\mathrm{Hom}^0_{R[G]}(R^\bullet, I^\bullet) \longrightarrow \mathrm{Hom}^1_{R[G]}(R^\bullet, I^\bullet)$ is equal to $\{(e,\sigma) \longmapsto -(\sigma(a^0) - a^0)\} \oplus \oplus a^1$. Hence we have an equality $[(e,\sigma) \longmapsto (\sigma(a^0) - a^0)] = a^1$ in $H^1(\mathrm{Hom}^\bullet_{R[G]}(R^\bullet, I^\bullet))$. Therefore the image of c by the composite morphism (5.2.1) is equal to $(-1) \times \{(e,\sigma) \longmapsto (\sigma(a^0) - a^0)\} = \{(e,\sigma) \longmapsto -(\sigma(a^0) - a^0)\}$. Since b' is G-invariant, $\alpha^0(\sigma(a^0) - a^0) = \iota_J(b) - \sigma(\iota_J(b))$. Now we see that the desired cocycle is represented by $\sigma \longmapsto \alpha^{-1}(\sigma(b) - b)$. □

REMARK 5.3. Let the notations be as in [RZ, p. 26]. To obtain the equality $H^1(G, \Lambda(1)) = \mathrm{Hom}_{\mathrm{D}^+\Lambda[G]}(\Lambda(1), \Lambda(1)[1])$ in [loc. cit], we have to make the convention on signs about the Hom-complex, e.g., as in [Co] (=the Conventions (9), (10) in this paper), which was not made in [RZ]. Furthermore, (5.2) is missing in [RZ]. Note that the analogue of (5.2) for the Hom-complex in [SaT, p. 586] is also missing in [loc. cit.]. This analogue is necessary for the proof of [SaT, (1.6) (2)]. Consequently the proofs of [RZ, (1.2)] and [SaT, (1.6) (2)] are not complete in arguments on signs.

LEMMA 5.4. *Let the notations and the assumptions be as in* (5.1). *Let*

$$d\colon \mathcal{H}^{r+1}(L^\bullet)\{-r-1\} \longrightarrow \mathcal{H}^r(L^\bullet)\{-r\}[1]$$

be the boundary morphism of the following triangle

$$\mathrm{gr}^\tau_r L^\bullet \longrightarrow (\tau_{r+1}/\tau_{r-1})(L^\bullet) \longrightarrow \mathrm{gr}^\tau_{r+1} L^\bullet \xrightarrow{+1}$$

by using the Convention (4). *Then the following diagram*

(5.4.1)
$$\begin{array}{ccc} \mathcal{H}^{r+1}(L^\bullet)\{-r-1\} & \xrightarrow{d} & \mathcal{H}^r(L^\bullet)\{-r\}[1] \\ {\scriptstyle (5.0.1)}\downarrow\simeq & & {\scriptstyle (5.0.1)}\downarrow\simeq \\ (\mathbb{Z}/l^n)_{\overset{\circ}{X}{}^{(r+1)}}(-r-1)\{-r-1\} & \xrightarrow{G} & (\mathbb{Z}/l^n)_{\overset{\circ}{X}{}^{(r)}}(-r)\{-r\}[1] \end{array}$$

is commutative. Here $G := \sum_{\underline{i}\in I_r} \sum_{j=0}^{r} (-1)^j G^{\underline{i}_j}_{\underline{i}}$, *where*

(5.4.2)
$$G^{\underline{i}_j}_{\underline{i}}\colon (\mathbb{Z}/l^n(-1))_{X_{\underline{i}}}\{-1\} \longrightarrow (\mathbb{Z}/l^n)_{X_{\underline{i}_j}}[1]$$

is the Gysin morphism of the closed immersion $X_{\underline{i}} \xrightarrow{\subset} X_{\underline{i}_j}$.

PROOF. (5.4) is nothing but (4.10) for the case $m = l^n$. □

COROLLARY 5.5. *The boundary morphism* $d_1^{-k,h+k}\colon E_{1,l}^{-k,h+k} \longrightarrow E_{1,l}^{-k+1,h+k}$ *of the spectral sequence* (2.0.7; l) *is identified with the following morphism*:

$$\sum_{j\geq\max\{-k,0\}} \{(-1)^j G + (-1)^{j+k}\rho\}. \tag{5.5.1}$$

PROOF. Let j and k be the indexes in (2.0.7; l). By the commutative diagram (5.4.1), we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}^{2j+k+1}(L^\bullet)(j+1)\{-2j-k-1\} & \xrightarrow{d} & \mathcal{H}^{2j+k}(L^\bullet)(j+1)\{-2j-k\}[1] \\ {\scriptstyle (5.0.1)}\downarrow\simeq & & {\scriptstyle (5.0.1)}\downarrow\simeq \\ (\mathbb{Z}/l^n)_{\overset{\circ}{X}{}^{(2j+k+1)}}(-j-k)\{-2j-k-1\} & \xrightarrow{G} & (\mathbb{Z}/l^n)_{\overset{\circ}{X}{}^{(2j+k)}}(-j-k+1)\{-2j-k\}[1] \\ {\scriptstyle (-1)^{j+k}}\downarrow\simeq & & {\scriptstyle (-1)^{j+k-1}}\downarrow\simeq \\ (\mathbb{Z}/l^n)_{\overset{\circ}{X}{}^{(2j+k+1)}}(-j-k)\{-2j-k-1\} & \xrightarrow{-G} & (\mathbb{Z}/l^n)_{\overset{\circ}{X}{}^{(2j+k)}}(-j-k+1)\{-2j-k\}[1]. \end{array} \tag{5.5.2}$$

Hence we have the part $(-1)^{j+1}(-G)$ in (5.5.1) by the diagram (5.0.3). By the commutative diagram (5.1.1; l), we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}^{2j+k+1}(L^\bullet)(j+1) & \xrightarrow{\theta} & \mathcal{H}^{2j+k+2}(L^\bullet)(j+2) \\ {\scriptstyle (5.0.1)}\downarrow\simeq & & {\scriptstyle (5.0.1)}\downarrow\simeq \\ (\mathbb{Z}/l^n)_{\overset{\circ}{X}{}^{(2j+k+1)}}(-j-k) & \xrightarrow{\rho} & (\mathbb{Z}/l^n)_{\overset{\circ}{X}{}^{(2j+k+2)}}(-j-k) \\ {\scriptstyle (-1)^{j+k}}\downarrow\simeq & & {\scriptstyle (-1)^{(j+1)+(k-1)}}\downarrow\simeq \\ (\mathbb{Z}/l^n)_{\overset{\circ}{X}{}^{(2j+k+1)}}(-j-k) & \xrightarrow{\rho} & (\mathbb{Z}/l^n)_{\overset{\circ}{X}{}^{(2j+k+2)}}(-j-k). \end{array} \tag{5.5.3}$$

Hence we obtain the part $(-1)^{j+k}\rho$ in (5.5.1) by (5.1) and by the diagram (5.0.3) for the case $i+j+1 = 2j+k+1$, i.e., $i = j+k$. □

Next, in the following propositions (5.6) and (5.7), we compare (5.5) with a correction of Rapoport-Zink's work ([RZ, (2.10)]).

PROPOSITION 5.6. *Assume that* $\overset{\circ}{X}$ *is the special fiber of a semistable family* $\overset{\circ}{\mathcal{X}}$ *over a complete discrete valuation ring* A *of mixed characteristics with residue field* κ. *Endow* $\overset{\circ}{X}$ *with the canonical log structure obtained from* $\overset{\circ}{\mathcal{X}}$ *and let* X *be the resulting log scheme. Let* $\mathfrak{G}$ *be the absolute Galois group of the fraction field of* A. *Let* $\overset{\circ}{\mathcal{X}}_\eta$ *be the generic fiber* *of* $\overset{\circ}{\mathcal{X}}$. *Let* $\overset{\circ}{i}\colon \overset{\circ}{X} \longrightarrow \overset{\circ}{\mathcal{X}}$ *and* $\overset{\circ}{j}\colon \overset{\circ}{\mathcal{X}}_{\eta} \longrightarrow \overset{\circ}{\mathcal{X}}$ *be the natural closed immersion and the natural open immersion, respectively. Let* $R\overset{\circ}{\Psi}(\mathbb{Z}/l^n)$ *be the classical nearby cycle sheaf in* $\mathrm{D}^+(\overset{\circ}{X}, \mathfrak{G}, \mathbb{Z}/l^n)$. *Then there exists the following natural commutative diagram*

(5.6.1)
$$\begin{array}{ccc} R\epsilon_*(\mathbb{Z}/l^n) & \xrightarrow{\sim} & s((K^\bullet, d) \overset{T-1}{\longrightarrow} (K^\bullet, -d)) \\ \simeq\uparrow & & \uparrow\simeq \\ \overset{\circ}{i}{}^* R\overset{\circ}{j}_*(\mathbb{Z}/l^n) & \xrightarrow{\sim} & s((R\overset{\circ}{\Psi}(\mathbb{Z}/l^n), d) \overset{T-1}{\longrightarrow} (R\overset{\circ}{\Psi}(\mathbb{Z}/l^n), -d)). \end{array}$$

Proof. Let $R\Psi(\mathbb{Z}/l^n)$ be the log nearby cycle sheaf in $\mathrm{D}^+(X_{\bar{s}}, \mathfrak{G}, \mathbb{Z}/l^n)$ ([Nak2, §3]). Then we have $K^\bullet \underset{\sim}{=} R\Psi(\mathbb{Z}/l^n)$ (cf. the latter part of the proof of [Nak3, (1.9)]). Let $\varepsilon'\colon \widetilde{X_{\bar{s}}}{}^{\log}_{\mathrm{et}} \longrightarrow \overset{\circ}{X}_{\mathrm{et}}$ be the natural functor. By [Nak2, (3.2) (ii), (iv), (v)],

(5.6.2)
$$R\overset{\circ}{\Psi}(\mathbb{Z}/l^n) = R\varepsilon'_* R\Psi(\mathbb{Z}/l^n) = R\varepsilon'_*(\mathbb{Z}/l^n).$$

Let $\mathcal{X}$ be the log scheme $\overset{\circ}{\mathcal{X}}$ with canonical log structure. Let $i\colon X \longrightarrow \mathcal{X}$ and $j\colon \mathcal{X}_\eta := (\overset{\circ}{\mathcal{X}}_\eta, \mathcal{O}^*_{\overset{\circ}{\mathcal{X}}_\eta}) \longrightarrow \mathcal{X}$ be the natural exact closed immersion and the natural open immersion, respectively. Then we have the following commutative diagram

(5.6.3)
$$\begin{array}{ccc} R\epsilon_*(\mathbb{Z}/l^n) & \longrightarrow & R\epsilon'_*(\mathbb{Z}/l^n) \\ \downarrow & & \downarrow\simeq \\ R\epsilon_* i^* Rj_*(\mathbb{Z}/l^n) & \longrightarrow & R\epsilon'_* R\Psi(\mathbb{Z}/l^n). \end{array}$$

(We leave the reader to the proof of the commutativity of (5.6.3) by using the adjoint property of morphisms of ringed topoi.) By [FK, (3.1)] ([Il4, (7.4)]), the adjunction morphism $\mathbb{Z}/l^n \longrightarrow Rj_*(\mathbb{Z}/l^n)$ is an isomorphism. Hence the left vertical morphism in (5.6.3) is an isomorphism. Furthermore there exists a natural morphism $\overset{\circ}{i}{}^* R\overset{\circ}{j}_* \longrightarrow R\varepsilon_* i^* Rj_*$ of functors. Indeed, let $\varepsilon''\colon \widetilde{\mathcal{X}}{}^{\log}_{\mathrm{et}} \longrightarrow \overset{\circ}{\mathcal{X}}_{\mathrm{et}}$ be also the forgetting log morphism. Then $\overset{\circ}{i}{}^* R\overset{\circ}{j}_* = \overset{\circ}{i}{}^* R\varepsilon''_* Rj_*$. Since $L\overset{\circ}{i}{}^* = \overset{\circ}{i}{}^*$, it suffices to construct a natural morphism $R\varepsilon''_* \longrightarrow R\overset{\circ}{i}_* R\varepsilon_* i^*$. Since $\overset{\circ}{i} \circ \varepsilon = \varepsilon'' \circ i$, $R\overset{\circ}{i}_* R\varepsilon_* i^* = R\varepsilon''_* Ri_* i^*$. Consequently the adjunction morphism $\mathrm{id} \longrightarrow Ri_* i^*$ gives the morphism $\overset{\circ}{i}{}^* R\overset{\circ}{j}_* \longrightarrow R\varepsilon_* i^* Rj_*$. Because the equality (5.6.2) is similarly obtained by adjunction morphisms, the following diagram

$$\begin{array}{ccc} R\epsilon_* i^* Rj_*(\mathbb{Z}/l^n) & \longrightarrow & R\epsilon'_* R\Psi(\mathbb{Z}/l^n) \\ \uparrow & & \| \\ \overset{\circ}{i}{}^* R\overset{\circ}{j}_*(\mathbb{Z}/l^n) & \longrightarrow & R\overset{\circ}{\Psi}(\mathbb{Z}/l^n). \end{array} \tag{5.6.4}$$

is commutative. By the commutative diagrams (5.6.3) and (5.6.4), we obtain (5.6). □

Let the notations and the assumptions be as in (5.6). Then, by the following triangle

$$\longrightarrow \mathrm{gr}^{\tau}_{r-1} R\varepsilon_*(\mathbb{Z}/l^n) \longrightarrow (\tau_r/\tau_{r-2}) R\varepsilon_*(\mathbb{Z}/l^n) \longrightarrow \mathrm{gr}^{\tau}_r R\varepsilon_*(\mathbb{Z}/l^n) \xrightarrow{+1} \quad (r \in \mathbb{Z}_{\geq 2}), \tag{5.6.5}$$

we have the following boundary morphism

$$d\colon R^r\varepsilon_*(\mathbb{Z}/l^n)\{-r\} \longrightarrow R^{r-1}\varepsilon_*(\mathbb{Z}/l^n)\{-r+1\}[1]. \tag{5.6.6}$$

Similarly, we have the following boundary morphism

$$d\colon \overset{\circ}{i}{}^* R^r \overset{\circ}{j}_*(\mathbb{Z}/l^n)\{-r\} \longrightarrow \overset{\circ}{i}{}^* R^{r-1} \overset{\circ}{j}_*(\mathbb{Z}/l^n)\{-r+1\}[1]. \tag{5.6.7}$$

By Gabber's purity [Fu2, §8, third Consequence],

$$\overset{\circ}{i}{}^* R^r \overset{\circ}{j}_*(\mathbb{Z}/l^n) \overset{\sim}{\longleftarrow} (\mathbb{Z}/l^n)_{\overset{\circ}{X}{}^{(r)}}(-r). \tag{5.6.8}$$

The isomorphism (5.6.8) is obtained from the isomorphism

$$\overset{\circ}{i}{}^* R^1 \overset{\circ}{j}_*(\mathbb{Z}/l^n) \overset{\sim}{\longleftarrow} (\mathbb{Z}/l^n)_{\overset{\circ}{X}{}^{(1)}}(-1) \tag{5.6.9}$$

and the cup product

$$\bigwedge^r \overset{\circ}{i}{}^* R^1 \overset{\circ}{j}_*(\mathbb{Z}/l^n) \overset{\sim}{\longrightarrow} \overset{\circ}{i}{}^* R^r \overset{\circ}{j}_*(\mathbb{Z}/l^n). \tag{5.6.10}$$

The isomorphism (5.6.9) is obtained by the Kummer sequence

$$0 \longrightarrow \mathbb{Z}/l^n(1) \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 0 \tag{5.6.11}$$

in $\overset{\sim}{\overset{\circ}{\mathcal{X}}}_{\eta,\mathrm{et}}$. Here the section $(1,\dots,1)$ on the right hand side of (5.6.9)(1) goes to $\overset{\circ}{i}{}^*(\partial(\pi))$, where π is a uniformizer of A and $\partial\colon R^0\overset{\circ}{j}_*(\mathbb{G}_m) \longrightarrow R^1\overset{\circ}{j}_*(\mathbb{Z}/l^n(1))$ is the boundary morphism of (5.6.11).

Proposition 5.7. *The induced morphism* $\overset{\circ}{i}{}^* R^r \overset{\circ}{j}_*(\mathbb{Z}/l^n) \longrightarrow R^r\varepsilon_*(\mathbb{Z}/l^n)$ *by the left vertical isomorphism in* (5.6.1) *fits into the following commutative diagram:*

$$\begin{array}{ccc} \bigwedge^r(\mathcal{M}_X^{\mathrm{gp}}/\mathcal{O}_X^*) \otimes_{\mathbb{Z}} \mathbb{Z}/l^n(-r) & \xrightarrow[\sim]{(2.0.3;l^n)} & R^r\epsilon_*(\mathbb{Z}/l^n) \\ \| & & \uparrow \simeq \\ (\mathbb{Z}/l^n)_{\overset{\circ}{X}{}^{(r)}}(-r) & \xrightarrow[\sim]{(5.6.8)} & \overset{\circ}{i}{}^* R^r \overset{\circ}{j}_*(\mathbb{Z}/l^n). \end{array} \tag{5.7.1}$$

Proof. Using the adjunction morphism $\mathrm{id} \longrightarrow Rj_*j^*$, we have a canonical morphism

$$R\varepsilon_*i^* \longrightarrow R\varepsilon_*i^*Rj_*j^* \tag{5.7.2}$$

of functors. We have also obtained a canonical morphism

$$\overset{\circ}{i}{}^* R\overset{\circ}{j}_*j^* \longrightarrow R\varepsilon_*i^*Rj_*j^* \tag{5.7.3}$$

of functors in the proof of (5.6). Let $\mathcal{M}_{\mathcal{X},\log}$ be the log structure in $\widetilde{\mathcal{X}}^{\log}_{\mathrm{et}}$. Using (5.7.2) and (5.7.3) for $\mathcal{M}^{\mathrm{gp}}_{\mathcal{X},\log}$ and $\mathbb{Z}/l^n(1)$, we have the following natural commutative diagram

$$\begin{array}{ccc} \mathcal{M}_X^{\mathrm{gp}} & \longrightarrow & R^1\epsilon_*(\mathbb{Z}/l^n(1)) \\ \downarrow & & \| \\ \epsilon_*(\mathcal{M}^{\mathrm{gp}}_{X,\log}) & \longrightarrow & R^1\epsilon_*(\mathbb{Z}/l^n(1)) \\ \uparrow & & \| \\ \mathcal{H}^0(R\epsilon_*i^*Rj_*j^*(\mathcal{M}^{\mathrm{gp}}_{\mathcal{X},\log})) & \longrightarrow & \mathcal{H}^1(R\epsilon_*i^*Rj_*j^*(\mathbb{Z}/l^n(1))) \\ \uparrow & & \uparrow \\ \overset{\circ}{i}{}^*\overset{\circ}{j}_*(\mathbb{G}_m) & \longrightarrow & \overset{\circ}{i}{}^*R^1\overset{\circ}{j}_*(\mathbb{Z}/l^n(1)). \end{array} \tag{5.7.4}$$

Here we have used the fact $Rj_*(\mathbb{Z}/l^n) = \mathbb{Z}/l^n$ ([FK, (3.1)], [Il4, (7.4)]). Hence we have the commutative diagram (5.7.1). □

The following remark is only for the reader; in this paper I shall not use the facts in the remark.

Remark 5.8. (1) It is better to replace the boundary morphisms of the double complex $\overline{C}$ in [RZ, p. 38] by the following boundary morphisms in

(5.8.1) below (cf. [Nakk3, (5.1.2)]): let $\{d^\bullet\}$ be the boundary morphisms of $I^\bullet$ in [RZ, p. 37]. First set $\overline{D}^{ij} := \overline{C}^{ji}$. Then $\overline{D}^{\bullet\bullet}$ becomes a double complex with the following boundary morphisms:

(5.8.1)
$$\begin{array}{ccc} \overline{D}^{i,j+1} & & \\ {\scriptstyle (-1)^i d^j}\big\uparrow & & \\ \overline{D}^{ij} & \xrightarrow{\ G\ } & \overline{D}^{i+1,j}. \end{array}$$

Here G is the Čech-Gysin morphism in (5.4). The two boundary morphisms in (5.8.1) are different from the two boundary morphisms in [loc. cit., p. 25, p. 38]. Rapoport-Zink's boundary morphisms in [loc. cit.] are not good since each Čech Gysin morphism in the double complex $\overline{C}$ in [loc. cit.] has different signs which depend on the parity of the degrees of $I^\bullet$.

If we follow the convention in [RZ, §1] and if we use the identification

(5.8.2) $$a_{r*}b_r^! I^\bullet = a_{r*}Rb_r^!(\mathbb{Z}/l^n) = a_{r*}(\mathbb{Z}/l^n)(-r)[-2r] \quad (r \in \mathbb{Z}_{>0})$$

in [loc. cit., p. 37], the following diagram

(5.8.3)
$$\begin{array}{ccccc} \overline{C}^{p+2(q-1),-(q-1)} & \longrightarrow & \overline{C}^{p+2q-1,-(q-1)} & \longrightarrow & \overline{C}^{p+2q,-(q-1)} \\ & & & & {\scriptstyle (-1)^{p+2q}\sum_j(-1)^j\delta_{j*}}\big\uparrow \\ & & & & \overline{C}^{p+2q,-q} \end{array}$$

tells us that we have to replace the formula

$$d_1' = \sum(-1)^j\delta_{j*} : H^p(Y^{(q)}, \mathbb{Z}/l^n(-q)) \longrightarrow H^{p+2}(Y^{(q-1)}, \mathbb{Z}/l^n(-(q-1)))$$

in [loc. cit., p. 39] (δ_{j*} in [loc. cit., p. 39] is mistaken; the right δ_{j*} is in [loc. cit., p. 36]) by

(5.8.4) $$d_1' = (-1)^{p+2q}\sum(-1)^j\delta_{j*} = (-1)^p\sum(-1)^j\delta_{j*}.$$

As a result, we also have to replace the sign $(-1)^k$ before d_1' in [loc. cit., 2.10] by

(5.8.5) $$(-1)^{(k+1)+(q-r-2k)} = (-1)^{q+k+r+1}.$$

Here, note that we have to replace $(-1)^k d_1'$ in [loc. cit., p. 31] by $(-1)^{k+1}d_1'$.

Since our Čech Gysin morphism G is equal to $-\sum_j(-1)^j\delta_{j*}$ in [RZ], the part of the Gysin morphism in (5.5.1) is accidentally the same as that of [RZ, (2.10)] if we correct the sign $(-1)^k$ in [loc. cit.] before d_1' by $(-1)^{k+1}$ as in (5.8.5).

(2) Let the notations be as in (5.6). There are at least two canonical isomorphisms $\overset{\circ}{i}{}^*R^1\overset{\circ}{j}_*(\mathbb{Z}/l^n)(1) \overset{\sim}{\longleftarrow} (\mathbb{Z}/l^n)_{\overset{\circ}{X}{}^{(1)}}$. One is obtained by the Kummer sequence on $\overset{\circ}{X}_\eta$ as in (5.6.9). The other is obtained by a canonical boundary isomorphism $\overset{\circ}{i}{}^*R^1\overset{\circ}{j}_*(\mathbb{Z}/l^n) \overset{\partial}{\underset{\sim}{\longrightarrow}} R^2\overset{\circ}{i}{}^!(\mathbb{Z}/l^n)$ which is obtained by the localization sequence and by a canonical isomorphism $(\mathbb{Z}/l^n)_{\overset{\circ}{X}{}^{(1)}} \overset{\sim}{\longrightarrow} R^2\overset{\circ}{i}{}^!(\mathbb{Z}/l^n)(1)$ which is obtained by the cycle classes of $X_{i_0} \cap X_{i_1}$ $((i_0, i_1) \in I_2)$ on X_{i_0} and X_{i_1}. These two isomorphisms have a gap of signs by the anti-commutative diagram in [SGA $4\frac{1}{2}$, Cycle (2.1.3)].

I do not understand the proof for the part $(-1)^{r+k}\theta$ in [RZ, (2.10)]. In the proof of [RZ, (2.9)], we have to use the resolution in [RZ, (2.6)] of $i^*j_*j^*I^\bullet$ in addition to the formula (5.8.2). I do not understand where the resolution for the description of the part $(-1)^{r+k}\theta$ has been used; we have to describe how θ acts on the resolution.

Using the better complex $\overline{D}^{\bullet\bullet}$, we correct the proof of [RZ, (2.10)] as follows.

Let (S, s, η) be the base henselian discrete valuation ring with closed point and generic point in [RZ] and let $i_b\colon s \overset{\subset}{\longrightarrow} S$ (resp. $j_b\colon \eta \overset{\subset}{\longrightarrow} S$) be the natural closed (resp. open) immersion. Let $I_b^\bullet$ be an injective resolution of $\mathbb{Z}/l^n$ on S. Then we have the following exact sequence

$$0 \longrightarrow i_b^!(I_b^\bullet) \longrightarrow i_b^*(I_b^\bullet) \longrightarrow i_b^*j_{b*}j_b^*(I_b^\bullet) \longrightarrow 0$$

since $i_b^*i_{b*} = \mathrm{id}$. Then $\mathcal{H}^2(i_b^!(I_b^\bullet)) = \mathbb{Z}/l^n(-1)$ and $\mathcal{H}^q(i_b^!(I_b^\bullet)) = 0$ $(q \neq 2)$. By the boundary morphism $d\colon \mathcal{H}^1(i_b^*j_{b*}j_b^*(I_b^\bullet))(1) \longrightarrow \mathcal{H}^2(i_b^!(I_b^\bullet))(1) = \mathbb{Z}/l^n$, the class $\theta \in \mathcal{H}^1(i_b^*j_{b*}j_b^*(I_b^\bullet))(1)$ is mapped to the minus cycle class of the closed point s of S by (5.2) and [SGA $4\frac{1}{2}$, Cycle (2.1.3)]. By the functoriality, this minus class defines a cohomology class in $a_{1*}R^2b_1^!(\mathbb{Z}/l^n)$. Hence, if we use the double complex $\overline{D}^{\bullet\bullet}$ and if we use the following identification

(5.8.6)
$$H^h(Y, (a_{r*}Rb_r^!(\mathbb{Z}/l^n), (-1)^r d)) = H^h(Y, (a_{r*}b_r^!I^\bullet, (-1)^r d)) =$$
$$H^h(Y, (a_{r*}b_r^!I^\bullet, d)) = H^{h-2r}(Y, a_{r*}(\mathbb{Z}/l^n)(-r)) \quad (h \in \mathbb{Z}, r \in \mathbb{Z}_{>0})$$

by using the Convention (6) in the case where r is odd, we have the following description

(5.8.7)
$$d_1 = \sum_{j \geq \max\{-k, 0\}} \{(-1)^{j+1}G + (-1)^{j+k+1}\rho\}$$

instead of the description d_1 in [RZ, (2.10)]. If we use the double complex $\overline{C}$ and the formula (5.8.2), the correction of [RZ, (2.10)] is as follows:

(5.8.8)
$$d_1 = \sum_{k \geq \max\{-r, 0\}} \{(-1)^{q+k+r+1}G_{\mathrm{RZ}} + (-1)^{k+r+1}\rho\},$$

where $G_{RZ} := \sum_j (-1)^j \delta_{j*}$ is the Čech-Gysin morphism in [RZ, (2.8)], which is $-G$ in (5.4). However we do not use the double complexes $\overline{D}$ and $\overline{C}$ in this paper; we do not use these descriptions (5.8.7) and (5.8.8) either in this paper; we use only the description (5.5.1).

Last but not least in (2), Rapoport and Zink have used two identifications(=(5.8.2) and the identification in the proof of [RZ, (2.8)]) of the E_1-terms with classical etale cohomologies at the same time. When we give the description of the boundary morphisms between the E_1-terms, we have to fix only one identification until the end of the proof for the description.

(3) Note that the morphism $R^1 j_*(\mathbb{Z}/l^n(1)) \longrightarrow i_* R^2 i^!(\mathbb{Z}/l^n(1))$ considered in [SaT, (1.5) (1)] is the opposite morphism of the traditional boundary morphism obtained from the localization sequence; we consider only the traditional boundary morphism as in (2) above.

As pointed out in (5.3) and (5.8), the weight spectral sequence in [RZ, (2.10)] is an incomplete and mistaken spectral sequence. The proof for [Nak3, (1.9)] that Nakayama's weight spectral sequence in [loc. cit.] is isomorphic to the mistaken weight spectral sequence in [RZ, (2.10)] is incomplete because we need the commutative diagrams (5.6.1) and (5.7.1) for the proof of [Nak3, (1.9)] (Nakayama has proved (5.9) (1) below). Hence we would like to give a correction of the weight spectral sequence of [RZ, (2.10)] and to establish a relation between the weight spectral sequence (2.0.7; l) and the corrected weight spectral sequence as follows.

Let $\overset{\circ}{\mathcal{X}}$ be a proper strict semistable family over a complete discrete valuation ring A. Let η be the generic point of $\operatorname{Spec} A$. Let $\overset{\circ}{X}$ be the special fiber of $\overset{\circ}{\mathcal{X}}$. Let j, k be two indexes in the direct factor of the E_1-term of (2.0.7; l). Then we have the following isomorphism

(5.8.9)
$$(\mathbb{Z}/l^n)(-2j-k-1)_{\overset{\circ}{X}{}^{(2j+k+1)}} \overset{(-1)^{j+k}}{\underset{\sim}{\longrightarrow}} (\mathbb{Z}/l^n)(-2j-k-1)_{\overset{\circ}{X}{}^{(2j+k+1)}} \overset{(5.6.8)}{\underset{\sim}{\longrightarrow}} \overset{\circ}{i}{}^* R^{2j+k+1} \overset{\circ}{j}_*(\mathbb{Z}/l^n).$$

Hence we have the following spectral sequence

$$(5.8.10)\quad E_{1,l}^{-k,h+k} = \bigoplus_{j \geq \max\{-k,0\}} H^{h-2j-k}_{\mathrm{et}}(\overset{\circ}{\overline{X}}{}^{(2j+k+1)}, \mathbb{Z}/l^n)(-j-k) \Longrightarrow H^h_{\mathrm{et}}(\overset{\circ}{\mathcal{X}}_{\overline{\eta}}, \mathbb{Z}/l^n).$$

Taking the projective limit of (5.8.10) with respect to n, we have the following weight spectral sequence

$$(5.8.11)\quad E_{1,l}^{-k,h+k} = \bigoplus_{j \geq \max\{-k,0\}} H^{h-2j-k}_{\mathrm{et}}(\overset{\circ}{\overline{X}}{}^{(2j+k+1)}, \mathbb{Z}_l)(-j-k) \Longrightarrow H^h_{\mathrm{et}}(\overset{\circ}{\mathcal{X}}_{\overline{\eta}}, \mathbb{Z}_l).$$

The following is worth stating (cf. [FN, §4] for the ∞-adic analogue):

COROLLARY 5.9 [cf. [Nak3, (1.9)]]. *Let the assumptions be as in* (5.6). *Then the following hold*:

(1) ([Nak3, (1.9)]) *The complex* $A^{\bullet}_{X,l,n}$ *is canonically isomorphic to the complex* $A^{\bullet}$ *constructed in* [RZ].

(2) *The preweight spectral sequence* (2.0.7; l) *is canonically isomorphic to the weight spectral sequence* (5.8.11).

PROOF. (1): (1) has been proved in the proof of [Nak3, (1.9)].
(2): (2) immediately follows from (5.6) and (5.7). □

Next let us consider the ∞-adic case.

Let X be a SNCL analytic variety over a log point $s = ((\operatorname{Spec}\mathbb{C})_{\mathrm{an}}, \mathbb{N}\oplus\mathbb{C}^*)$. If X is algebraic, then the boundary morphism between the E_1-terms of (2.0.7; ∞) is the obvious ∞-analogue of (5.5.1) by (5.5) and by the comparison theorem of (2.0.7; ∞) with (2.0.7; l) ([FN, (7.1)]). In fact, we need not assume that X is algebraic. Let us prove it briefly. We follow the formulation of [RZ].

Let the notations be as in §2. Let $\varepsilon_X\colon X^{\log}\longrightarrow \overset{\circ}{X}$ be the real blow up of X and let $\pi_X\colon X_\infty\longrightarrow X^{\log}$ be the projection defined in §2. Set $J^{\bullet} := J^{\bullet}_{\mathbb{Z}}$.

Let $L^{\bullet}_{\infty}$ be the single complex $s((J^{\bullet},d)\xrightarrow{\delta}(J^{\bullet}(-1),-d))$. Then $\theta\colon L^{\bullet}_{\infty}\longrightarrow L^{\bullet}_{\infty}(1)[1]$ in (2.0.1; ∞) is the vertical morphism in the following diagram:

(5.9.1)
$$\begin{array}{ccccc}
(J^{\bullet}(1),-d(1)) & \xrightarrow{-\delta} & (J^{\bullet},d) & & \\
 & & \Big\uparrow{\scriptstyle -\mathrm{id}} & & \\
 & & (J^{\bullet},d) & \xrightarrow{\delta} & (J^{\bullet}(-1),-d(-1)).
\end{array}$$

Assume that $X = s$. Then the isomorphism (3.2.6) for $r=1$ is equal to

(5.9.2) $$\mathbb{Z}=\mathcal{M}^{\mathrm{gp}}_s/\mathcal{O}^*_s\xrightarrow{\sim}R^1\varepsilon_{s*}(\mathbb{Z}(1))=H^1_{\mathrm{sing}}(\mathbb{S}^1,\mathbb{Z}(1))=H^1(\pi_1(s^{\log}),\mathbb{Z}(1)).$$

Let T be an automorphism of $\mathbb{R}$ over $\mathbb{S}^1$ defined by $x\longmapsto x+1$ ($x\in\mathbb{R}$). Since $s_\infty=\mathbb{R}$, the following sequence

$$0\longrightarrow\Gamma(s_\infty,\mathbb{Z}(1))\longrightarrow\Gamma(s_\infty,\pi_s^{-1}(\mathcal{L}_{s^{\log}}))\xrightarrow{\exp}\Gamma(s_\infty,\pi_s^{-1}\varepsilon_s^{-1}(\mathcal{M}^{\mathrm{gp}}_s))\longrightarrow 0$$

is exact. Let $e=(1,1)$ be a section of $\Gamma(s^{\log},\varepsilon_s^{-1}(\mathcal{M}^{\mathrm{gp}}_s))=\mathbb{Z}\oplus\mathbb{C}^*$. Let f be an element of $\Gamma(s_\infty,\pi_s^{-1}\mathrm{Cont}_{s^{\log}}(\ ,\sqrt{-1}\mathbb{R}))$ defined by $s_\infty=\mathbb{R}\ni x\longmapsto 2\pi\sqrt{-1}x\in\sqrt{-1}\mathbb{R}$. Let $h\colon\mathbb{Z}\longrightarrow\mathbb{S}^1$ be a point of $s^{\log}$. Then, by using the identification $s^{\log} \ni h \longmapsto h(1) \in \mathbb{S}^1$, the image of e by the morphism $\Gamma(s^{\log}, \varepsilon_s^{-1}(\mathcal{M}_s^{\mathrm{gp}})) \longrightarrow \mathrm{Cont}(s^{\log}, \mathbb{S}^1) = \mathrm{Cont}(\mathbb{S}^1, \mathbb{S}^1)$ is the identity. Since the following diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f} & \sqrt{-1}\mathbb{R} \\ {\scriptstyle \pi_s}\downarrow & & \downarrow{\scriptstyle \exp} \\ \mathbb{S}^1 & \xrightarrow{\mathrm{id}} & \mathbb{S}^1 \end{array}$$

is obviously commutative, the pair (f, e) is indeed an element of $\Gamma(s_\infty, \pi_s^{-1}(\mathcal{L}_{s^{\log}}))$. Set $l(1) := (f, e)$. Obviously we have $\exp(l(1)) = e$. As in the l-adic case, consider the following 1-cocycle

$$\pi_1(s^{\log}) \ni T \longmapsto T(l(1)) - l(1) = 2\pi\sqrt{-1} \in \mathbb{Z}(1). \tag{5.9.3}$$

and identify $T(l(1)) - l(1)$ with T. The map (5.9.3) is an isomorphism. By (5.2), the 1-cocycle (5.9.3) corresponds to the following extension

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathbb{Z} \underset{1}{\oplus} \mathbb{Z}(1) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where the $\pi_1(s^{\log})$-actions on $\mathbb{Z}(1)$ and $\mathbb{Z}$ are trivial and the $\pi_1(s^{\log})$-action on the middle term is given by $T(x, y) = (x, y + x \otimes 2\pi\sqrt{-1}) = (x, y + x \otimes T)$ ($x \in \mathbb{Z}$, $y \in \mathbb{Z}(1)$). Set $L'^{\bullet}_{\infty} := s((J^{\bullet}, d) \overset{T-1}{\longrightarrow} (J^{\bullet}, -d))$. By the same proof as that of (5.1), the left cup product of the cocycle (5.9.3): $\mathcal{H}^r(L'^{\bullet}_{\infty}) \longrightarrow \mathcal{H}^r(L'^{\bullet}_{\infty}(1)[1])$ is induced by the following vertical morphism of complexes

$$\begin{array}{ccccc} (J^{\bullet}(1), -d(1)) & \xrightarrow{-(T-1)} & (J^{\bullet}(1), d(1)) & & \\ & & {\scriptstyle \mathrm{id}\otimes T}\uparrow & & \\ & & (J^{\bullet}, d) & \xrightarrow{T-1} & (J^{\bullet}, -d). \end{array}$$

Since T is identified with $2\pi\sqrt{-1}$ by the isomorphism (5.9.3) and since we have the following commutative diagram

$$\begin{array}{ccc} J^{\bullet} & \xrightarrow{T-1} & J^{\bullet} \\ \| & & \downarrow{\scriptstyle -(2\pi\sqrt{-1})^{-1}} \\ J^{\bullet} & \xrightarrow{\delta} & J^{\bullet}(-1), \end{array}$$

the left cup product $\mathcal{H}^r(L^{\bullet}_{\infty}) \longrightarrow \mathcal{H}^{r+1}(L^{\bullet}_{\infty}(1))$ by the image of the cocycle (5.9.3) in $R^1\varepsilon_{X*}(\mathbb{Z}(1))$ is induced from a morphism (5.9.1). Hence we have the

∞-adic analogue of (5.1) for a SNCL analytic variety as in (5.1) by using the exponential sequence (3.2.5) for $X = s$ instead of the log Kummer sequence (5.1.8).

In (4.6) we have already proved the ∞-adic analogue of (5.4). Therefore we obtain the following (1) by the argument in the proof of (5.5):

THEOREM 5.10. (1) *The boundary morphism* $d_1^{-k,h+k}\colon E_{1,\infty}^{-k,h+k} \longrightarrow E_{1,\infty}^{-k+1,h+k}$ *of the spectral sequence* (2.0.7; ∞) *is identified with the following morphism*:

$$(5.10.1) \qquad \sum_{j \geq \max\{-k,0\}} \{(-1)^j G + (-1)^{j+k}\rho\}.$$

(2) *The boundary morphism* $d_1^{-k,h+k}\colon E_{1,\infty}^{-k,h+k} \longrightarrow E_{1,\infty}^{-k+1,h+k}$ *of the spectral sequence* (2.1.10) *is identified with the following morphism*:

$$(5.10.2) \qquad \sum_{j \geq \max\{-k,0\}} \{(-1)^j G + (-1)^{j+k}\rho\}.$$

PROOF. We have only to prove (2). Because the spectral sequence (2.0.8; ∞) is isomorphic to the weight spectral sequence (2.1.10) ([FN, (6.5)]) and because the endomorphism

$$\log T/(T-1) = 1 - 2^{-1}(T-1) + \cdots$$

of $I^{\bullet}_{\mathbb{Q}}$ is the identity, (2) immediately follows from (1). □

REMARK 5.11. (1) We have another proof of (5.10) (2): we may apply the tensor product $\otimes_{\mathbb{Q}}\mathbb{C}$ to the E_1-terms of (2.1.10). Using the isomorphism (3.6.9) and the complex $A^{\bullet}_{X/\mathbb{C}}$, we have (5.10) (2) by the same proof as that for the description (2.0.8.3; p) in [Nakk3, (10.1)]. Consequently we have (5.10) (1) again by [FN, (6.5)].

(2) In general, we use the boundary morphisms in (2.0.8.3; p), (5.5.1), (5.10.1) and (5.10.2). However, to keep the symmetry for specific examples in §6, §7 and §11, we consider an order which does not satisfy the transitive law in (6.1) (1) below. Consequently, the signs before G and ρ will change for specific examples.

PROPOSITION 5.12. *Let r be a positive integer and k a non-negative integer. Let* $\{E_r^{**}(\delta)\}$ (resp. $\{E_r^{**}(N_\infty)\}$) *be the E_r-terms of the spectral sequence* (2.0.7; ∞) (resp. (2.1.10)). *Then the morphism* $\upsilon_\infty^k\colon E_2^{-k,h+k}(\delta) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow E_2^{k,h-k}(\delta)(-k) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is an isomorphism if and only if so is* $N_\infty^k\colon E_2^{-k,h+k}(N_\infty) \longrightarrow E_2^{k,h-k}(N_\infty)(-k)$.

Proof. Let the notations be as in the ∞-adic case in §2. Let $E_r^{**}(\delta_{\mathbb{Q}})$ $(r \in \mathbb{Z}_{>0})$ be the E_r-terms of the weight spectral sequence obtained from the use of the mapping fiber $\mathrm{MF}(\delta_{\mathbb{Q}})$ of $\delta_{\mathbb{Q}}\colon B(J^\bullet_{\mathbb{Q}}) \longrightarrow B(J^\bullet_{\mathbb{Q}})(-1)$. Then we have the following three morphisms:

(1) $\nu^k_\infty\colon E_2^{-k,h+k}(\delta) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow E_2^{k,h-k}(\delta)(-k) \otimes_{\mathbb{Z}} \mathbb{Q}$,
(2) $\nu^k_\infty\colon E_2^{-k,h+k}(\delta_{\mathbb{Q}}) \longrightarrow E_2^{k,h-k}(\delta_{\mathbb{Q}})(-k)$,
(3) $N^k_\infty\colon E_2^{-k,h+k}(N_\infty) \longrightarrow E_2^{k,h-k}(N_\infty)(-k)$.

Since we have the following commutative diagram

$$\begin{array}{ccc} B(J^\bullet_{\mathbb{Q}}) & \xrightarrow{\subset} & J^\bullet_{\mathbb{Q}} \\ {\scriptstyle \delta_{\mathbb{Q}}}\big\downarrow & & \big\downarrow{\scriptstyle \delta\otimes\mathbb{Q}} \\ B(J^\bullet_{\mathbb{Q}})(-1) & \xrightarrow{\subset} & J^\bullet_{\mathbb{Q}}(-1) \end{array}$$

and since the morphism $E_1^{-k,h+k}(\delta_{\mathbb{Q}}) \longrightarrow E_1^{-k,h+k}(\delta) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism, the morphism (1) is identified with the morphism (2). Next, by noting the obvious commutative diagram

$$\begin{array}{ccc} \mathrm{MF}(\delta_{\mathbb{Q}}) & \xrightarrow{\pm 1} & \mathrm{MF}(\delta_{\mathbb{Q}}) \\ \cup\big\uparrow & & \big\uparrow\cup \\ \epsilon_*(I^\bullet_{\mathbb{Q}}) & \xrightarrow{\pm 1} & \epsilon_*(I^\bullet_{\mathbb{Q}}) \\ \cap\big\downarrow & & \big\downarrow\cap \\ \mathrm{MF}_{\mathbb{Q}}(N_\infty) & \xrightarrow{\pm 1} & \mathrm{MF}_{\mathbb{Q}}(N_\infty), \end{array}$$

it is easy to see that the morphism $\nu^k_\infty\colon E_1^{-k,h+k}(\delta_{\mathbb{Q}}) \longrightarrow E_1^{k,h-k}(\delta_{\mathbb{Q}})(-k)$ is equal to the morphism $N^k_\infty\colon E_1^{-k,h+k}(N_\infty) \longrightarrow E_1^{k,h-k}(N_\infty)(-k)$ (cf. [RZ, (1.7)]). By (5.10.1) and (5.10.2), we have $E_2^{-k,h+k}(\delta_{\mathbb{Q}}) = E_2^{-k,h+k}(N_\infty)$. Hence the morphism (2) is equal to the morphism (3). □

Corollary 5.13. *If $\mathring{X}$ is projective, then the following four filtrations coincide*:

(1) *the weight filtration on $H^h(X_\infty, \mathbb{Q})$ by the spectral sequence* (2.0.8; ∞),

(2) *the monodromy filtration on $H^h(X_\infty, \mathbb{Q})$ defined by the isomorphism $H^h(X_\infty, \mathbb{Q}) \simeq H^h(X, A^\bullet_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q})$ and the monodromy operator* (2.1.1),

(3) *the weight filtration on $H^h(X_\infty, \mathbb{Q})$ by the spectral sequence* (2.1.10)

and

(4) *the monodromy filtration on* $H^h(X_\infty, \mathbb{Q})$ *defined by the monodromy operator* (2.1.3).

PROOF. The coincidence of (3) and (4) follows from the argument due to M. Saito [SaM, §4] (cf. [St2, p. 117]). By using the monodromy operator (2.1.1), the morphism

$$-(2\pi\sqrt{-1})^{-1}\log T\colon H^h(X_\infty,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}\longrightarrow H^h(X_\infty,\mathbb{Z})(-1)\otimes_{\mathbb{Z}}\mathbb{Q}$$

is well-defined. Because the natural inclusion $B(J^\bullet_{\mathbb{Q}}) \xrightarrow{\subset} J^\bullet_{\mathbb{Q}}$ is a quasi-isomorphism ([FN, (3.5)]), the coincidence of (2) and (4) follows. By the proof of (5.12), the morphism (3) in the proof of (5.12) is equal to the morphism (2) in that of (5.12). Because it is easy to see that the morphism (2) in the proof of (5.12) is equal to the following morphism

$$N^k_\infty\colon E_2^{-k,h+k}(\delta_{\mathbb{Q}})\longrightarrow E_2^{k,h-k}(\delta_{\mathbb{Q}})(-k), \tag{5.13.1}$$

we see that the morphism (5.13.1) is an isomorphism. The weight spectral sequence (2.0.8; ∞) degenerates at E_2. Now the coincidence of (1) and (2) follows from the definition of the monodromy filtration of (2). □

REMARK 5.14. (1) The incomplete construction of the $\mathbb{Z}$-structure of the weight filtration in [St2] makes no sense as pointed out in §2 and §3. Even if the $\mathbb{Z}$-structure is proved to be well-defined, I do not know whether the $\mathbb{Z}$-structure is equal to the $\mathbb{Z}$-structure induced by the weight spectral sequence (2.0.7; ∞).

(2) The explanation in [Il4, p. 312] that one recovers the degeneration at E_2 of the weight spectral sequences for the algebraic cases in [St1] and [St2] from that of the l-adic weight spectral sequence in [Nak3] by using the classical comparison theorem is incomplete because nothing about the boundary morphisms between the E_1-terms of the weight spectral sequences is mentioned. We complete the explanation as follows. For the former case [St1], by the same proof as that in [Nakk3, (10.1)], the boundary morphism between the E_1-terms of the weight spectral sequence in [St1] (with the variant of Steenbrink's double complexes (cf. (2.1.6))) has the description (5.10.2) (cf. [GN, (1.8), (2.7)]). Hence, by using the explicit description (5.5.1), the dimensions of the E_2-terms are equal to the dimensions of the E_2-terms of the l-adic weight spectral sequence. Therefore the desired degeneration follows from the classical comparison theorem. For the latter case [St2], the desired degeneration follows from

(a) Fujisawa-Nakayama's comparison theorem ([FN, (5.8)]) between their weight spectral sequence and Steenbrink's weight spectral sequence in [St2],

(b) their another comparison theorem (the proof of [FN, (7.1)]) between the filtered Steenbrink complex $(A^\bullet_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_l, P)$ and the projective system $\{(A^\bullet_{X,l,n}, P)\}_{n\in\mathbb{N}}$ in §5 of filtered complexes (it is easy to check that the family $\{(A^\bullet_{X,l,n}, P)\}_{n\in\mathbb{N}}$ is indeed a natural projective system)

and

(c) the explicit descriptions (5.5.1) and (5.10.2)

or

(c)′ $(A^\bullet_{\mathbb{Q}}, P) \simeq (A^\bullet_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}, P \otimes_{\mathbb{Z}} \mathbb{Q})$ ([FN, (6.5)]).

We define convenient notations for later sections and the remark (5.16) below. Set

$$H^h_\star(Z) := \begin{cases} H^h_{\text{et}}(\overline{Z}, \mathbb{Q}_l) & (\star = l), \\ H^h_{\text{crys}}(Z/W) \otimes_W K_0 & (\star = p), \\ H^h(Z_{\text{an}}, \mathbb{Q}) & (\star = \infty). \end{cases}$$

for a proper (smooth) variety Z over κ, where κ is a field, a perfect field of characteristic $p > 0$ and the complex number field $\mathbb{C}$, respectively. Set also

$$H^h_{\log,\star}(X) := \begin{cases} H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l) & (\star = l), \\ H^h_{\text{log-crys}}(X/W) \otimes_W K_0 & (\star = p), \\ H^h((X_{\text{an}})_\infty, \mathbb{Q}) & (\star = \infty) \end{cases}$$

for a proper SNCL variety X/s. When we consider (log) crystalline cohomologies and the Witt ring $W := W(\kappa)$ of κ, we always assume that the base field κ is a perfect field of characteristic $p > 0$. We often use the notations $H^h_{\log,\star}(X)$ and $H^h_\star(Z)$ in order to avoid giving statements with respect to $\star = l$, p and ∞ repeatedly, though we admit that the notations are confusing in the l-adic case because we always consider the (log) l-adic cohomologies of $\overline{Z}$ and $X_{\overline{s}}$ (neither Z nor X) in this paper. We shall also use the following notations in §7 and §11:

$$\underline{H}^h_\star(Z) := \begin{cases} H^h_{\text{et}}(\overline{Z}, \mathbb{Z}_l) & (\star = l), \\ H^h_{\text{crys}}(Z/W) & (\star = p), \\ H^h(Z_{\text{an}}, \mathbb{Z}) & (\star = \infty), \end{cases}$$

$$\underline{H}^h_{\log,\star}(X) := \begin{cases} H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Z}_l) & (\star = l), \\ H^h_{\text{log-crys}}(X/W) & (\star = p), \\ H^h((X_{\text{an}})_{\infty}, \mathbb{Z}) & (\star = \infty). \end{cases}$$

In the case $\star = \infty$, we use the same notations $H^h_\star(Z)$, $\underline{H}^h_\star(Z)$, $H^h_{\log,\star}(X)$ and $\underline{H}^h_{\log,\star}(X)$ for a proper (smooth) analytic variety Z over $\mathbb{C}$ and a proper SNCL analytic variety $X/(\operatorname{Spec}\mathbb{C}, \mathbb{C}^* \oplus \mathbb{N})$.

Set

$$\mathbf{1}_\star := \begin{cases} \mathbb{Q}_l & (\star = l), \\ K_0 & (\star = p), \\ \mathbb{Q} & (\star = \infty) \end{cases}$$

and

$$\underline{\mathbf{1}}_\star := \begin{cases} \mathbb{Z}_l & (\star = l), \\ W & (\star = p), \\ \mathbb{Z} & (\star = \infty). \end{cases}$$

Proposition 5.15. *Assume that* $\overset{\circ}{X}$ *is of pure dimension* d. *Let* $\{E^{\bullet\bullet}_{r,\star}\}_{r\geq 1}$ $(\star = l, p, \infty)$ *be the* E_r*-terms of the weight spectral sequence* (2.0.8;$\star$) *or* (2.1.10). *Then the Poincaré duality pairing*

$$(5.15.1) \qquad \langle\ ,\ \rangle : E^{-k,2d-h-k}_{1,\star} \otimes_{\mathbf{1}_\star} E^{k,h+k}_{1,\star} \longrightarrow \mathbf{1}_\star(-d)$$

induces the following perfect pairing

$$(5.15.2) \qquad \langle\ ,\ \rangle : E^{-k,2d-h-k}_{2,\star} \otimes_{\mathbf{1}_\star} E^{k,h+k}_{2,\star} \longrightarrow \mathbf{1}_\star(-d).$$

Proof. In the p-adic case, we have proved (5.15) in [Nakk3, (10.5) (2)]. In the l-adic case and the two ∞-adic cases, the proof is the same as that in the p-adic case by using (5.5.1), (5.10.1) and (5.10.2), respectively, and only by replacing crystalline cohomologies in [loc. cit.] with l-adic and Betti cohomologies, respectively. □

At last we can conclude the part I of this paper by giving the following remarks on signs of the $\star$-adic weight spectral sequence $(\star = l, p, \infty)$.

Remark 5.16. If we do not make the twist of the identification by the signs in $(2.0.6; l)$ (resp. $(2.0.6; \infty)$), the description of the boundary morphism

$$d_1^{-k,h+k}\colon E_{1,\star}^{-k,h+k}=\bigoplus_{j\geq\max\{-k,0\}}\underline{H}_{\star}^{h-2j-k}(\overset{\circ}{X}{}^{(2j+k+1)})(-j-k)\longrightarrow E_{1,\star}^{-k+1,h+k}\quad(\star{=}l,\infty)$$

of the E_1-terms of (2.0.5; l) (resp. (2.0.4; ∞) by the use of (2.0.4.1; ∞)) is

$$\sum_{j\geq\max\{-k,0\}}\{(-1)^{j+1}G+(-1)^{j+k}\rho\}. \tag{5.16.1}$$

If we consider the anti-commutative diagram in [SGA $4\frac{1}{2}$, Cycle (2.1.3)], it is natural to make the following twist by a sign

$$(-1)^{2j+k+1}\times\colon\underline{H}_{\star}^{h-2j-k}(\overset{\circ}{X}{}^{(2j+k+1)})(-j-k)\longrightarrow\underline{H}_{\star}^{h-2j-k}(\overset{\circ}{X}{}^{(2j+k+1)})(-j-k). \tag{5.16.2}$$

Then the boundary morphism $d_1^{-k,h+k}$ of the E_1-terms of (2.0.5; l) (resp. (2.0.4; ∞) by the use of (2.0.4.1; ∞)) with the twist above is described by

$$\sum_{j\geq\max\{-k,0\}}\{(-1)^{j}G+(-1)^{j+k+1}\rho\}. \tag{5.16.3}$$

Note that the twist in (5.16.2) for $\star=\infty$ is compatible with (3.2.3) and (3.2.4) in the following sense: for the integer $i:=2j+k+1$, the following diagram

$$\begin{array}{ccc}
\mathbb{C}_{\overset{\circ}{X}{}^{(i)}}(-i)\{-i\} & \xrightarrow{-G} & \mathbb{C}_{\overset{\circ}{X}{}^{(i-1)}}(-(i-1))\{-(i-1)\}[1] \\
{\scriptstyle (-1)^i}\downarrow & & \downarrow{\scriptstyle (-1)^{i-1}} \\
\mathbb{C}_{\overset{\circ}{X}{}^{(i)}}(-i)\{-i\} & & \mathbb{C}_{\overset{\circ}{X}{}^{(i-1)}}(-(i-1))\{-(i-1)\}[1] \\
{\scriptstyle (3.2.3)}\downarrow & & \downarrow{\scriptstyle (3.2.3)} \\
\Omega^{\bullet}_{\overset{\circ}{X}{}^{(i)}/\mathbb{C}}\{-i\} & & \Omega^{\bullet}_{\overset{\circ}{X}{}^{(i-1)}/\mathbb{C}}\{-(i-1)\}[1] \\
\cup\uparrow & & \uparrow\cup \\
\mathbb{C}_{\overset{\circ}{X}{}^{(i)}}(-i)\{-i\} & \xrightarrow{-G} & \mathbb{C}_{\overset{\circ}{X}{}^{(i-1)}}(-(i-1))\{-(i-1)\}[1]
\end{array} \tag{5.16.4}$$

in $\mathrm{D}^+(\mathbb{C}_{\overset{\circ}{X}})$ is commutative. The sign in (5.16.2) corresponds to the sign in (3.2.4). Note also that the composite sign by the twist (5.16.2) and the sign $(-1)^{j+1}$ appearing in (3.6.5) is equal to

$$(-1)^{2j+k+1+j+1}=(-1)^{j+k} \tag{5.16.5}$$

which appears in (2.0.6; l) and (2.0.6; ∞). By the sign $(-1)^{j+1}$ and by the obvious ∞-adic analogue of (5.5.2), one sees the reason why the sign before

G in (5.16.3) is the same as that in (5.10.1) and (5.10.2). Similarly, by the obvious ∞-adic analogue of (5.5.3), one sees the reason why the sign before ρ in (5.16.3) is the multiplication of that in (5.10.1) by -1.

As a summary, we give only the following seven types of the explicit descriptions of the boundary morphisms of the E_1-terms of the weight spectral sequences of proper SNCL algebraic and analytic varieties over a log point:

(Type I) (5.5.1)=[Nakk3, (10.1)] =(5.10.1)=(5.10.2),
(Type II) (5.16.1),
(Type III) (5.16.3),
(Type IV) (5.8.7),
(Type V) [GN, (2.7)] (cf. [Nakk3, 10.4.3; $\star$]),
(Type VI) [SaT, (2.10)],
(Type VII) (5.8.8).

Part II. Weight spectral sequences of log surfaces.

6. Weight spectral sequences of analytic reductions of rigid analytic elliptic surfaces.

In this section we give examples of proper SNCL surfaces over a log point $s = (\operatorname{Spec}\kappa, \mathcal{M}_s)$ whose first and third $\star$-adic ($\star = l,\ p,\ \infty$) log cohomologies have different monodromy filtrations and weight filtrations.

Let π be a global section of $\mathcal{M}_s$ whose image in $\Gamma(s, \mathcal{M}_s/\mathcal{O}_s^*)$ is the generator. Let q be a global section of $\mathcal{M}_s$ such that the image in κ is 0. Let E_q be the q-Tate curve over s ([Kk2, §2.2], cf. [DR, VII 1], [Il3, 3.1]): E_q represents the following functor from the category of the fs(=fine and saturated) log schemes over s to (Sets):

$$\text{(6.0.1)} \qquad \Big(Z \longmapsto \bigcup_{n\in\mathbb{Z}} \{f \in \Gamma(Z, \mathcal{M}_Z^{\mathrm{gp}}) \mid \pi^n | f | \pi^{n+1}\}/q^{\mathbb{Z}}\Big)^a,$$

where a means the sheafification and $f|g$ ($f, g \in \Gamma(Z, \mathcal{M}_Z^{\mathrm{gp}})$) means $f^{-1}g \in \Gamma(Z, \mathcal{M}_Z)$. The multiplicative group $\mathbb{G}_{m/\overset{\circ}{s}}$ over $\overset{\circ}{s}$ naturally acts on E_q. Let Y be an fs log scheme over s and let $\mathcal{L}$ be an invertible sheaf on the underlying scheme $\overset{\circ}{Y}$. Consider the following log scheme which is obtained by twisting $E_q \times_s Y$ by $\mathcal{L}$: let $Y = \bigcup_i U_i$ be an open covering of Y such that $\mathcal{L}$ is trivialized on $\overset{\circ}{U}_i$. Let (g_{ij}) $(g_{ij} \in \mathbb{G}_m(\overset{\circ}{U}_i \cap \overset{\circ}{U}_j))$ be a cocycle representing $\mathcal{L}$. To fix our idea, we follow the rule in [SGA $4\frac{1}{2}$, Cycle (1.1.2)] for the $\mathbb{G}_m$-torsors on $\overset{\circ}{Y}$. Let $E_{q,Y}(\mathcal{L})$ be a union $\bigcup_{i \in I} (U_i \times_s E_q)$ whose gluing on $(U_i \times_s E_q) \cap \cap (U_j \times_s E_q)$ is given by an automorphism $f \bmod q^{\mathbb{Z}} \longmapsto g_{ij} f \bmod q^{\mathbb{Z}}$ with respect to E_q. This construction of $E_{q,Y}(\mathcal{L})$ is due to K. Kato ([Kk3]).

Write q as the following form: $q = u\pi^e$ $(u \in \kappa^*,\ e \in \mathbb{Z}_{\geq 1})$. Assume that $e \geq 2$ in this paper. Then the fibers of the underlying scheme of $E_{q,Y}(\mathcal{L})$ over $\overset{\circ}{Y}$ are the e-gon of $\mathbb{P}^1_\kappa$ and $E_{q,Y}(\mathcal{L})$ is a SNCL variety over s.

We can also construct $E_{q,Y}(\mathcal{L})$ concretely without using (6.0.1). Indeed, consider the e-gon $\overset{\circ}{P}$ over κ. The scheme $\overset{\circ}{P}$ has a log structure of SNCL type (cf. [Kf, (11.7) 2]), and we have an fs log scheme P over s. Consider the following disjoint union of fs log schemes: $\coprod_i (U_i \times_s P)$. Set $U_{ij} := U_i \cap U_j$. Let $f_{ij} \colon U_j|_{U_{ij}} \xrightarrow{\sim} U_i|_{U_{ij}}$ be the gluing isomorphism ($f_{ii} = \mathrm{id}$ and $f_{ij} \circ f_{jk} = f_{ik}$) of the open covering $Y = \bigcup_i U_i$. Let u be a local section of $\mathcal{O}_{U_{ij}}$ and x, y homogeneous coordinates of $\mathbb{P}^1_\kappa$. Then we have an isomorphism $h_{ij} \colon (U_j|_{U_{ij}}) \times_s P \xrightarrow{\sim} (U_i|_{U_{ij}}) \times_s P$ such that $h^*_{ij}(u \otimes 1) = f^*_{ij}(u)$, $h^*_{ij}(1 \otimes x) = = 1 \otimes x$ and $h^*_{ij}(1 \otimes y) = g_{ij} \otimes y$ in the ring $\mathcal{O}_{U_{ij}} \otimes_\kappa \mathcal{O}_P$. The family $\{h^*_{ij}\}$ satisfies the cocycle condition, and thus we have a scheme, in fact, a proper SNCL variety $E_{q,Y}(\mathcal{L})$.

Remark 6.1. (1) The following convention is important. First consider the case $e = 2$. Then we fix an order $0 < 1$ in the elements of $\mathbb{Z}/2$. Next consider the case $e > 2$. Then we fix the following two-term relation for the elements of $\mathbb{Z}/e$

$$0 < 1,\ 1 < 2,\ \ldots,\ e-2 < e-1,\ e-1 < 0.$$

Note that this two-term relation does not satisfy the transitive law. By using this relation, for example in the p-adic case, we define the Poincaré residue isomorphism, and G and ρ in [Mo, 4.10] and [Mo, 4.12], respectively. Set $X = E_{q,Y}(\mathcal{L})$ and let X_i $(i = 0, 1, \ldots, e-1)$ be the irreducible components of $\overset{\circ}{X}$. In the l-adic and the ∞-adic cases, we use this relation for the isomorphism $\bigwedge^r (\mathcal{M}^{\mathrm{gp}}_X / \mathcal{O}^*_X) \simeq \bigoplus_{i<j} \mathbb{Z}_{X_i \cap X_j}$ (cf. the proof of [Nak3, (1.8.3)]).

(2) If Y is a proper smooth curve C over κ with a log structure which is the pull-back of that of s, $E_{q,C}(\mathcal{L})$ is expressed as follows:

(a): If $\kappa = \mathbb{C}$, $E_{q,C}(\mathcal{L})$ is the degeneration of a proper strict semistable family of non-Kähler elliptic surfaces over a unit disk. (We do not use this fact in this paper.)

(b): Let κ be a not necessarily perfect field of characteristic $p > 0$. Let W be a Cohen ring of κ and K_0 the fraction field of W. Then $E_{q,C}(\mathcal{L})$ is the special fiber of a formal proper strict semistable family of surfaces over Spf W with the canonical log structure in the sense of (6.2) below. (Only as to the underlying scheme $\overset{\circ}{E}_{q,C}(\mathcal{L})$, this fact is obtained by [Ue, §6 a)] and by the analytic reduction with respect to a pure covering which produces $E_{q,C}(\mathcal{L})$.) Because there is no appropriate reference for this fact and because we shall use this fact in (7.10), (7.11) and (7.12) below, we construct this formal proper strict semistable family in (6.3) below.

For (6.3) below, we need the log structure associated to a formal strict semistable family, which is a vertical version of the log structure defined in [NS, §8].

Let V be a complete discrete valuation ring. Let $\overset{\circ}{\mathfrak{Y}}$ be a formal strict semistable family over Spf V. Let $\{Y_j\}_{j=1}^m$ be the smooth irreducible components of the special fiber $\overset{\circ}{Y}$ of $\overset{\circ}{\mathfrak{Y}}$. Let $\mathrm{Div}(\overset{\circ}{\mathfrak{Y}})_{\geq 0}$ be the monoid of effective Cartier divisors on $\overset{\circ}{\mathfrak{Y}}$. Let $\mathrm{Div}_{\overset{\circ}{Y}}(\overset{\circ}{\mathfrak{Y}})_{\geq 0}$ be a submonoid of $\mathrm{Div}(\overset{\circ}{\mathfrak{Y}})_{\geq 0}$ consisting of effective Cartier divisors E's on $\overset{\circ}{\mathfrak{Y}}$ such that there exists an open covering $\overset{\circ}{\mathfrak{Y}} = \bigcup_{i\in I} \overset{\circ}{\mathcal{U}}_i$ of $\overset{\circ}{\mathfrak{Y}}$ such that $E|_{\overset{\circ}{\mathcal{U}}_i}$ is contained in the submonoid of $\mathrm{Div}(\overset{\circ}{\mathcal{U}}_i)_{\geq 0}$ generated by $Y_j|_{\overset{\circ}{\mathcal{U}}_i}$ $(1 \leq j \leq m)$.

The scheme $\overset{\circ}{\mathfrak{Y}}$ gives a natural fs(= fine and saturated) log structure in $\tilde{\overset{\circ}{\mathfrak{Y}}}_{\mathrm{zar}}$ as follows (cf. [Kk1, p. 222–223], [Fa, §2]).

Let $\mathcal{M}'$ be a presheaf of monoids in $\tilde{\overset{\circ}{\mathfrak{Y}}}_{\mathrm{zar}}$ defined as follows: for an open subscheme $\mathcal{V}$ of $\overset{\circ}{\mathfrak{Y}}$, set

$$\text{(6.1.1)}\quad \Gamma(\mathcal{V}, \mathcal{M}') := \{(E, a) \in \mathrm{Div}_{\overset{\circ}{Y}|_{\mathcal{V}}}(\mathcal{V})_{\geq 0} \times \Gamma(\mathcal{V}, \mathcal{O}_{\overset{\circ}{\mathfrak{Y}}}) \mid a \text{ is a generator of } \Gamma(\mathcal{V}, \mathcal{O}_{\overset{\circ}{\mathfrak{Y}}}(-\overset{\circ}{Y}))\}$$

and with a monoid structure defined by $(E, a) \cdot (E', a') := (E + E', aa')$. The second projection $\mathcal{M}' \longrightarrow \mathcal{O}_{\overset{\circ}{\mathfrak{Y}}}$ induces a morphism $\mathcal{M}' \longrightarrow (\mathcal{O}_{\overset{\circ}{\mathfrak{Y}}}, *)$ of presheaves of monoids in $\tilde{\overset{\circ}{\mathfrak{Y}}}_{\mathrm{zar}}$. The log structure $\mathcal{M}$ is, by definition, the associated log structure to the sheafification of $\mathcal{M}'$.

Definition 6.2. We call $\mathcal{M}$ the canonical log structure of $\overset{\circ}{\mathfrak{Y}}/\mathrm{Spf}\, V$.

Proposition 6.3. *Assume that κ is a perfect field of characteristic $p > 0$. Let W be the Witt ring of κ. Let Y be a proper smooth scheme over κ.*

Assume that there exists a pair $(\mathcal{Y}, \mathfrak{L})$ *of a formal proper smooth scheme with an invertible sheaf such that* $\mathcal{Y} \otimes_W \kappa = Y$ *and* $\mathfrak{L} \otimes_W \kappa = \mathcal{L}$. *Then there exists a formal proper strict semistable family* $\mathfrak{Y}$ *over* Spf W *with canonical log structure such that* $\mathfrak{Y} \otimes_W \kappa \simeq E_{q,Y}(\mathcal{L})$.

In particular, for a proper smooth curve C *over* κ, *there exists a formal proper strict semistable family* $\mathfrak{X}$ *over* Spf W *with canonical log structure such that* $\mathfrak{X} \otimes_W \kappa \simeq E_{q,C}(\mathcal{L})$.

PROOF. Let $\{\mathcal{U}_i\}_i$ be an affine open covering of $\mathcal{Y}$ such that there exists an isomorphism $\mathfrak{L}|_{\mathcal{U}_i} \simeq \mathcal{O}_{\mathcal{U}_i}$. Let $\mathbb{L} := \underline{\mathrm{Spf}}_{\mathcal{Y}}(\mathrm{Sym}\, \mathfrak{L})$ be an affine line bundle over $\mathcal{Y}$ associated to $\mathfrak{L}$. Let $\{0\}$ be the zero section of $\mathbb{L}$. Then $(\mathbb{L} \setminus \{0\})|_{\mathcal{U}_i} \simeq \mathcal{U}_i \widehat{\times}_{\mathrm{Spf}\, W} \hat{\mathbb{G}}_{m/W}$. Consider the following rigid analytic spaces for integers $j = 1, 2, \ldots, e$:

$$\begin{aligned} V_j \big(&= \text{“}\{z \in K_0 \mid |p|^j \leq |z| \leq |p|^{j-1}\}\text{”} = \text{“}\mathrm{Spm}\, K_0\{z/p^{j-1}, p^j/z\}\text{”}\big) \\ &:= \mathrm{Spm}\, K_0\{z, u_j, v_j\}/(z - p^{j-1}u_j, zv_j - p^j), \\ &\simeq \mathrm{Spm}\, K_0\{u_j, v_j\}/(u_j v_j - p). \end{aligned}$$

Natural inclusions

$$K_0\{z/p^{j-1}, p^j/z\} \overset{\subset}{\longrightarrow} K_0\{z/p^j, p^j/z\} \overset{\supset}{\longleftarrow} K_0\{z/p^j, p^{j+1}/z\} \quad (j = 1, \ldots, e-1)$$

and two inclusions

$$K_0\{z/p^{e-1}, p^e/z\} \overset{\subset}{\longrightarrow} K_0\{z, 1/z\} \overset{\supset}{\longleftarrow} K_0\{z, p/z\}$$

gives the patching V_j and V_{j+1} $(j = 1, \ldots, e-1)$ along $V_{j,j+1} :=$ $:= \mathrm{Spm}\, K_0\{z/p^j, p^j/z\}$, and V_e and V_1 along $V_{e,1} := \mathrm{Spm}\, K_0\{z, 1/z\}$. Here the last $\overset{\subset}{\longrightarrow}$ is given by $z/p^{e-1} \longmapsto pz$ and $p^e/z \longmapsto 1/z$ and the last $\overset{\supset}{\longleftarrow}$ is the natural inclusion. The union $\bigcup_{j=1}^{e} V_j$ is, what is called, the Tate curve $\mathbb{G}_{m/K_0}/(p^e)^{\mathbb{Z}}$.

Each V_j $(j \in \mathbb{Z}/e)$ has a formal model $\mathcal{V}_j := \mathrm{Spf}\, W\{u_j, v_j\}/(u_j v_j - p)$, which has strict semistable reduction over Spf W. Set $\mathcal{V}_{j,j+1} :=$ $\mathrm{Spf}\, W\{u_{j+1}, v_j\}/(u_{j+1}v_j - 1)$, and patch $\mathcal{V}_j$ and $\mathcal{V}_{j+1}$ along $\mathcal{V}_{j,j+1}$ as follows by imitating the patching above:

$$W\{u_j, v_j\}/(u_j v_j - p) \overset{\subset}{\longrightarrow} W\{u_{j+1}, v_j\}/(u_{j+1}v_j - 1) \overset{\supset}{\longleftarrow} W\{u_{j+1}, v_{j+1}\}/(u_{j+1}v_{j+1} - p). \tag{6.3.1}$$

The left morphism in (6.3.1) is given by $u_j \longmapsto pu_{j+1}$, $v_j \longmapsto v_j$ and the right one is given by $u_{j+1} \longleftarrow u_{j+1}$, $pv_j \longleftarrow v_{j+1}$. Set $\mathcal{U}_{ij} := \mathcal{U}_i \cap \mathcal{U}_j$. Let $\widetilde{g}_{ij} \in \Gamma(\mathcal{U}_{ij}, \hat{\mathbb{G}}_{m/W})$ be a cocycle representing the invertible sheaf $\mathfrak{L}$.

Consider a product $\mathcal{U}_i \widehat{\times}_{\operatorname{Spf} W}\big(\bigcup_{k=1}^{e} \mathcal{V}_k\big)$. Patch $\mathcal{U}_i \widehat{\times}_{\operatorname{Spf} W}\big(\bigcup_{k=1}^{e} \mathcal{V}_k\big)$ and $\mathcal{U}_j \widehat{\times}_{\operatorname{Spf} W}\big(\bigcup_{k=1}^{e} \mathcal{V}_k\big)$ by the following isomorphism

$$\mathcal{O}_{\mathcal{U}_j|_{\mathcal{U}_{ij}}} \hat{\otimes}_W \mathcal{O}_{\mathcal{V}_k} \overset{\sim}{\longrightarrow} \mathcal{O}_{\mathcal{U}_i|_{\mathcal{U}_{ij}}} \hat{\otimes}_W \mathcal{O}_{\mathcal{V}_k}$$

$$x \otimes 1 \longmapsto x \otimes 1, \quad 1 \otimes u_k \longmapsto \widetilde{g}_{ij} \otimes u_k, \quad 1 \otimes v_k \longmapsto \widetilde{g}_{ij}^{-1} \otimes v_k \quad (x \in \mathcal{O}_{\mathcal{U}_j|_{\mathcal{U}_{ij}}}).$$

Let $\overset{\circ}{\mathfrak{Y}}$ be the resulting formal scheme over $\operatorname{Spf} W$. Then $\overset{\circ}{\mathfrak{Y}}$ has strict semistable reduction over $\operatorname{Spf} W$. Endow $\overset{\circ}{\mathfrak{Y}}$ with the canonical log structure and let $\mathfrak{Y}$ be the resulting log scheme. Then $\mathfrak{Y}$ is a desired formal log scheme.

The latter statement immediately follows from the existence of a formal lift of $(C, \mathcal{L})$ ([SGA 1, III (6.10)]). □

Remark 6.4. By the construction in the proof of (6.3), the generic fiber of $\mathfrak{X}$ is nothing but the rigid analytic space in [Ue, §6 a)] and the special fiber of $\mathfrak{X}$ is $E_{q,C}(\mathcal{L})$.

Theorem 6.5. *Let C be a proper smooth curve over κ endowed with log structure which is the pull-back of that of s. Let $\mathcal{L}$ be an invertible sheaf on C with non-zero degree d. Let $\{E_{2,\star}^{\bullet\bullet}\}$ $(\star = l, p, \infty)$ be the E_2-terms of the weight spectral sequence* (2.0.8; $\star$) *for $X := E_{q,C}(\mathcal{L})$. Then $E_{2,\star}^{-12} = 0$, but $E_{2,\star}^{10}$ is a* 1*-dimensional vector space over $\mathbf{1}_\star$. In particular, the $\star$-adic monodromy operator on $H^1_{\log,\star}(X)$ does not induce an isomorphism from $E_{2,\star}^{-12}$ to $E_{2,\star}^{10}(-1)$.*

Proof. We may assume that κ is algebraically closed. Because the dual graph of $\overset{\circ}{X}$ is a circle, the claim on $E_{2,\star}^{10}$ is obvious. We prove that $E_{2,\star}^{-12} = 0$. Let $\{X_k\}_{k \in \mathbb{Z}/e}$ be the irreducible components of $\overset{\circ}{X}$ such that $C_k :=$ $:= X_k \cap X_{k+1} \neq \phi$. The double curves C_k and C_{k-1} are sections of X_k. Let $C = \bigcup_i U_i$ be an open covering of U_i such that $\mathcal{L}|_{U_i} \simeq \mathcal{O}_{U_i}$. Let $P_k(\simeq \mathbb{P}^1_\kappa)$ be the irreducible component of the e-gon $\overset{\circ}{P}/\kappa$ corresponding to X_k. We have an isomorphism $\gamma_k\colon X_k \overset{\sim}{\longrightarrow} \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L})$ over C because the ratio of relative homogeneous coordinates of X_k over C changes by the multiplication g_{ij} with respect to the open covering $\{U_i \times_s P_k\}_i$ of X_k; we identify X_k with $\mathbb{P}(\mathcal{O}_C \oplus \mathcal{L})$ by γ_k. Let $\pi_k\colon X_k \longrightarrow C$ be the projection. Let $D_{k,1}$ (resp. $D_{k,2}$) be a section of X_k which corresponds to the projection $\mathcal{O}_C \oplus \mathcal{L} \overset{\mathrm{pr}_1}{\longrightarrow} \mathcal{O}_C$ (resp. $\mathcal{O}_C \oplus \mathcal{L} \overset{\mathrm{pr}_2}{\longrightarrow} \mathcal{L}$). Then, by [Ha2, V (2.6)], $\pi_k^*(\mathcal{L}) = \mathcal{O}_{X_k}(1) \otimes \mathcal{O}_{X_k}(-D_{k,1})$ and $\pi_k^*(\mathcal{O}_C) = \mathcal{O}_{X_k}(1) \otimes \mathcal{O}_{X_k}(-D_{k,2})$. Thus $\pi_k^*(\mathcal{L}) = \mathcal{O}_{X_k}(D_{k,2} - D_{k,1})$. Because

$\{C_k, C_{k-1}\} = \{D_{k,1}, D_{k,2}\}$,

$$\pi_k^*(\mathcal{L}) = \mathcal{O}_{X_k}(\pm(C_k - C_{k-1})). \tag{6.5.1}$$

Consider the following commutative diagram

$$\begin{array}{ccc} \operatorname{Pic} C & \xrightarrow{\pi_k^*} & \operatorname{Pic} X_k \\ {\scriptstyle c_{1,C}}\downarrow & & \downarrow{\scriptstyle c_{1,X_k}} \\ H^2_\star(C)(1) & \xrightarrow{\pi_k^*} & H^2_\star(X_k)(1), \end{array}$$

where the vertical morphisms are first Chern classes. By (6.5.1) we have $c_{1,X_k}(C_k) - c_{1,X_k}(C_{k-1}) = \pm\pi_k^*(c_{1,C}(\mathcal{L}))$. Since $\deg\mathcal{L} \neq 0$, $c_{1,C}(\mathcal{L}) \neq 0$, and since $\pi_k^*\colon H^2_\star(C) \longrightarrow H^2_\star(X_k)$ is injective, $\pi_k^*(c_{1,C}(\mathcal{L})) \neq 0$. Hence $c_{1,X_k}(C_k) - c_{1,X_k}(C_{k-1}) \neq 0$.

The boundary morphism

$$d_1^{-12}\colon H^0_\star(X^{(2)})(-1) \longrightarrow H^2_\star(X^{(1)})$$

is obtained by the first Chern classes c_{1,X_k} of X_k as follows by ((2.0.8.3; p), (5.5.1), (5.10.1), (5.10.2)):

$$(a_k)_{k\in\mathbb{Z}/e} \longmapsto -(a_k c_{1,X_k}(C_k) - a_{k-1}c_{1,X_k}(C_{k-1}))_{k\in\mathbb{Z}/e} \tag{6.5.2}$$
$$(a_k \in H^0_\star(X_k \cap X_{k+1})(-1)).$$

Assume that $(a_k)_{k\in\mathbb{Z}/e} \in \operatorname{Ker} d_1^{-12}$. Using the compatibility of the cup products of $\star$-adic cohomologies with intersection theory [Mi, VI (10.7)], [DI1, (3.3)] and [GH, p. 470], respectively, we see that $a_k = a_{k-1}$ ($\forall i \in \mathbb{Z}/e$) by considering the intersection with a fiber of X_k/C. Because $c_{1,X_k}(C_k) - c_{1,X_k}(C_{k-1}) \neq 0$, $a_k = 0$. □

REMARK 6.6. In [Kk3], Kato suggested that the weight filtration and the monodromy filtration on the first log l-adic cohomology of the degeneration of a Hopf surface ($C = \mathbb{P}^1$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(1)$) are different.

Though $X = E_{q,C}(\mathcal{L})$ is the special fiber of a formal proper strict semistable family ((6.3)), X is not the special fiber of an algebraic proper strict semistable family. This has been suggested by T. Saito:

COROLLARY 6.7. *Assume that κ is a field of characteristic $p \geq 0$. Then the following hold*:

(1) $X = E_{q,C}(\mathcal{L})$ *cannot be the special fiber of an algebraic proper strict*

semistable family over a complete discrete valuation ring with residue field κ.

(2) *Assume that the genus of* C *is positive. Then there does not exist an algebraic proper strict semistable family* $\mathcal{Y}$ *over a complete discrete valuation ring such that there is a morphism* f *with* $\deg f \neq 0$ *from the special fiber of* $\mathcal{Y}$ *with canonical log structure to* X (*See* [Nak4] *for the definition of* $\deg f$).

Proof. (1): If X were the special fiber of a proper algebraic strict semistable family above, then (6.5) would contradict [RZ, (2.13)] (cf. (8.1) below).

(2): Assume that $\mathcal{Y}$ in (6.7) existed. Let Y be the special fiber of $\mathcal{Y}$ with canonical log structure. Then, by [Nak4], $H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l)$ would be a direct factor of $H^h_{\text{log-et}}(Y_{\overline{s}}, \mathbb{Q}_l)$ ($h \in \mathbb{Z}$) and we have the following commutative diagram for $h \in \mathbb{Z}$:

(6.7.1)
$$\begin{array}{ccc} H^h_{\text{log-et}}(Y_{\overline{s}}, \mathbb{Q}_l) & \xrightarrow{N_l} & H^h_{\text{log-et}}(Y_{\overline{s}}, \mathbb{Q}_l)(-1) \\ \cup\uparrow & & \uparrow\cup \\ H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l) & \xrightarrow{N_l} & H^h_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l)(-1). \end{array}$$

By (8.1) below, the graded pieces of the monodromy filtration on $H^1_{\text{log-et}}(Y_{\overline{s}}, \mathbb{Q}_l)$ are pure in the sense of [SGA 7-I, I, (6.3)] (cf. the proof of (8.1) below), and hence those for $H^1_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l)$ are also. Because $E_{2,l}^{-12} = 0$ by (6.5), $N_l = 0$ on $H^1_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l)$. Hence the monodromy filtration $\{M_k\}_{k\in\mathbb{Z}}$ on $H^1_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l)$ is as follows: $M_0 = 0$ and $M_1 = H^1_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l)$. Because $E_{2,l}^{10} \neq 0$ and $E_{2,l}^{01} = H^1_{\text{et}}(\overline{C}, \mathbb{Q}_l) \neq 0$ (see (7.1) below), M_1/M_0 is not pure. This is a contradiction. □

Remark 6.8. (1) The coincidence of the monodromy filtrations and the weight filtrations on the 0, 2, 4-th's $\star$-adic ($\star = l$, p, ∞, respectively) log cohomologies of a proper SNCL surface follows from the E_2-degeneration of (2.0.8; $\star$) ([Nak3, (2.1)] [Nakk3, (3.6)], the theory of weight in Hodge theory, respectively) and the argument of Rapoport-Zink ([Mo, §6]).

(2) Let $\{E_{2,p}^{\bullet\bullet}\}$ be the E_2-terms of the spectral sequence (2.0.8; p). In [Mo, 4.15 (ii)], Mokrane claimed that there exists a perfect pairing between $E_{2,p}^{-k,h+k}$ and $E_{2,p}^{-k,2d-h+k}$ for a proper SNCL variety over κ. The log scheme $X = E_{q,C}(\mathcal{L})$ in (6.5) is also a counter-example of this duality. Indeed, we first note that $E_{2,p}^{k,h+k}$ and $E_{2,p}^{-k,2d-h-k}$ are dual by [Nakk3, (10.5) (2)] ((5.15)).

Hence $E_{2,p}^{12} = 0$ by (6.5); however $E_{2,p}^{10} \simeq K_0$ since the dual graph of $\overset{\circ}{X}$ is a circle. Therefore $E_{2,p}^{12}$ and $E_{2,p}^{10}$ are not dual. Note that, if one neglects the torsion in [Mo, 4.15 (ii)] and if (2.0.9;p) holds for a projective SNCL variety, the duality in [Mo, 4.15 (ii)] holds.

(3) Since the dual graph of $\overset{\circ}{X} = \overset{\circ}{E}_{q,C}(\mathcal{L})$ is a circle, $E_{2,\star}^{-14}$ is a 1-dimensional vector space. By (2), $E_{2,\star}^{12}(-1) = 0$. In particular, the induced morphism $N_\star \colon E_{2,\star}^{-14} \longrightarrow E_{2,\star}^{12}(-1)$ is not an isomorphism for X in (6.5).

(4) Considering (6.5) and (3) in this remark into account, one knows that [Mo, 6.2.1] for the proof of the statement that the morphisms $N_\star \colon E_{2,p}^{-14} \longrightarrow E_{2,p}^{12}$ and $N_\star \colon E_{2,p}^{-12} \longrightarrow E_{2,p}^{10}$ are isomorphisms is incomplete because the projectivity or the existence of the semistable family have not been used for the proof (cf. [loc. cit., 6.2.4]). However Mokrane let me know a fact (by an e-mail) that, by [Mo] and [R2] or [CI], the monodromy filtrations and the weight filtrations on the log crystalline cohomologies of the special fiber of an algebraic proper strict semistable family of surfaces over a complete discrete valuation ring of mixed characteristics with finite residue field coincide. We generalize it for any complete discrete valuation ring with any perfect residue field of positive characteristic in (8.3) below.

Next, we consider a part of the p-adic analogue of Clemens-Schmid (exact) sequence (cf. [Nakk2]).

Let κ be a perfect field of characteristic $p > 0$ and W the Witt ring of κ. In the introduction of [Ch], Chiarellotto has conjectured that the sequence

$$(6.8.1;p) \quad H^h(Y, \operatorname{Ker} \upsilon_p) \otimes_W K_0 \longrightarrow H^h_{\text{log-crys}}(Y/W) \otimes_W K_0 \overset{\upsilon_p}{\longrightarrow} H^h_{\text{log-crys}}(Y/W)(-1) \otimes_W K_0$$

is exact for a proper SNCL variety Y/s, where υ_p is the canonical morphism in §2 (We need the Tate twist (-1) in the right term of (6.8.1;p); it is forgotten in [Ch].). Note that $H^h(Y, \operatorname{Ker} \upsilon_p) \otimes_W K_0 = H^h_{\text{rig}}(\overset{\circ}{Y}/K_0)$ ([Ch, (3.6)]). In (6.11) below we show that this conjecture does not hold in general and that the l-adic and the ∞-analogues of the above do not hold in general either (cf. [Cl, p. 229]).

Proposition 6.9. *Let κ be a field of characteristic $p \geq 0$. Then the following hold*:

(1) *Let $l \neq p$ be a prime number. Let Y/s be a* (*not necessarily proper*) *SNCL variety. Then there exists a natural sequence*

$$(6.9.1;l) \qquad H^h(\overset{\circ}{\overline{Y}}, \mathbb{Z}_l) \longrightarrow H^h_{\text{log-et}}(Y_{\overline{s}}, \mathbb{Z}_l) \overset{\upsilon_l}{\longrightarrow} H^h_{\text{log-et}}(Y_{\overline{s}}, \mathbb{Z}_l)(-1).$$

(2) *Assume that κ is a perfect field of characteristic $p > 0$. Let Y/s be a (not necessarily proper) SNCL variety. Then there exists a natural sequence*

$$(6.9.1;p) \qquad H^h(Y, \operatorname{Ker} \upsilon_p) \longrightarrow H^h_{\text{log-crys}}(Y/W) \xrightarrow{\upsilon_p} H^h_{\text{log-crys}}(Y/W)(-1).$$

(3) *Let $Y/((\operatorname{Spec} \mathbb{C})_{\text{an}}, \mathbb{N} \oplus \mathbb{C}^*)$ be a SNCL analytic variety. Then there exists a natural sequence*

$$(6.9.1;\infty) \qquad H^h(\overset{\circ}{Y}, \mathbb{Z}) \longrightarrow H^h(Y_\infty, \mathbb{Z}) \xrightarrow{\upsilon_\infty} H^h(Y_\infty, \mathbb{Z})(-1).$$

(4) *Let $Y/((\operatorname{Spec} \mathbb{C})_{\text{an}}, \mathbb{N} \oplus \mathbb{C}^*)$ be a SNCL analytic variety. Then there exists a natural sequence*

$$(6.9.2;\infty) \qquad H^h(\overset{\circ}{Y}, \mathbb{Q}) \longrightarrow H^h(Y_\infty, \mathbb{Q}) \xrightarrow{N_\infty} H^h(Y_\infty, \mathbb{Q})(-1).$$

If the irreducible components of $\overset{\circ}{Y}$ are compact and Kähler or the analytifications of proper smooth schemes over $\mathbb{C}$, then (6.9.2; ∞) *is a sequence of mixed Hodge structures.*

PROOF. (1): Let the notations be as in the l-adic case in §5. Let $\upsilon_{l,n}$ be the quasi-monodromy operator for $A^\bullet_{Y,l,n}$: $\upsilon_{l,n} = (-1)^{i+j+1}\text{proj.}: A^{ij}_{Y,l,n} \longrightarrow \longrightarrow A^{i-1,j+1}_{Y,l,n}(-1)$. Then

$$\begin{aligned}(6.9.3;l)\ \operatorname{Ker} \upsilon_{l,n} &= (\cdots \longrightarrow (\tau_{j+1}L^\bullet(j+1))^{i+j+1}/(\tau_j L^\bullet(j+1))^{i+j+1} \longrightarrow \cdots)_{i,j\in\mathbb{Z}} \\ &= (\cdots \xrightarrow{\theta} \mathcal{H}^{j+1}(L^\bullet)(j+1) \xrightarrow{\theta} \cdots) \\ &= (\cdots \longrightarrow R^{j+1}\varepsilon_{Y*}(\mathbb{Z}/l^n)(j+1) \longrightarrow \cdots) \\ &= (\cdots \longrightarrow (\mathbb{Z}/l^n)_{\overset{\circ}{Y}^{(j+1)}} \longrightarrow \cdots).\end{aligned}$$

Here note that

$$\begin{array}{ccc} L^{j+1}(j+2)/(\tau_{j+1}L^\bullet(j+2))^{j+1} & \xrightarrow{(-1)^{j+2}d} & (\tau_{j+2}L^\bullet(j+2))^{j+2} \\ {\scriptstyle -\theta}\uparrow & & \uparrow{\scriptstyle \theta} \\ L^j(j+1)/(\tau_j L^\bullet(j+1))^j & \xrightarrow{(-1)^{j+1}d} & (\tau_{j+1}L^\bullet(j+1))^{j+1} \end{array}$$

is anti-commutative, however note that the natural projection

$$(\tau_{j+1}L^\bullet(j+1))^{j+1} \longrightarrow \mathcal{H}^{j+1}(L^\bullet)(j+1) \quad (j \in \mathbb{Z})$$

gives the second equality in (6.9.3; l). We also have to note that the sign by the twist in (2.0.6; l) is $(-1)^{j+(-j)} = 1$ (cf. (5.5.3)). Hence the boundary morphism

$d\colon (\mathbb{Z}/l^n)_{\overset{\circ}{\overline{Y}}{}^{(j+1)}} \longrightarrow (\mathbb{Z}/l^n)_{\overset{\circ}{\overline{Y}}{}^{(j+2)}}$ is equal to ρ by (5.1). Hence there exists a canonical isomorphism $(\mathbb{Z}/l^n)_{\overset{\circ}{\overline{Y}}} \overset{\sim}{\longrightarrow} \mathrm{Ker}\, v_{l,n}$. Therefore $\varprojlim_n H^h_{\mathrm{et}}(\overset{\circ}{\overline{Y}}, \mathrm{Ker}\, v_{l,n}) = H^h_{\mathrm{et}}(\overset{\circ}{\overline{Y}}, \mathbb{Z}_l)$. Hence we obtain (1).

(2): (2) is obvious.

(3): By using a fact $R^r\varepsilon_*(\mathbb{Z}_{Y^{\log}}) = \mathbb{Z}(-r)_{\overset{\circ}{Y}{}^{(r)}}$ $(r \in \mathbb{Z}_{>0})$ ([KN, (1.5)]), we have only to replace $L^\bullet$ in (1) with $L^\bullet_\infty$ in the ∞-adic case in §5.

(4): We replace $L^\bullet$ in (1) by $s((B(J^\bullet_{\mathbb{Q}}), d) \overset{N_\infty}{\longrightarrow} (B(J^\bullet_{\mathbb{Q}})(-1), d))$ in §2. By the proof of (1), we have

$$\mathrm{Ker}\, N_\infty = \mathrm{Ker}\, v_\infty = (\cdots \longrightarrow \mathbb{Q}_{\overset{\circ}{Y}{}^{(j+1)}} \longrightarrow \cdots). \tag{6.9.4}$$

The induced weight filtration on $\mathrm{Ker}\, N_\infty$ is equal to the increasing stupid filtration on

$$\{(\mathbb{Q}_{\overset{\circ}{Y}{}^{(-k+1)}} \longrightarrow \mathbb{Q}_{\overset{\circ}{Y}{}^{(-k+2)}} \longrightarrow \cdots)\{k\}\}_{k\in\mathbb{Z}},$$

which induces the weight filtration on $H^h(\overset{\circ}{Y}, \mathbb{Q})$ (cf. [St1, (3.5)]). Hence (4) for the weight filtration follows.

As to the Hodge filtration, we prove (4) as follows. The complex $A^\bullet(N_\infty) \otimes_{\mathbb{Q}} \mathbb{C}$ is isomorphic to $s(A^{\bullet\bullet}_{Y/\mathbb{C}})$ ((3.7)); the (i,j)-component of the double complex $A^{\bullet\bullet}_{Y/\mathbb{C}}$ $(i,j \in \mathbb{N})$ is $\widetilde{\Lambda}^{i+j+1}_{Y/\mathbb{C}}/P_j\widetilde{\Lambda}^{i+j+1}_{Y/\mathbb{C}}$. Then, by using the Poincaré residue isomorphism, we have an isomorphism

$$\mathrm{Res}\colon P_{j+1}\widetilde{\Lambda}^{i+j+1}_{Y/\mathbb{C}}/P_j\widetilde{\Lambda}^{i+j+1}_{Y/\mathbb{C}} \overset{\sim}{\longrightarrow} \Omega^i_{\overset{\circ}{Y}{}^{(j+1)}/\mathbb{C}}.$$

Hence

$$\mathrm{Ker}(v_\infty\colon A^\bullet_{\mathbb{Z}} \longrightarrow A^\bullet_{\mathbb{Z}}(-1)) \otimes_{\mathbb{Q}} \mathbb{C} = \mathrm{Ker}(N_\infty\colon A^\bullet(N_\infty) \longrightarrow A^\bullet(N_\infty)(-1)) \otimes_{\mathbb{Q}} \mathbb{C}$$

is isomorphic to the single complex of the following double complex (cf. the proof of [Nakk3, (10.1)])

$$\begin{array}{ccc} \Omega^i_{\overset{\circ}{Y}{}^{(j+2)}/\mathbb{C}} & & \\ \rho\big\uparrow & & \\ \Omega^i_{\overset{\circ}{Y}{}^{(j+1)}/\mathbb{C}} & \xrightarrow{(-1)^{j+1}d} & \Omega^{i+1}_{\overset{\circ}{Y}{}^{(j+1)}/\mathbb{C}}. \end{array} \tag{6.9.5}$$

Since the Hodge filtration $\{\mathrm{Fil}^i_{\mathrm{H}}\}_{i\in\mathbb{Z}}$ on $A^{\bullet\bullet}_{Y/\mathbb{C}}$ is defined by $\{A^{\bullet\geq i,\bullet}_{Y/\mathbb{C}}\}_{i\in\mathbb{Z}}$, $\mathrm{Fil}^i_{\mathrm{H}} = [(\Omega^\bullet_{\overset{\circ}{Y}{}^{(j+1)}/\mathbb{C}}, (-1)^{j+1}d)]_{\bullet\geq i, j\in\mathbb{N}}$. This filtration induces the Hodge filtration on $H^h(\overset{\circ}{Y}, \mathbb{C})$ (cf. [St1, (3.5)]). □

Proposition 6.10. *Let the notations be as in* (6.9). *Then the following hold*:

(1) *There exists a spectral sequence*

$$(6.10.1;l)\qquad E_1^{-k,h+k}(\mathrm{Ker}\,\upsilon_l)=H^{h+k}_{\mathrm{et}}(\overset{\circ}{Y}{}^{(-k+1)},\mathbb{Z}_l)\Longrightarrow H^h_{\mathrm{et}}(\overset{\circ}{Y},\mathrm{Ker}\,\upsilon_l).$$

(2) *There exists a spectral sequence*

$$(6.10.1;p)\qquad E_1^{-k,h+k}(\mathrm{Ker}\,\upsilon_p)=H^{h+k}_{\mathrm{crys}}(\overset{\circ}{Y}{}^{(-k+1)}/W)\Longrightarrow H^h(\overset{\circ}{Y},\mathrm{Ker}\,\upsilon_p).$$

(3) *There exists a spectral sequence*

$$(6.10.1;\infty)\qquad E_1^{-k,h+k}(\mathrm{Ker}\,\upsilon_\infty)=H^{h+k}(\overset{\circ}{Y}{}^{(-k+1)},\mathbb{Z})\Longrightarrow H^h(\overset{\circ}{Y},\mathrm{Ker}\,\upsilon_\infty).$$

Proof. (1): By considering the stupid filtration on the right hand side of $(6.9.3;l)$ for $j=-k$, we obtain the spectral sequence $(6.10.1;l)$.

(2): Consider the following filtration on $\mathrm{Ker}\,\upsilon_p$.

$$P_k(\mathrm{Ker}\,\upsilon_p)=$$
$$=(\cdots\longrightarrow((P_{2j+k+1}\cap P_{j+1})W\widetilde{\Lambda}^{i+j+1}_X+P_jW\widetilde{\Lambda}^{i+j+1}_Y)/P_jW\widetilde{\Lambda}^{i+j+1}_Y\longrightarrow\cdots)_{i,j\geq 0}.$$

Here we used the notation $W\widetilde{\Lambda}^{\bullet}_Y$ for $W\widetilde{\omega}^{\bullet}_Y$ in [Mo, §3]. By [Nakk3, (8.6) (5), (9.12)], we obtain a canonical isomorphism

$$\mathrm{gr}^P_k(\mathrm{Ker}\,\upsilon_p)\ \xrightarrow{\sim}\ W\Omega^{\bullet}_{\overset{\circ}{Y}{}^{(-k+1)}}\{k\},\quad (k\leq 0)$$

which is compatible with the Frobenius. Therefore we obtain (2).

(3): The proof of (3) is the same as that of (1) by using (6.9.4). □

Proposition 6.11. *The three sequences*

$$(6.11.1;l)\qquad H^3(\overset{\circ}{\overline{X}},\mathbb{Q}_l)\longrightarrow H^3_{\text{log-et}}(X_{\overline{s}},\mathbb{Q}_l)\ \xrightarrow{\upsilon_l}\ H^3_{\text{log-et}}(X_{\overline{s}},\mathbb{Q}_l)(-1),$$

$$(6.11.1;p)\quad H^3(X,\mathrm{Ker}\,\upsilon_p)\otimes_W K_0\longrightarrow$$
$$H^3_{\text{log-crys}}(X/W)\otimes_W K_0\ \xrightarrow{\upsilon_p}\ H^3_{\text{log-crys}}(X/W)(-1)\otimes_W K_0$$

and

$$(6.11.1;\infty)\qquad H^3(\overset{\circ}{X},\mathbb{Q})\longrightarrow H^3(X_\infty,\mathbb{Q})\ \xrightarrow{N_\infty}\ H^3(X_\infty,\mathbb{Q})(-1)$$

are not exact for $X=E_{q,C}(\mathcal{L})$ *over the log point of a field of characteristic* $p\neq l$, *a perfect field of characteristic* $p>0$, $\mathbb{C}$, *respectively.*

PROOF. We give the proof only for the p-adic case; the proof for the l-adic and the ∞-adic cases is the same.

We have

$$\dim_{K_0}(\operatorname{Ker} \upsilon_p : H^3_{\text{log-crys}}(X/W) \otimes_W K_0 \longrightarrow H^3_{\text{log-crys}}(X/W)(-1) \otimes_W K_0) = \dim_{K_0} E_{2,p}^{0,3} + 1$$

by (6.8) (2), (3). However

$$\dim_{K_0} H^3(X, \operatorname{Ker} \upsilon_p) \otimes_W K_0 = \dim_{K_0} E_2^{03}(\operatorname{Ker} \upsilon_p) \otimes_W K_0 = \dim_{K_0} E_{2,p}^{0,3}$$

by (6.8) (2) and (6.10.1; p). Hence (6.11.1; p) is not exact. □

REMARK 6.12. If (2.0.9; l) holds, then (6.9.1; l) is exact; in fact, the l-adic generalized Clemens-Schmid sequence for a projective SNCL variety X is exact by the argument of [Z, (7.5), (7.6)] and the strict compatibility of the l-adic weight filtrations; this strict compatibility can be checked by reducing that to the case where the base field κ is a finite field ([Nak3]).

The idea of considering direct products in (6.13) below is due to T. Saito.

COROLLARY 6.13. *Set* $X := E_{q,C}(\mathcal{L})$ *in* (6.5). *Let* Y *be a proper smooth scheme over* κ *endowed with log structure which is the pull-back of that of* s. *Set* $n := \dim Y$. *Let* $Z := Y \times_s X$ *be the product of* Y *and* X. *If* $H^n_\star(Y) \neq 0$ *or* $H^{n-2}_\star(Y) \neq 0$ $(\star = l,\ p,\ \infty)$, *then* $N_\star : E_{2,\star}^{-1,n+2} \longrightarrow E_{2,\star}^{1n}(-1)$ *and* $N_\star : E_{2,\star}^{-1,n+4} \longrightarrow E_{2,\star}^{1,n+2}(-1)$ *are not isomorphisms. In particular, the monodromy filtrations and the weight filtrations on* $H^{n+1}_{\log,\star}(Z)$ *and* $H^{n+3}_{\log,\star}(Z)$ *do not coincide.*

PROOF. Since $\mathring{X}^{(j)} = \phi$ $(j \geq 3)$, $\mathring{Z}^{(j)} = \phi$ and hence $E_{1,\star}^{-k,h+k} = 0$ for $|k| \geq 2$. The boundary morphism $d_1^{-1,n+2} : E_{1,\star}^{-1,n+2} \longrightarrow E_{1,\star}^{0,n+2}$ is

$$\bigoplus_{h=0}^{2} (H^{n-h}_\star(Y) \otimes H^h_\star(X^{(2)})(-1)) \xrightarrow{(\mathrm{id}\otimes G)^{\oplus 3}} \bigoplus_{h=0}^{4} (H^{n+2-h}_\star(Y) \otimes H^h_\star(X^{(1)})).$$

First, assume that $H^n_\star(Y) \neq 0$. The restriction of $N_\star : E_{2,\star}^{-1,n+2} \longrightarrow E_{2,\star}^{1n}(-1)$ to $H^n_\star(Y) \otimes \operatorname{Ker}(H^0_\star(X^{(2)})(-1) \xrightarrow{G} H^2_\star(X^{(1)}))$ is the zero by (6.5), while the target of this restriction is $H^n_\star(Y) \otimes \operatorname{Coker}(H^0_\star(X^{(1)}) \xrightarrow{\rho} H^0_\star(X^{(2)}))$, which is not 0. In particular, N is not an isomorphism. Next, assume that $H^{n-2}_\star(Y) \neq 0$. In this case we have only to consider the restriction of N to $H^{n-2}_\star(Y) \otimes \operatorname{Ker}(H^2_\star(X^{(2)})(-1) \xrightarrow{G} H^4_\star(X^{(1)}))$. Indeed,

$H^{n-2}_{\star}(Y) \otimes \mathrm{Ker}(H^2_{\star}(X^{(2)})(-1) \xrightarrow{G} H^4_{\star}(X^{(1)})) \simeq H^{n-2}_{\star}(Y) \neq 0$ but $H^{n-2}_{\star}(Y) \otimes$ $\otimes \mathrm{Coker}(H^2_{\star}(X^{(1)}) \xrightarrow{\rho} H^2_{\star}(X^{(2)})) = 0$ by (6.8) (2). The duality (5.15) shows the rest of (6.13). □

REMARK 6.14. In [It2] (cf. [D4, (1.8.4)]), Ito has proved the l-adic monodromy-weight conjecture for an algebraic proper strict semistable family over a complete discrete valuation ring of equal characteristic. Hence, by the proof of (6.7), (6.7) for Z in (6.13) holds if the complete discrete valuation ring in (6.7) is of equal characteristic.

We conclude this section with a remark which is related to the log hard Lefschetz conjecture (9.5) below; (6.5) does not contradict (9.5) by the following:

PROPOSITION 6.15. *Under the assumption of* (6.5), $E_{q,C}(\mathcal{L})$ *is not projective over* κ.

PROOF. We may assume that κ is algebraically closed. We keep the notations in (6.5). However we replace k in (6.5) by $i \in \mathbb{Z}/e$ in this proof.

By [Ha2, V (2.6)], $\pi_i^*(\mathcal{O}_C) \simeq \mathcal{O}_{X_i}(1) \otimes \mathcal{O}_{X_i}(-D_{i,2})$, and thus $(D_{i,2})^2 = |\deg \mathcal{L}| = |d|$ by [loc. cit., V (2.9), (2.11.3)]. Let F_i be a fiber of $X_i \longrightarrow C$. Since $\{C_i, C_{i+1}\} = \{D_{i,1}, D_{i,2}\}$ and $C_i \cap C_{i+1} = \phi$, $D_{i,1} = D_{i,2} - |d|F_i$ in $\mathrm{Num}\, X_i$. Let $\mathcal{N}$ be an invertible sheaf on X. Set $\mathcal{N}_i := \mathcal{N}|_{X_i}$. We take integers a_i and b_i such that $\mathcal{N}_i$ is numerically equivalent to $\mathcal{O}_{X_i}(a_i D_{i,2} + b_i F_i)$. By the patching condition of the $\mathcal{N}_i$'s, we have the following equalities:

$$(a_i D_{i,2} + b_i F_i) \cdot D_{i,2} = (a_{i-1} D_{i-1,2} + b_{i-1} F_{i-1}) \cdot D_{i-1,1} \quad (i \in \mathbb{Z}/e)$$

or

$$(a_i D_{i,2} + b_i F_i) \cdot D_{i,1} = (a_{i-1} D_{i-1,2} + b_{i-1} F_{i-1}) \cdot D_{i-1,2} \quad (i \in \mathbb{Z}/e).$$

Summing up all the equalities above in either case, we obtain $\left(\sum_{i \in \mathbb{Z}/e} a_i\right)|d| = 0$, and hence we have $\sum_{i \in \mathbb{Z}/e} a_i = 0$ since $d \neq 0$. If $\mathcal{N}$ were ample, then $\mathcal{N}_i$ ($\forall i \in \mathbb{Z}/e$) would be so. Hence $a_i = (a_i D_{i,2} + b_i F_i) \cdot F_i > 0$ by the Nakai-Moishezon criterion [loc. cit., V (1.10)]. This contradicts the equation $\sum_{i \in \mathbb{Z}/e} a_i = 0$. Therefore there does not exist an ample invertible sheaf on $\overset{\circ}{X}$. □

7. Integral log $\star$-adic cohomologies of analytic reductions of rigid analytic elliptic surfaces.

This section is a continuation of §6. In this section we study the log $\star$-adic ($\star = l, p, \infty$) cohomologies of the degeneration X in (6.5) of the formal elliptic surface $\mathfrak{X}$ in (6.3) in detail. If one neglects the torsions of the cohomologies above, then the arguments in this section become much simpler. However we consider the torsions because they are interesting (e.g., (7.3) (1), (2) below). In addition, we study the log Hodge(-Witt) cohomologies and the log de Rham cohomologies of the degeneration.

Let X be the proper SNCL surface in (6.5), g the genus of C in (6.5) and $d(\neq 0)$ the degree of $\mathcal{L}$. In this section we use (2.0.7; $\star$) ($\star = l, p, \infty$). Let $P_{\bullet}$ be the weight filtration on $\underline{H}^h_{\log,\star}(X)$ obtained from the weight spectral sequence (2.0.7; $\star$).

THEOREM 7.1. (1) *The integral log $\star$-adic cohomologies of X are as follows*:

(a) $\underline{H}^0_{\log,\star}(X) = \underline{\mathbf{1}}_{\star}$.

(b) $P_0\underline{H}^1_{\log,\star}(X) = \underline{\mathbf{1}}_{\star}$, $\mathrm{gr}^P_1\underline{H}^1_{\log,\star}(X) = \underline{H}^1_{\star}(C)$, $\mathrm{gr}^P_2\underline{H}^1_{\log,\star}(X) = 0$.

(c) $P_1\underline{H}^2_{\log,\star}(X) = \underline{H}^1_{\star}(C)$, $\mathrm{gr}^P_2\underline{H}^2_{\log,\star}(X) = (\underline{\mathbf{1}}_{\star}/d)(-1)$, $\mathrm{gr}^P_3\underline{H}^2_{\log,\star}(X) \subset \subset \underline{H}^1_{\star}(C)(-1)$, $\mathrm{gr}^P_3 H^2_{\log,\star}(X) = H^1_{\star}(C)(-1)$.

(d) $P_2\underline{H}^3_{\log,\star}(X)$: *a quotient of* $(\underline{\mathbf{1}}_{\star}/d)(-1)$, $\mathrm{gr}^P_3\underline{H}^3_{\log,\star}(X) = \underline{H}^1_{\star}(C)(-1)$, $\mathrm{gr}^P_4\underline{H}^3_{\log,\star}(X) = \underline{\mathbf{1}}_{\star}(-2)$.

(e) $\underline{H}^4_{\log,\star}(X) = \underline{\mathbf{1}}_{\star}(-2)$.

(2) *Assume that κ is a perfect field of characteristic $p>0$. As F-isocrystals, there exists the following isomorphisms*: $H^1_{\text{log-crys}}(X/W)\otimes_W K_0 \simeq K_0 \oplus \oplus (H^1_{\text{crys}}(C/W)\otimes_W K_0)$ *(for our memory)*, $H^2_{\text{log-crys}}(X/W)\otimes_W K_0 \simeq \simeq (H^1_{\text{crys}}(C/W)\otimes_W K_0) \oplus (H^1_{\text{crys}}(C/W)(-1)\otimes_W K_0)$, $H^3_{\text{log-crys}}(X/W)\otimes_W K_0 \simeq \simeq (H^1_{\text{crys}}(C/W)(-1)\otimes_W K_0) \oplus K_0(-2)$.

PROOF. We keep the notations of (6.5) and (6.15). For simplicity we assume that κ is algebraically closed.

(1): The assertion about $h = 0, 4$ is obvious by (2.0.7; $\star$). Let $E^{\bullet\bullet}_{r,\star}$ ($r \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$) be the E_r-terms of the weight spectral sequence (2.0.7; $\star$) for X. Since the dual graph of $\overset{\circ}{X}$ is a circle, $E^{-14}_{2,\star} \simeq \underline{\mathbf{1}}_{\star}(-2)$ and $E^{10}_{2,\star} \simeq \underline{\mathbf{1}}_{\star}$.

Let $\iota_{j,i}\colon C_i \overset{\subset}{\longrightarrow} X_j$ ($j = i, i+1$) be the closed immersion from a double curve C_i to irreducible components of $\overset{\circ}{X}$. By the construction of X, we obtain $\pi_i \circ \iota_{i,i} = \pi_{i+1} \circ \iota_{i+1,i}$. Hence the following diagram

$$(7.1.1)\qquad \begin{array}{ccc} \underline{H}^1_\star(C_i) & \xleftarrow{\iota^*_{i+1,i}} & \underline{H}^1_\star(X_{i+1}) \\ {\scriptstyle \iota^*_{i,i}}\uparrow & & \uparrow{\scriptstyle \pi^*_{i+1}} \\ \underline{H}^1_\star(X_i) & \xleftarrow{\pi^*_i} & \underline{H}^1_\star(C) \end{array}$$

is commutative. We identify $\underline{H}^1_\star(X_i)$ (resp. $\underline{H}^1_\star(C_i)$) with $\underline{H}^1_\star(C)$ by using $(\pi^*_i)^{-1}$ (resp. $(\iota^*_{i,i} \circ \pi^*_i)^{-1}$. The boundary morphism $d^{01}_{1,\star}$ is the following morphism

$$d^{01}_{1,\star}\colon \bigoplus_{i\in\mathbb{Z}/e} \underline{H}^1_\star(X_i) \ni (a_i)_{i\in\mathbb{Z}/e} \longmapsto (\iota^*_{i+1,i}(a_{i+1}) - \iota^*_{i,i}(a_i))_{i\in\mathbb{Z}/e} \in \bigoplus_{i\in\mathbb{Z}/e} \underline{H}^1_\star(C_i).$$

By using the identifications of $\underline{H}^1_\star(X_i)$ and $\underline{H}^1_\star(C_i)$ with $\underline{H}^1_\star(C)$ and by using the commutativity of (7.1.1), it is easy to see that $d^{01}_{1,\star}$ is identified with the following morphism

$$\bigoplus_{\mathbb{Z}/e} \underline{H}^1_\star(C) \ni (a_i)_{i\in\mathbb{Z}/e} \longmapsto (a_{i+1} - a_i)_{i\in\mathbb{Z}/e} \in \bigoplus_{\mathbb{Z}/e} \underline{H}^1_\star(C).$$

Thus $E^{01}_{2,\star} = E^{11}_{2,\star} = \underline{H}^1_\star(C)$. By the proof of the duality in (5.15), we have $E^{03}_{2,\star} = E^{-13}_{2,\star} = \underline{H}^1_\star(C)(-1)$ because C is a proper smooth curve over κ.

Next, we determine $E^{12}_{2,\star}$. Since $\underline{H}^*_\star(Z) = \underline{H}^*_\star(C) \oplus \underline{H}^{*-2}_\star(C)\mathcal{O}_Z(1)$ for a relatively minimal ruled surface $Z \longrightarrow C$, we immediately see that $\operatorname{Num} Z \otimes_{\mathbb{Z}} \underline{\mathbf{1}}_\star = \underline{H}^2_\star(Z)(1)$. Hence the boundary morphism $d^{02}_{1,\star}(1)$ is a morphism from $\bigoplus_{i\in\mathbb{Z}/e} \operatorname{Num} X_i \otimes_{\mathbb{Z}} \underline{\mathbf{1}}_\star$ to $\bigoplus_{i\in\mathbb{Z}/e} \underline{H}^2_\star(C_i)(1)$. The restriction of $d^{02}_{1,\star}(1)$ to $\operatorname{Num} X_i \otimes_{\mathbb{Z}} \underline{\mathbf{1}}_\star$ is the following morphism

$$\operatorname{Num} X_i \otimes_{\mathbb{Z}} \underline{\mathbf{1}}_\star \ni D \longmapsto (-D\cdot C_i, D\cdot C_{i-1}) \in \underline{H}^2_\star(C_i)(1) \oplus \underline{H}^2_\star(C_{i-1})(1).$$

As shown in the proof of (6.15), $(D_{i,2})^2 = |\deg \mathcal{L}| = |d|$, $D_{i,1} = D_{i,2} - |d|F_i$ in $\operatorname{Num} X_i$ and $\{C_i, C_{i+1}\} = \{D_{i,1}, D_{i,2}\}$. The image of the restriction of $d^{02}_{1,\star}(1)$ to $\operatorname{Num} X_i \otimes_{\mathbb{Z}} \underline{\mathbf{1}}_\star$ is $\underline{\mathbf{1}}_\star(1,-1) \oplus \underline{\mathbf{1}}_\star(|d|, 0)$ as $\underline{\mathbf{1}}_\star$-modules up to the permutation. By (7.2) below we see that $E^{12}_{2,\star} \simeq \underline{\mathbf{1}}_\star/d(-1)$. (We may assume that $C_i = D_{i,2}$ on X_i, and we apply (7.2) by setting $v_i = F_i$, $w_i = C_i$, $H^{-1} = \underline{H}^0(X^{(2)})(-1)$, $H^0 = \bigoplus_{i\in\mathbb{Z}/e} \operatorname{Num} X_i \otimes_{\mathbb{Z}} \underline{\mathbf{1}}_\star$, $H^1 = \underline{H}^2(X^{(2)})$, and $\alpha = |d|$ (cf. (6.5.2)).) Again, by the lemma (7.2) below, we see that $E^{-12}_{2,\star} = 0$ and $E^{02}_{2,\star} = \underline{\mathbf{1}}_\star/d(-1)$. Thus we have proved the claim on $\underline{H}^h_{\log,\star}(X)$ $(h = 0,1,2,3,4)$.

(2): By the determination of $\operatorname{gr}^P_i H^2_{\text{log-crys}}(X/W)$ $(i \in \mathbb{N})$, it is easy to see that the following sequence is exact:

$$0 \longrightarrow P_2 H^2_{\text{log-crys}}(X/W)/\text{torsion} \longrightarrow H^2_{\text{log-crys}}(X/W)/\text{torsion} \longrightarrow \mathrm{gr}^P_3 H^2_{\text{log-crys}}(X/W) \longrightarrow 0.$$

By [Kz, (1.3.4)] we have

$$H^2_{\text{log-crys}}(X/W) \otimes_W K_0 \simeq (P_1 H^2_{\text{log-crys}}(X/W) \otimes_W K_0) \oplus (\mathrm{gr}^P_3 H^2_{\text{log-crys}}(X/W) \otimes_W K_0)$$

since $P_1 H^2_{\text{log-crys}}(X/W) \otimes_W K_0 = P_2 H^2_{\text{log-crys}}(X/W) \otimes_W K_0$. Hence (2) for $H^2_{\text{log-crys}}(X/W) \otimes_W K_0$ follows; (2) for $H^3_{\text{log-crys}}(X/W) \otimes_W K_0$ also follows as above. □

LEMMA 7.2. *Let m be a positive integer greater than* 1. *Let R be a commutative ring with unit element. Let α be an element of R. Assume that α is not a zero-divisor of R. Let $H^{-1} := \bigoplus_{j \in \mathbb{Z}/m} Re_j$, $H^0 := \bigoplus_{j \in \mathbb{Z}/m} (Rv_j \oplus Rw_j)$ and $H^1 := \bigoplus_{j \in \mathbb{Z}/m} Rf_j$ be free R-modules of rank m, $2m$ and m, respectively. Let $G\colon H^{-1} \longrightarrow H^0$ (resp. $F\colon H^0 \longrightarrow H^1$) be a morphism of R-modules defined by $e_j \longmapsto (w_{j+1} - \alpha v_{j+1}) - w_j$ (resp. $v_j \longmapsto -(f_j - f_{j-1})$ and $w_j \longmapsto -\alpha f_j$). Then G is injective, $\operatorname{Ker} F/\operatorname{Im} G \simeq R/\alpha$ and $\operatorname{Coker} F \simeq R/\alpha$. The image of $\sum_{j \in \mathbb{Z}/m} v_j$ in $\operatorname{Ker} F/\operatorname{Im} G$ (resp. f_{m-1} in $\operatorname{Coker} F$) is a generator of $\operatorname{Ker} F/\operatorname{Im} G$ (resp. $\operatorname{Coker} F$).*

PROOF. First, we find a basis of $\operatorname{Ker} F$. Set $x_j := \Sigma_{i=0}^{j} v_i$ $(0 \leq j \leq m-1)$. Let $A \in M_{m,2m}(R)$ be a matrix defined by the following equality:

(7.2.1) $$F(x_0, x_1, \ldots, x_{m-1}, w_0, w_1, \ldots, w_{m-1}) = (f_0, f_1 \ldots, f_{m-1})A.$$

Then $A = (B \ (-\alpha)E_m)$, where

$$B := \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \in M_m(R).$$

Hence

(7.2.2) $$F(x_0, \ldots, x_{m-1}, w_0, \ldots, w_{m-1}) = (z_0, \ldots, z_{m-1})A',$$

where

$$(7.2.3)\qquad A' := \begin{bmatrix} -1 & 0 & \cdots & 0 & 0 & -\alpha & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 & 0 & -\alpha & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots & -\alpha & 0 \\ 0 & 0 & \cdots & 0 & 0 & -\alpha & -\alpha & \cdots & -\alpha & -\alpha \end{bmatrix},$$

$z_j := f_j - f_{m-1}$ $(0 \leq j \leq m-2)$ and $z_{m-1} := f_{m-1}$. Multiplying (7.2.2) by

$$\left[\begin{array}{c|c} E_m & -\alpha E_m \\ \hline O & E_m \end{array}\right]$$

from the right side, we obtain

$$(7.2.4)\qquad F(x_0, \ldots, x_{m-1}, y_0, \ldots, y_{m-1}) = (z_0, \ldots, z_{m-1})(C\ D),$$

where

$$C := \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},\quad D := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ -\alpha & -\alpha & -\alpha & \cdots & -\alpha & -\alpha \end{bmatrix}$$

and $y_j := -\alpha x_j + w_j$ $(0 \leq j \leq m-1)$. By (7.2.4) the following equation

$$(7.2.5)\qquad \sum_{j=0}^{m-1} \alpha_j F(x_j) + \sum_{j=0}^{m-1} \beta_j F(y_j) = 0 \quad (\alpha_j, \beta_j \in R)$$

is equivalent to $\alpha_j = 0$ $(0 \leq j \leq m-2)$ and $\beta_{m-1} = -\Sigma_{j=0}^{m-2}\beta_j$ since α is not a zero-divisor of R. Hence $\{x_{m-1}, \{y_j - y_{m-1}\}_{j=0}^{m-2}\} = \{\Sigma_{j=0}^{m-1} v_j, \{y_j - y_{m-1}\}_{j=0}^{m-2}\}$ is a basis of $\mathrm{Ker}\, F$.

Next, we give a matrix expression of G. It is easy to see that

$$(7.2.6)\quad G(e_0, e_1, \ldots, e_{m-1}) = (y_0 - y_{m-1}, y_1 - y_{m-1}, \ldots, y_{m-2} - y_{m-1}, x_{m-1})E,$$

where

$$E := \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha \end{bmatrix}.$$

By taking other bases of H^{-1} and H^0, E is transformed into

$$E' := \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha \end{bmatrix}. \tag{7.2.7}$$

Now the injectivity of G is obvious since α is not a zero-divisor of R and we see that the morphism $R/\alpha \ni r \longmapsto rx_{m-1} \in \operatorname{Ker} F/\operatorname{Im} G$ is an isomorphism.

By (7.2.3) it is easy to see that the morphism $R/\alpha \ni r \longmapsto rf_{m-1} \in \operatorname{Coker} F$ is an isomorphism. □

REMARK 7.3. (1) We do not know whether the morphism $d_2^{-13}\colon E_{2,\star}^{-13} \longrightarrow E_{2,\star}^{12}$ ($\star = l, p, \infty$) for $X = E_{q,C}(\mathcal{L})$ is the zero. We conjecture that this is the zero.

(2) If C is a proper smooth rational curve, then the components of X and the double curves are rational. In this case $E_{1,\star}^{ij} = 0$ ($j = 1, 3,\ i = 0, \pm 1$). Thus $P_2\underline{H}^3_{\log,\star}(X) = \underline{1}_\star/d(-1)$ by the proof of (7.1) and (7.2).

COROLLARY 7.4. *Let κ be the finite field $\mathbb{F}_q$ with q-elements. Let*

$$Z(H^h_{\log,\star}(X), t) = \det(1 - t\Phi^*|H^h_{\log,\star}(X)^{N=0}) \quad (\star = l, p)$$

be the h-th zeta function of X, where Φ is the q-th power Frobenius of X. Then

$$Z(H^0_{\log,\star}(X), t) = 1 - t,$$

$$Z(H^1_{\log,\star}(X), t) = (1 - t)\det(1 - t\Phi^*|H^1(C)),$$

$$Z(H^2_{\log,\star}(X), t) = \det(1 - t\Phi^*|H^1(C))\det(1 - qt\Phi^*|H^1(C)),$$

$$Z(H^3_{\log,\star}(X), t) = (1 - q^2t)\det(1 - qt\Phi^*|H^1(C)),$$

$$Z(H^4_{\log,\star}(X), t) = 1 - q^2t.$$

PROOF. (7.4) follows from (7.1). □

Let Y/s be a proper SNCL variety. As in [Il1, II (3.1.1)], we have the following slope spectral sequence:

$$E_1^{i,h-i} = H^{h-i}(Y, W\Lambda^i_Y) \Longrightarrow H^h_{\text{log-crys}}(Y/W).$$

We have the slope filtration F on $H^h_{\text{log-crys}}(Y/W)$:

$$F^iH^h_{\text{log-crys}}(Y/W) := \mathrm{Im}(H^h(Y, W\Lambda_Y^{\bullet\geq i}) \longrightarrow H^h_{\text{log-crys}}(Y/W)) \quad (i \in \mathbb{Z}).$$

Let n be a positive integer. In [Nakk3, (4.1)] we have constructed the following spectral sequence of the log Hodge-Witt cohomologies of Y/s:

$$(7.4.1; p; n) \quad E_1^{-k,h+k}(i;p)_n =$$

$$\bigoplus_{j\geq\max\{-k,0\}} H^{h-i-j}(\overset{\circ}{Y}{}^{(2j+k+1)}, W_n\Omega^{i-j-k}_{\overset{\circ}{Y}{}^{(2j+k+1)}})(-j-k) \Longrightarrow H^{h-i}(Y, W_n\Lambda^i_Y).$$

We give a filtration P on $H^{h-i}(Y, W_n\Lambda^i_Y)$ such that $\mathrm{gr}^P_{h+k}H^{h-i}(Y, W_n\Lambda^i_Y) = E_\infty^{-k,h+k}(i;p)_n$. By taking the projective limit of $(7.4.1; p; n)$ with respect to projections ([Nakk3, (8.6) (2)]), we obtain the following weight spectral sequence ([loc. cit. (4.1)])

$$(7.4.1; p) \quad E_1^{-k,h+k}(i;p) =$$

$$\bigoplus_{j\geq\max\{-k,0\}} H^{h-i-j}(\overset{\circ}{Y}{}^{(2j+k+1)}, W\Omega^{i-j-k}_{\overset{\circ}{Y}{}^{(2j+k+1)}})(-j-k) \Longrightarrow H^{h-i}(Y, W\Lambda^i_Y).$$

We also obtain a filtration P on $H^{h-i}(Y, W\Lambda^i_Y)$, which is called the weight filtration on $H^{h-i}(Y, W\Lambda^i_Y)$.

THEOREM 7.5. *Let κ be a perfect field of characteristic $p > 0$.*

(1) *The graded pieces of the slope filtration on $H^*_{\text{log-crys}}(X/W)$ are as follows*:

(a) $H^0(X, W\mathcal{O}_X) = W$, $P_0H^1(X, W\mathcal{O}_X) = W$, $\mathrm{gr}^P_1H^1(X, W\mathcal{O}_X) = H^1(C, W\mathcal{O}_C)$, $H^2(X, W\mathcal{O}_X) = H^1(C, W\mathcal{O}_C)$.

(b) $H^0(X, W\Lambda^1_X) = H^0(C, W\Omega^1_C)$, $P_1H^1(X, W\Lambda^1_X) = H^0(C, W\Omega^1_C)$, $\mathrm{gr}^P_2H^1(X, W\Lambda^1_X) = (W/d)(-1)$, $\mathrm{gr}^P_3H^1(X, W\Lambda^1_X) \subset H^1(C, W\mathcal{O}_C)(-1)$, $\mathrm{gr}^P_3H^1(X, W\Lambda^1_X) \otimes_W K_0 = H^1(C, W\mathcal{O}_C)(-1) \otimes_W K_0$, $P_2H^2(X, W\Lambda^1_X)$: *a quotient of* $(W/d)(-1)$, $\mathrm{gr}^P_3H^2(X, W\Lambda^1_X) = H^1(C, W\mathcal{O}_C)(-1)$.

(c) $H^0(X, W\Lambda^2_X) = H^0(C, W\Omega^1_C)(-1)$, $P_3H^1(X, W\Lambda^2_X) = H^0(C, W\Omega^1_C)(-1)$, $\mathrm{gr}^P_4H^1(X, W\Lambda^2_X) = W(-2)$, $H^2(X, W\Lambda^2_X) = W(-2)$.

(2) $\widehat{\mathrm{Pic}}\, X = \widehat{\mathbb{G}}_m \times \widehat{\mathrm{Jac}}(C)$, $\widehat{\mathrm{Br}}\, X = \widehat{\mathrm{Jac}}(C)$.

(3) *The slope spectral sequence of X*

$$E_1^{ij} = H^j(X, W\Lambda^i_X) \Longrightarrow H^{i+j}_{\text{log-crys}}(X/W)$$

degenerates at E_1.

PROOF. (1): (1) follows from $(7.4.1;p)$ and the same argument of (7.1). For example, the isomorphism $\mathrm{gr}_2^P H^1(X, W\Lambda_X^1) \xrightarrow{\sim} (W/d)(-1)$ is obtained by $(7.4.1;p)$ and (7.2.7). We leave the detail to the reader.

(2): (2) follows from (1) (a).

(3): By (1), $H^2(X, W\mathcal{O}_X)$ is a finitely generated W-module. Since $\dim \overset{\circ}{X} = 2$, the slope spectral sequence degenerates at E_1 by the obvious log version of [Il1, II (3.14)]. □

COROLLARY 7.6. *Let κ be a field of characteristic $p \geq 0$. Then the following hold*:

(1) $H^h_{\text{log-dR}}(X/\kappa) \simeq \kappa \quad (h = 0, 4)$.

(2) *Assume that p divides d. Then*

(a) $2g + 1 \leq \dim_\kappa H^1_{\text{log-dR}}(X/\kappa) = \dim_\kappa H^3_{\text{log-dR}}(X/\kappa) \leq 2g + 2$.

(b) $4g \leq \dim_\kappa H^2_{\text{log-dR}}(X/\kappa) \leq 4g + 2$.

(3) *Assume that p does not divide d. Then*

(a) $H^1_{\text{log-dR}}(X/\kappa) \simeq \kappa \oplus \kappa^{2g} \simeq H^3_{\text{log-dR}}(X/\kappa)$.

(b) $H^2_{\text{log-dR}}(X/\kappa) \simeq \kappa^{2g} \oplus \kappa^{2g}$.

(4) *The log Hodge-de Rham spectral sequence of X/s*

$$E_1^{ij} = H^j(X, \Lambda^i_{X/\kappa}) \Longrightarrow H^{i+j}_{\text{log-dR}}(X/\kappa)$$

degenerates at E_1.

(5) *If p divides d (resp. p does not divide d), then the log Hodge numbers $h^{ij} = \dim_\kappa H^j(X, \Lambda^i_{X/\kappa})$ are as follows*:

$$\begin{matrix} g & \alpha & 1 \\ g+1 & 2\alpha & g+1 \\ 1 & \alpha & g \end{matrix} \qquad \left(\textit{resp.} \qquad \begin{matrix} g & g & 1 \\ g+1 & 2g & g+1 \\ 1 & g & g \end{matrix} \right),$$

where $\alpha := g$ or $g + 1$.

PROOF. We give the proof only for the case $p > 0$ except (4); the proof for the case $p = 0$ is similar.

Consider the following spectral sequence (cf. [Mo, 3.23])

$$(7.6.1) \qquad E_1^{-k,h+k} = \bigoplus_{j \geq \max\{-k,0\}} H^{h-2j-k}_{\text{dR}}(\overset{\circ}{X}{}^{(2j+k+1)}/\kappa) \Longrightarrow H^h_{\text{log-dR}}(X/\kappa).$$

(1): (1) is obvious by (7.6.1) or the log Serre duality of Tsuji ([Ts2, (2.21)]).

(2): Assume that p divides d. Then, as in the proof of (7.1), we see that $E_2^{12} \simeq \kappa$; by the duality [Nakk3, (10.5)] (or the direct computation as in (7.1)), $E_2^{-12} \simeq \kappa$. Consider the following sequence: $E_1^{-12} \longrightarrow E_1^{02} \longrightarrow E_1^{12}$. Because

$\dim_\kappa E_1^{02} = \dim_\kappa E_1^{12} + \dim_\kappa E_1^{-12}$, $\dim_\kappa E_2^{02} = \dim_\kappa E_2^{12} + \dim_\kappa E_2^{-12} = 2$. By the proof of (7.1), we obtain $E_2^{01} \simeq \kappa^{2g} \simeq E_2^{11}$ and $E_2^{03} \simeq \kappa^{2g} \simeq E_2^{-13}$. Therefore we obtain (2) by the log Serre duality of Tsuji ([Ts2, (2.21)]).

(3): Assume that p does not divide d. Then we see that $E_2^{12} = 0$ by the proof of (7.1). By the same proof as that of (2), we see that $E_2^{-12} = 0 = E_2^{02}$. The rest of the proof is similar to that of (7.1).

(4): Assume that $p > 0$. We may assume that κ is perfect. By (7.5), $H^2(X, W\mathcal{O}_X)$ is finitely generated. Hence (4) follows from the obvious log version of [Ny, (2.7)] or [Il1, II (5.17)].

Next, we assume that $p = 0$. Then we obtain (4) by [Kk1, (4.12)] and the proof of [DI2, (2.7)].

(5): (5) follows from $(7.4.1; p; n)$ for $n = 1$ as in (7.1). □

REMARK 7.7. We do not know whether $\alpha = g + 1$ in (7.6) (5).

The following is the ∞-adic analogue of (7.5).

PROPOSITION 7.8. *The log Hodge cohomologies of X with Hodge filtrations and weight filtrations are as follows*:

(a) $H^0(X, \mathcal{O}_X) = \mathbb{C}, H^1(X, \mathcal{O}_X) = \mathbb{C} \oplus H^1(C, \mathcal{O}_C), H^2(X, \mathcal{O}_X) = H^1(C, \mathcal{O}_C)$.

(b) $H^0(X, \Lambda^1_{X/\mathbb{C}}) = H^0(C, \Omega^1_{C/\mathbb{C}})$, $P_1 H^1(X, \Lambda^1_{X/\mathbb{C}}) = P_2 H^1(X, \Lambda^1_{X/\mathbb{C}}) = H^0(C, \Omega^1_{C/\mathbb{C}})$, $\mathrm{gr}_3^P H^1(X, \Lambda^1_{X/\mathbb{C}}) = H^1(C, \mathcal{O}_C)(-1)$, $H^2(X, \Lambda^1_{X/\mathbb{C}}) = H^1(C, \mathcal{O}_C)(-1)$.

(c) $H^0(X, \Lambda^2_{X/\mathbb{C}}) = H^0(C, \Omega^1_{C/\mathbb{C}})(-1)$, $P_3 H^1(X, \Lambda^2_{X/\mathbb{C}}) = H^0(C, \Omega^1_{C/\mathbb{C}})(-1)$, $\mathrm{gr}_4^P H^1(X, \Lambda^2_{X/\mathbb{C}}) = \mathbb{C}(-2)$, $H^2(X, \Lambda^2_{X/\mathbb{C}}) = \mathbb{C}(-2)$.

PROOF. In [Nakk3, (4.8)], we have constructed the following weight spectral sequence of the log Hodge cohomologies of a proper SNCL variety $Y/s = (\mathrm{Spec}\, \mathbb{C}, \mathcal{M}_s)$:

$(7.8.1; \infty)$ $E_1^{-k, h+k}(i; \infty) =$

$$\bigoplus_{j \geq \max\{-k, 0\}} H^{h-i-j}(\overset{\circ}{Y}{}^{(2j+k+1)}, \Omega^{i-j-k}_{\overset{\circ}{Y}{}^{(2j+k+1)}/\mathbb{C}})(-j-k) \Longrightarrow H^{h-i}(Y, \Lambda^i_{Y/\mathbb{C}}).$$

(7.8) follows from $(7.8.1; \infty)$. □

Assume that κ is a perfect field of characteristic $p > 0$. Let V be a complete discrete valuation ring of mixed characteristics with residue field κ and fraction field K.

THEOREM 7.9. *Let $\mathfrak{Y}$ be a formal proper strict semistable family over* Spf V *with canonical log structure in* (6.2). *Let Y be the special fiber of $\mathfrak{Y}$. Let $\mathfrak{Y}_{\mathrm{rig}}$ be the Raynaud generic fiber of $\mathfrak{Y}$* {([R1], cf. [BL])}. *Then there exists a canonical isomorphism*

$$H^h_{\text{log-crys}}(Y/W) \otimes_W K \xrightarrow{\sim} H^h_{\mathrm{dR}}(\mathfrak{Y}_{\mathrm{rig}}/K) \quad (h \in \mathbb{N}). \tag{7.9.1}$$

PROOF. Let $\mathcal{U} := \{\mathfrak{U}_i\}_{i\in I}$ be an affine open covering of $\mathfrak{Y}$. Then $\mathcal{U}$ is a Leray covering of $\mathfrak{Y}$ for a quasi-coherent $\mathcal{O}_{\mathfrak{Y}}$-module. Hence $H^h_{\mathrm{dR}}(\mathfrak{Y}/V)$ is calculated by the Čech cohomology of $\mathcal{U}$:

$$H^h_{\mathrm{dR}}(\mathfrak{Y}/V) = H^h(s(\oplus_{ij}\, C^i(\mathfrak{U}, \Omega^j_{\mathfrak{Y}/V}))). \tag{7.9.2}$$

Here we define the signs of the boundary morphisms of the double complex $\oplus_{ij} C^i(\mathfrak{U}, \Omega^j_{\mathfrak{Y}/V})$ as in (3.2.7). Set $\mathcal{U}_{\mathrm{rig}} := \{(\mathfrak{U}_i)_{\mathrm{rig}}\}_{i\in I}$. Then, by Tate's acyclicity ([Ta, Theorem 8.2, Theorem 8.7], [BGR, 8.2, Corollary 5]), $\mathcal{U}_{\mathrm{rig}}$ is a Leray covering of $\mathfrak{Y}_{\mathrm{rig}}$ for a quasi-coherent $\mathcal{O}_{\mathfrak{Y}_{\mathrm{rig}}}$-module. Hence

$$H^h_{\mathrm{dR}}(\mathfrak{Y}_{\mathrm{rig}}/K) = H^h(s(\oplus_{ij}\, C^i(\mathfrak{U}, \Omega^j_{\mathfrak{Y}_{\mathrm{rig}}/K}))). \tag{7.9.3}$$

Therefore we obtain a canonical isomorphism

$$H^h_{\mathrm{dR}}(\mathfrak{Y}/V) \otimes_V K \xrightarrow{\sim} H^h_{\mathrm{dR}}(\mathfrak{Y}_{\mathrm{rig}}/K). \tag{7.9.4}$$

(In fact, there is the following equivalence of categories of coherent modules for a V-adic formal scheme $\mathfrak{Z}$:

$$\mathrm{Coh}(\mathcal{O}_{\mathfrak{Z}} \otimes_V K) \xrightarrow{\sim} \mathrm{Coh}(\mathcal{O}_{\mathfrak{Z}_{\mathrm{rig}}}).$$

Indeed, the essential surjectivity has been proved in [BL, (5.6), (5.7)]. Because the full faithfulness is a local problem, we have only to prove

$$\mathrm{Hom}_{\mathcal{O}_{\mathfrak{Z}}\otimes_V K}((\mathcal{O}_{\mathfrak{Z}} \otimes_V K)^n, (\mathcal{O}_{\mathfrak{Z}} \otimes_V K)^m) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{\mathfrak{Z}_{\mathrm{rig}}}}((\mathcal{O}_{\mathfrak{Z}_{\mathrm{rig}}})^n, (\mathcal{O}_{\mathfrak{Z}_{\mathrm{rig}}})^m) \quad (m, n \in \mathbb{N}),$$

which is clear by the definition of $\mathfrak{Z}_{\mathrm{rig}}$.)

By the proof of [HK, (5.1)] for which some results in [Nakk3, §7] are necessary (see [Nakk3, §7] for details), we obtain a canonical isomorphism

$$H^h_{\text{log-crys}}(Y/W) \otimes_W K \xrightarrow{\sim} H^h_{\mathrm{dR}}(\mathfrak{Y}/V) \otimes_V K. \tag{7.9.5}$$

By (7.9.4) and (7.9.5) we have a canonical isomorphism (7.9.1). □

Let $\mathfrak{X}$ be a formal proper strict semistable family over Spf V with special fiber X ((6.3)). Let $\mathfrak{X}_{\mathrm{rig}}$ be the Raynaud generic fiber of $\mathfrak{X}$ ([R1], cf. [BL]). Then the following hold:

COROLLARY 7.10. *$\dim_K H^h_{\mathrm{dR}}(\mathfrak{X}_{\mathrm{rig}}/K) = 1,\ 2g+1,\ 4g,\ 2g+1$ and 1 for $h = 0,\ 1,\ 2,\ 3,\ 4$, respectively.*

PROOF. (7.10) immediately follows from (7.1) and (7.9). □

Let $h^{ij}_{\mathrm{rig}} := \dim_K H^j(\mathfrak{X}_{\mathrm{rig}}, \Omega^i_{\mathfrak{X}_{\mathrm{rig}}/K})$ be the Hodge numbers of $\mathfrak{X}_{\mathrm{rig}}/K$. In [Ue, (6.1) 3)], Ueno has proved that $h^{10}_{\mathrm{rig}} = g$ and $h^{01}_{\mathrm{rig}} = g+1$. We determine the h^{ij}_{rig}'s:

THEOREM 7.11. *The Hodge numbers of $\mathfrak{X}_{\mathrm{rig}}$ are as follows:*

$$\begin{matrix} g & g & 1 \\ g+1 & 2g & g+1 \\ 1 & g & g. \end{matrix}$$

PROOF. Let $\mathfrak{C}/\mathrm{Spf}\, W$ be a formal lift of C which was used for the construction $\mathfrak{X}$. In [Ue, (6.1) 3)], Ueno has proved that $h^{10}_{\mathrm{rig}} = g$ and $h^{01}_{\mathrm{rig}} = g+1$. By the Serre duality ([V, (5.1)], cf. [Ue, p. 773]), we have $h^{12}_{\mathrm{rig}} = g$ and $h^{21}_{\mathrm{rig}} = g+1$.

Let us prove $h^{02}_{\mathrm{rig}} = g$. By the Leray spectral sequence

$$E_2^{st} = H^s(\mathfrak{C}_{\mathrm{rig}}, R^t f_*(\mathcal{O}_{\mathfrak{X}_{\mathrm{rig}}})) \Longrightarrow H^{s+t}(\mathfrak{X}_{\mathrm{rig}}, \mathcal{O}_{\mathfrak{X}_{\mathrm{rig}}})$$

of the elliptic fibration $f\colon \mathfrak{X}_{\mathrm{rig}} \longrightarrow \mathfrak{C}_{\mathrm{rig}}$ and by the fact $R^1 f_*(\mathcal{O}_{\mathfrak{X}_{\mathrm{rig}}}) = \mathcal{O}_{\mathfrak{C}_{\mathrm{rig}}}$ ([Ue, p. 790]), we have $H^2(\mathfrak{X}_{\mathrm{rig}}, \mathcal{O}_{\mathfrak{X}_{\mathrm{rig}}}) = H^1(\mathfrak{C}_{\mathrm{rig}}, R^1 f_*(\mathcal{O}_{\mathfrak{X}_{\mathrm{rig}}})) = H^1(\mathfrak{C}_{\mathrm{rig}}, \mathcal{O}_{\mathfrak{C}_{\mathrm{rig}}})$. Hence $h^{02}_{\mathrm{rig}} = g$; by the Serre duality, we have $h^{20}_{\mathrm{rig}} = g$.

By the Hodge-de Rham spectral sequence

$$E_1^{ij} = H^j(\mathfrak{X}_{\mathrm{rig}}, \Omega^i_{\mathfrak{X}_{\mathrm{rig}}/K}) \Longrightarrow H^{i+j}_{\mathrm{dR}}(\mathfrak{X}_{\mathrm{rig}}/K) \tag{7.11.1}$$

and by (7.9.1), we have $\sum_{i,j} (-1)^{i+j} h^{ij}_{\mathrm{rig}} = \sum_h (-1)^h \dim_{K_0}(H^h_{\text{log-crys}}(X/W) \otimes_W K_0)$. Hence we have $h^{11}_{\mathrm{rig}} = 2g$ by (7.1). □

COROLLARY 7.12. *The Hodge-de Rham spectral sequence* (7.11.1) *degenerates at E_1.*

PROOF. (7.12) immediately follows from (7.10) and (7.11). □

REMARK 7.13. In [Ue, (6.1) 3)], Ueno has proved that the boundary morphisms

$$d\colon H^0(\mathfrak{X}_{\mathrm{rig}}, \Omega^1_{\mathfrak{X}_{\mathrm{rig}}/K}) \longrightarrow H^0(\mathfrak{X}_{\mathrm{rig}}, \Omega^2_{\mathfrak{X}_{\mathrm{rig}}/K})$$

and

$$d\colon H^1(\mathfrak{X}_{\rm rig},\mathcal{O}_{\mathfrak{X}_{\rm rig}})\longrightarrow H^1(\mathfrak{X}_{\rm rig},\Omega^1_{\mathfrak{X}_{\rm rig}/K})$$

vanish. As a corollary, he has proved that $\dim_K H^1_{\rm dR}(\mathfrak{X}_{\rm rig}/K)=2g+1$. Our proof for this fact in (7.10) is different from his and is interesting in the following point: we have used the degeneration X and the log geometry for the calculation of the de Rham cohomology of the Raynaud generic fiber of $\mathfrak{X}$ over V.

Conversely, we can determine $E_{2,p}^{-12}$ and $E_{2,p}^{01}$ by using [Ue, (6.1) 3)] and (7.9.1). Indeed, it is easy to see that $\dim_{K_0}(E_{2,p}^{01}\otimes_W K_0)\leq 2g$ and $\dim_{K_0}(E_{2,p}^{-12}\otimes_W K_0)\leq 1$ (cf. the proof of (6.5)). We also see that $\dim_{K_0}(E_{2,p}^{01}\otimes_W K_0)$ is an even integer. Since $\dim_{K_0}(E_{2,p}^{10}\otimes_W K_0)=1$, $\dim_{K_0}(E_{2,p}^{01}\otimes_W K_0)=2g$ and $E_{2,p}^{-1,2}\otimes_W K_0=0$ by (7.9.1) and [Ue, (6.1) 3)]; in fact, $E_{2,p}^{01}\simeq W^{2g}$ and $E_{2,p}^{-12}=0$. (We can also determine $E_{2,l}^{01}$ and $E_{2,l}^{-12}$ by (7.9.1) and [Ue, (6.1) 3)] because the boundary morphisms $d_1^{01}\colon E_{1,\star}^{01}\longrightarrow E_{1,\star}^{11}$ and $d_1^{-12}\colon E_{1,\star}^{-12}\longrightarrow E_{1,\star}^{02}$ are motivic ($\star=l,p$) (cf. the proof of (8.3) below).)

8. Algebraic proper strict semistable families and the monodromy-weight conjectures.

For the completeness of this paper, we give the proof of (8.1) below:

THEOREM 8.1. *Let $\mathcal{X}$ be an algebraic proper strict semistable family of relative pure dimension d over a complete discrete valuation ring V with residue field κ of characteristic $p>0$. Let X/s be the special fiber of $\mathcal{X}$ with canonical log structure. Then the monodromy filtration and the weight filtration on $H^h_{\text{log-et}}(X_{\overline{s}},\mathbb{Q}_l)$ coincide for $h=1,2d-1$.*

PROOF. (cf. [RZ, (2.13)]) We may assume that κ is perfect. Let $E_{2,l}^{\bullet\bullet}$ be the E_2-term of the weight spectral sequence (2.0.8; l). By (5.15), $E_{2,l}^{-k,2d-h-k}$ and $E_{2,l}^{k,h+k}$ are dual. The morphisms $\upsilon_l\colon E_{2,l}^{-12}\longrightarrow E_{2,l}^{10}(-1)$ and $\upsilon_l\colon E_{2,l}^{-1,2d}\longrightarrow$ $\longrightarrow E_{2,l}^{1,2d-2}(-1)$ are also dual since they are the identities on the E_1-terms (cf. the p-adic case in [Nakk3, (11.7)]). Hence it suffices to prove that $\upsilon_l\colon E_{2,l}^{-12}\longrightarrow E_{2,l}^{10}(-1)$ is an isomorphism. Let K be the fraction field of V and let $\overline{K}$ be the separable closure of K. Let $\mathcal{X}_K$ (resp. $\mathcal{X}_{\overline{K}}$) be the generic (resp. generic geometric) fiber of $\mathcal{X}$. Set $\eta:=\operatorname{Spec}K$. We may replace K with a finite extension of K. Hence we can assume that $\mathcal{X}_K(K)\neq\phi$ and the action of $\operatorname{Gal}(\overline{K}/K)$ on $H^1_{\rm et}(\mathcal{X}_{\overline{K}},\mathbb{F}_l)$ is trivial. Let $\mathcal{A}_K$ be the Albanese variety of $\mathcal{X}_K$ with a morphism $\mathcal{X}_K \longrightarrow \mathcal{A}_K$ which induces the following isomorphism of $\mathrm{Gal}(\overline{K}/K)$-module:

$$H^1_{\mathrm{et}}(\mathcal{A}_{\overline{K}}, \mathbb{Z}_l) \xrightarrow{\sim} H^1_{\mathrm{et}}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_l). \tag{8.1.1}$$

By Neron's blow up ([A, (4.6)], [SGA 7-I, I (0.5)]), there exist a discrete valuation ring V_0 and rings $\{B_i\}_i$ satisfying the following properties:

(1) V_0 and B_i are subrings of V,

(2) the residue field of V_0 is a purely inseparable extension of a field of finite type over $\mathbb{F}_p$,

(3) $\varinjlim_i B_i = V$,

(4) B_i is a smooth henselian V_0-algebra which is essentially of finite type over V_0,

(5) there exists a uniformizer π of V_0 and V.

Set $S_i := \mathrm{Spec}\, B_i$, and let D_i be the closed subscheme of S_i defined by $\pi = 0$ and $\mathring{s}_i$ the closed point of D_i. Let s_i be the log point whose underlying scheme is $\mathring{s}_i$. As in [SGA 7-I, I §6], if i is large enough, then the action of $\mathrm{Gal}(\overline{\eta}/\eta)$ on $H^1_{\mathrm{et}}(\mathcal{A}_{\overline{K}}, \mathbb{Q}_l)$ factors through $\pi_1(S_i \setminus D_i, \overline{\eta})$ and the action of $\pi_1(S_i \setminus D_i, \overline{\eta})$ is trivial on $H^1_{\mathrm{et}}(\mathcal{A}_{\overline{K}}, \mathbb{F}_l)$. Let I (resp. I_i) be the inertia group of S (resp. S_i) and $\mathcal{P}$ (resp. $\mathcal{P}_i$) the wild part of I (resp. I_i). Then we have a natural isomorphism $I/\mathcal{P} \xrightarrow{\sim} I_i/\mathcal{P}_i$. Take a generator T_i of $\mathbb{Z}_l(1) = I_i/\mathcal{P}_i$. Set $N_i := (T_i - 1) \otimes \check{T}_i \colon H^1_{\mathrm{et}}(\mathcal{A}_{\overline{K}}, \mathbb{Q}_l) \longrightarrow H^1_{\mathrm{et}}(\mathcal{A}_{\overline{K}}, \mathbb{Q}_l)(-1)$. By [loc. cit.] there exists a weight filtration $Q_0 \subset Q_1 \subset Q_2 := H^1_{\mathrm{et}}(\mathcal{A}_{\overline{K}}, \mathbb{Q}_l)$ and N_i induce an isomorphism $N_i \colon Q_2/Q_1 \xrightarrow{\sim} Q_0(-1)$. On the other hand, by [Nak3, (2.1), (2.2)], the SNCL variety X gives a model X_i over s_i if i is large enough, and the spectral sequence (2.0.8; l) for X is isomorphic to that for X_i. Hence we have another filtration $P_\bullet$ on $H^1_{\text{log-et}}(X_{\overline{s}_i}, \mathbb{Q}_l)$. Since N_i on $P_1 := P_1 H^1_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l)$ is 0, P_1 is a $\mathrm{Gal}(\overline{\mathring{s}}_i/\mathring{s}_i)$-module. The weights in the sense of [SGA 7-I, I (6.3)] (applying [loc. cit.] for any subfield of κ which is of finite type over $\mathbb{F}_p$) of P_1/P_0 and P_0 are 1 and 0, respectively. By [FK] (cf. [Nak2, (4.2)]), there exists an isomorphism of $\mathrm{Gal}(\overline{K}/K)$-module:

$$H^1_{\mathrm{et}}(\mathcal{X}_{\overline{K}}, \mathbb{Q}_l) \xrightarrow{\sim} H^1_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l). \tag{8.1.2}$$

By (8.1.1) and (8.1.2), $Q_\bullet$ induces a filtration $Q'_\bullet$ on $H^1_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l)$. Hence we have two Frobenius weight filtrations $P_\bullet$ and $Q'_\bullet$ on $H^1_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l)$. Elementary linear algebra shows that $P_\bullet = Q'_\bullet$. Hence (8.1) follows. □

REMARK 8.2. Another proof of (8.1) is possible by [dJ, (8.2)], by the log hard Lefschetz theorem for the first log l-adic cohomology of a projective strict semistable family ([Ka]) (see (9.5) below for the statement of the log hard Lefschetz conjecture), and by (9.10) below. Indeed, by [dJ, (8.2)], if we make a finite extension of V, then there exists a projective strict semistable family $\mathcal{X}'$ over $\operatorname{Spec} V$ which is an alteration of $\mathcal{X}$. Let X' be the special fiber of $\mathcal{X}'$ with canonical log structure. By [Nak4], $H^h_{\text{log-et}}(X_{\bar{s}}, \mathbb{Q}_l)$ is a direct factor of $H^h_{\text{log-et}}(X'_{\bar{s}}, \mathbb{Q}_l)$. By the explanation before (9.6) below and by (9.10) below, the graded pieces of $H^1_{\text{log-et}}(X'_{\bar{s}}, \mathbb{Q}_l)$ by the l-adic monodromy filtration on $H^1_{\text{log-et}}(X'_{\bar{s}}, \mathbb{Q}_l)$ are pure and hence so is for $H^1_{\text{log-et}}(X_{\bar{s}}, \mathbb{Q}_l)$.

THEOREM 8.3. *Let $\mathcal{X}$ be an algebraic proper strict semistable family of pure relative dimension d over a complete discrete valuation ring V with perfect residue field κ of characteristic $p > 0$. Let X/s be the special fiber of $\mathcal{X}$ with canonical log structure. Then the monodromy filtration and the weight filtration on $H^h_{\text{log-crys}}(X/W) \otimes_W K_0$ coincide for $h = 1, 2d-1$.*

PROOF. We may assume that κ is algebraically closed. Let $E_{2,p}^{\bullet\bullet}$ be the E_2-term of the weight spectral sequence (2.0.8; p). By the duality ([Nakk3, (10.5), (11.7)]), it suffices to prove that $\upsilon_p \colon E_{2,p}^{-12} \longrightarrow E_{2,p}^{10}(-1)$ is an isomorphism.

The boundary morphism $d_1^{-12}(1)$ of (2.0.8; l) is decomposed as

$$H^0(\overset{\circ}{X}{}^{(2)}, \mathbb{Q}_l) \longrightarrow (\mathrm{NS}(\overset{\circ}{X}{}^{(1)}) \otimes_{\mathbb{Z}} \mathbb{Q}_l) \oplus H^0(\overset{\circ}{X}{}^{(3)}, \mathbb{Q}_l) \overset{\subset}{\longrightarrow} H^2(\overset{\circ}{X}{}^{(1)}, \mathbb{Q}_l)(1) \oplus H^0(\overset{\circ}{X}{}^{(3)}, \mathbb{Q}_l).$$

The left morphism above is given by the sum of Chern class morphisms (with signs) and the induced morphisms (with signs) by the closed immersions from the irreducible components of $X^{(3)}$ to those of $X^{(2)}$ ((5.5.1)). Hence $d_1^{-12}(1)$ is defined over $\mathbb{Q}$ and $E_{2,l}^{-12}(1) = \operatorname{Ker}(d_1^{-1,2}(1))$ has a $\mathbb{Q}$-structure. The term $E_{2,l}^{10}$ also has a $\mathbb{Q}$-structure, and $\upsilon_l(1) \colon E_{2,l}^{-12}(1) \longrightarrow E_{2,l}^{10}$ is defined over $\mathbb{Q}$ since $\upsilon_l(1) \colon E_{1,l}^{-12}(1) \longrightarrow E_{1,l}^{10}$ is the identity by the same argument as that of [Nakk3, (10.5), (11.7)] for the p-adic case.

As for $\upsilon_p(1)$, we need the injectivity of the following morphism

$$(\mathrm{NS}(\overset{\circ}{X}{}^{(1)}) \otimes_{\mathbb{Z}} \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K_0 \longrightarrow H^2_{\text{log-crys}}(\overset{\circ}{X}{}^{(1)}/W) \otimes_W K_0. \tag{8.3.1}$$

By [Il1, II (5.8.5), (5.5.3)],

$$\mathrm{NS}(\overset{\circ}{X}{}^{(1)}) \otimes_{\mathbb{Z}} \mathbb{Q}_p \subset H^2_{\mathrm{fl}}(\overset{\circ}{X}{}^{(1)}, \mathbb{Q}_p) = (H^2_{\text{log-crys}}(\overset{\circ}{X}{}^{(1)}/W) \otimes_W K_0)^{F=p}.$$

By a theorem of Dieudonné-Manin, the maximal sub F-crystal of $H^2_{\text{log-crys}}(\overset{\circ}{X}{}^{(1)}/W) \otimes_W K_0$ of slope 1 has a basis $\{e_j\}_j$ over K_0 such that

$F(e_j) = pe_j$. Hence a natural morphism

$$H^2_{\rm fl}(\overset{\circ}{X}{}^{(1)}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K_0 = (H^2_{\text{log-crys}}(\overset{\circ}{X}{}^{(1)}/W) \otimes_W K_0)^{F=p} \otimes_{\mathbb{Q}_p} K_0 \longrightarrow H^2_{\text{log-crys}}(\overset{\circ}{X}{}^{(1)}/W) \otimes_W K_0$$

is injective. Consequently the morphism (8.3.1) is injective. Now, by the same argument as that in the l-adic case, $\upsilon_p(1)$ is defined over $\mathbb{Q}$; the two morphisms $\upsilon_p(1)$ and $\upsilon_l(1)$ are scalar extensions of the same morphism over $\mathbb{Q}$. Hence (8.3) follows from (8.1). □

The following is a correction and a generalization of Mokrane's result (cf. (6.8) (1), (4)).

COROLLARY 8.4. *Let $\mathcal{X}$ an algebraic proper strict semistable family of surfaces over a complete discrete valuation ring V with perfect residue field κ of characteristic $p > 0$. Let X/s be the special fiber of $\mathcal{X}$ with canonical log structure. Then* (2.0.9; $\star$) ($\star = l, p$) holds for X/s.

PROOF. By (8.3) and by the argument of [Mo, §6] (see also (6.8) (1)), we obtain (8.4). □

REMARK 8.5. Note that V in (8.3) and (8.4) is not necessarily of mixed characteristics.

9. Log hard Lefschetz conjectures, the monodromy-weight conjectures and the log Hodge symmetry.

In this section, following K. Kato's idea, we deduce the coincidence of the l-adic monodromy filtration and the l-adic weight filtration on the first log l-adic cohomology of a projective SNCL variety X of pure dimension d over a log point s from the log l-adic hard Lefschetz conjecture for the first log l-adic cohomology of X. Assume that $\overset{\circ}{X}$ is projective in this section unless otherwise stated. Let $\mathcal{L}$ be an ample invertible sheaf on $\overset{\circ}{X}$. Let $\lambda_\star = \lambda^{(j)}_\star$ be the cohomology class of $\mathcal{L}|_{\overset{\circ}{X}{}^{(j)}}$ in $H^2_\star(\overset{\circ}{X}{}^{(j)})$ ($j \in \mathbb{Z}_{\geq 1}$). Let $\{E^{\bullet\bullet}_{r,\star}\}$ be the E_r-terms of (2.0.8; $\star$) ($\star = l, p$).

THEOREM 9.1. (1) *The $\star$-adic monodromy operator $N_\star\colon E^{-12}_{2,\star} \longrightarrow E^{10}_{2,\star}(-1)$ is injective*; $N_\star\colon E^{-1,2d}_{2,\star} \longrightarrow E^{1,2d-2}_{2,\star}(-1)$ *is surjective.*

(2) *If the induced morphism* $\lambda_\star^{d-1}\colon E_{2,\star}^{-12} \longrightarrow E_{2,\star}^{-1,2d}$ *by the morphism* $\lambda_\star^{d-1}\colon E_{1,\star}^{-12} \longrightarrow E_{1,\star}^{-1,2d}$ *is surjective, then* $N_\star\colon E_{2,\star}^{-12} \longrightarrow E_{2,\star}^{10}(-1)$ *and* $N_\star\colon E_{2,\star}^{-1,2d} \longrightarrow E_{2,\star}^{1,2d-2}(-1)$ *are isomorphisms.*

PROOF. We may assume that κ is algebraically closed. Let

$$\langle\ ,\ \rangle^{(j)}\colon H_\star^0(X^{(j)}) \times H_\star^{2d}(X^{(j)})(d) \longrightarrow \mathbf{1}_\star \quad (j \in \mathbb{Z}_{\geq 1})$$

be the Poincaré duality morphism of $H_\star^\bullet(X^{(j)})$.

(1): Since the restriction of $\mathcal{L}^{d-1}$ to the double varieties is positive, the morphism $H_\star^0(X^{(2)}) \overset{\lambda_\star^{d-1}}{\underset{\simeq}{\longrightarrow}} H_\star^{2d-2}(X^{(2)})(d-1)$ is an isomorphism. Consider the composite of the following pairings

$$(9.1.1)\quad \langle\ ,\ \rangle\colon H_\star^0(X^{(2)}) \times H_\star^0(X^{(2)}) \overset{\mathrm{id}\times\lambda_\star^{d-1}}{\longrightarrow} H_\star^0(X^{(2)}) \times H_\star^{2d-2}(X^{(2)})(d-1) \overset{\langle\ ,\ \rangle^{(2)}}{\longrightarrow} \mathbf{1}_\star.$$

The cohomology $H_\star^0(X^{(2)})$ has a $\mathbb{Q}$-structure, and (9.1.1) is positive definite with respect to the $\mathbb{Q}$-structure. Indeed, let D be an irreducible component of $X^{(2)}$. The pairing $H_\star^0(D) \times H_\star^0(D) \overset{\mathrm{id}\times\lambda_\star^{d-1}}{\longrightarrow} H_\star^0(D) \times H_\star^{2d-2}(D)(d-1) \longrightarrow \mathbf{1}_\star$ is a morphism $(x, y) \longmapsto \deg(\mathcal{L}^{d-1}|_D) \cdot xy$. Thus the positive definiteness is clear.

Let $\rho\colon H_\star^0(X^{(1)}) \longrightarrow H_\star^0(X^{(2)})$ be the morphism induced by closed immersions and let $G_i\colon H_\star^i(X^{(2)}) \longrightarrow H_\star^{i+2}(X^{(1)})(1)$ $(i \in \mathbb{N})$ be the sum of Gysin morphisms in (5.5.1) and (2.0.8.3; p). Then we have $\langle \rho(x), y\rangle^{(2)} = \langle x, G_{2d-2}(y)\rangle^{(1)}$ $(x \in H_\star^0(X^{(1)}),\ y \in H_\star^{2d-2}(X^{(2)})(d-1))$.

To prove that $N_\star\colon E_{2,\star}^{-12} \longrightarrow E_{2,\star}^{10}(-1)$ is injective, it suffices to prove that the induced morphism

$$(9.1.2)\quad \mathrm{Ker}(H_\star^0(X^{(2)}) \overset{G_0}{\longrightarrow} H_\star^2(X^{(1)})(1)) \longrightarrow \mathrm{Coker}(H_\star^0(X^{(1)}) \overset{\rho}{\longrightarrow} H_\star^0(X^{(2)}))$$

by the identity of $H_\star^0(X^{(2)})$ ([Nakk3, (11.7)] for the p-adic case) is injective. As shown in the proof of (8.3), G_0 factors through $H_\star^0(X^{(2)}) \longrightarrow \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbf{1}_\star$ with the injective morphism $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbf{1}_\star \overset{\subset}{\longrightarrow} H_\star^2(X^{(1)})(1)$, and this is defined over $\mathbb{Q}$. Assume that (9.1.2) were not injective. Then there exists a $\mathbb{Q}$-rational vector $w \in H_\star^0(X^{(1)})$ such that $v := \rho(w)$ is not the zero but $G_0(v) = 0$. Since v is not the zero and $\mathbb{Q}$-rational, $\langle v, v\rangle \neq 0$. On the other hand, we have the following commutative diagram below by the projection formula:

$$\begin{array}{ccc} H_\star^{2d-2}(X^{(2)})(d-1) & \overset{G_{2d-2}}{\longrightarrow} & H_\star^{2d}(X^{(1)})(d) \\ {\scriptstyle \lambda_\star^{d-1},\simeq}\uparrow & & \uparrow{\scriptstyle \lambda_\star^{d-1}} \\ H_\star^0(X^{(2)}) & \overset{G_0}{\longrightarrow} & H_\star^2(X^{(1)})(1). \end{array}$$

Hence

$$\begin{aligned}\langle v, v\rangle &= \langle v, \lambda_\star^{d-1}(v)\rangle^{(2)} = \langle \rho(w), \lambda_\star^{d-1}(v)\rangle^{(2)}\\ &= \langle w, G_{2d-2}(\lambda_\star^{d-1}(v))\rangle^{(1)} = \langle w, \lambda_\star^{d-1}(G_0(v))\rangle^{(1)}\\ &= 0.\end{aligned}$$

This is a contradiction. Consequently $N_\star\colon E_{2,\star}^{-12} \longrightarrow E_{2,\star}^{10}(-1)$ is injective.

Because $N_\star\colon E_{1,\star}^{-12} \longrightarrow E_{1,\star}^{10}(-1)$ and $N_\star\colon E_{1,\star}^{-1,2d} \longrightarrow E_{1,\star}^{1,2d-2}(-1)$ are the identities ([Nakk3, (11.7)] for the p-adic case) and hence dual, $N_\star\colon E_{2,\star}^{-1,2d} \longrightarrow E_{2,\star}^{1,2d-2}(-1)$ is surjective by the duality (5.15).

(2): Note that there is a well-defined morphism $\lambda_\star^{d-1}\colon E_{2,\star}^{-12} \longrightarrow E_{2,\star}^{-1,2d}(d-1)$ since the following diagram commutes by the projection formula:

$$\begin{array}{ccc} E_{1,\star}^{-1,2d}(d-1) & \xrightarrow{d_1^{-1,2d}} & E_{1,\star}^{0,2d}(d-1)\\ {\scriptstyle \lambda_\star^{d-1}}\Big\uparrow & & \Big\uparrow{\scriptstyle \lambda_\star^{d-1}}\\ E_{1,\star}^{-12} & \xrightarrow{d_1^{-12}} & E_{1,\star}^{02}. \end{array}$$

By (1) and by the assumption of (2), (2) follows from the duality between $E_{2,\star}^{-1,2d}(d)$ and $E_{2,\star}^{10}$. □

As a corollary to (9.1), we can give another proof of [CI, Theorem 1] (cf. [Mo, 5.9]):

COROLLARY 9.2. *Let V be a complete discrete valuation ring of mixed characteristics with fraction field K. Let A_K be an abelian variety over K. Then A_K has a good reduction if and only if the p-adic representation $H^1_{\mathrm{et}}(A_{\overline{K}}, \mathbb{Q}_p)$ is crystalline.*

PROOF. Assume that $H^1_{\mathrm{et}}(A_{\overline{K}}, \mathbb{Q}_p)$ is crystalline. By [Fo1, (7.5.3) i)], we can replace K with a finite extension of K. Hence we may assume that A_K has a projective strict semistable reduction over V by [SGA 7-I, IX (3.6)] and [Kü, (4.6)]. By [RZ, §2, §3] (cf. [Nak2, (0.1.1)]), the Galois action on $H^1(A_{\overline{K}}, \mathbb{Q}_l)$ is tame.

By the assumption, the monodromy operator on $H^1_{\text{log-crys}}(A/W) \otimes_W K_0$ is trivial by [Fo2, (5.5.1)] and by C_{st}, which has been proved in [Ts1, (0.2)]. Therefore $E_{2,p}^{-12} = 0$ by (9.1), and hence $E_{2,l}^{-12} = 0$ since the boundary morphism $d_1^{-12}\colon E_{1,\star}^{-12} \longrightarrow E_{1,\star}^{02}$ is motivic ($\star = l, p$) as shown in the proof of (8.3); the inertia group of K acts trivially on $H^1_{\mathrm{et}}(A_{\overline{K}}, \mathbb{Q}_l)$. The criterion of

Neron-Ogg-Shafarevich ([ST], cf. [SGA 7-I, I §6]) tells us that A_K has a good reduction.

The converse follows from a special case of [Ts1, (0.2)]. □

COROLLARY 9.3. *Let X be a projective SNCL surface over a log point s, and let $\Gamma(X)$ be the dual graph of $\overset{\circ}{X}$. Assume that $H^1(\Gamma(X), \mathbb{Q}) = 0$. Then* (2.0.9; $\star$) ($\star = l, p$) *holds.*

PROOF. By the assumption, we have $E_{2,\star}^{10} = 0$, and hence $E_{2,\star}^{-12} = 0$ by (9.1). By the duality ((5.15)), we have $E_{2,\star}^{12} = 0$. By the proof of [Mo, 6.2.2], (2.0.9; $\star$) holds (cf. (6.8) (1)).

(9.4) (2) below includes a conjecture of Chiarellotto ([Ch]) for the first log crystalline cohomology of a projective SNCL variety:

COROLLARY 9.4. *Let X be a projective SNCL variety of pure dimension d over a log point s. Then the following hold*:

(1) *The following sequences*

$$0 \longrightarrow H^1_{\text{et}}(\overset{\circ}{\overline{X}}, \mathbb{Q}_l) \longrightarrow H^1_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l) \xrightarrow{N_l} H^1_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l)(-1)$$

and

$$H^{2d-1}_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l) \xrightarrow{N_l} H^{2d-1}_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l)(-1) \longrightarrow (H^1_{\text{et}}(\overset{\circ}{\overline{X}}, \mathbb{Q}_l))^*(-d-1) \longrightarrow 0$$

are exact.

(2) *The following sequences*

$$0 \longrightarrow H^1_{\text{rig}}(X/K_0) \longrightarrow H^1_{\text{log-crys}}(X/W) \otimes_W K_0 \xrightarrow{N_p} H^1_{\text{log-crys}}(X/W)(-1) \otimes_W K_0$$

and

$$H^{2d-1}_{\text{log-crys}}(X/W) \otimes_W K_0 \xrightarrow{N_p} H^{2d-1}_{\text{log-crys}}(X/W)(-1) \otimes_W K_0 \longrightarrow$$
$$(H^1_{\text{rig}}(X/K_0))^*(-d-1) \longrightarrow 0$$

are exact.

PROOF. (1): By (9.1), we have the following exact sequence

$$0 \longrightarrow E_{2,l}^{10} \longrightarrow \mathrm{Ker}(N_l\colon H^1_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l) \longrightarrow H^1_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l)(-1)) \longrightarrow E_{2,l}^{01} \longrightarrow 0.$$

By noting $N_l = \upsilon_l$ on $H^1_{\text{log-et}}(X_{\overline{s}}, \mathbb{Q}_l)$ and by (6.10.1; l), we obtain the former part of (1). Then we obtain the latter part of (1) by the log Poincaré duality of Nakayama ([Nak1]).

(2): (2) follows from (9.1), (6.10.1; p) and [Ch, (3.6)] as above. □

Next, we give the log hard Lefschetz conjecture. To state it, we define the log cohomology classes of an invertible sheaf in second log cohomologies. Let Y be a log smooth scheme over a log point s. Let $\mathcal{L}$ be an invertible sheaf on $\overset{\circ}{Y}$. Let n be an integer which is invertible on $\overset{\circ}{Y}$.

In the l-adic case, by the Kummer sequence

$$0 \longrightarrow \mathbb{Z}/n(1) \longrightarrow \mathbb{G}_m \xrightarrow{\ n\ } \mathbb{G}_m \longrightarrow 0$$

in $\widetilde{Y}^{\log}_{\mathrm{et}}$, we obtain the log cohomology class $\lambda_l = c_{1,l}(\mathcal{L})$ of $\mathcal{L}$ in $H^2_{\text{log-et}}(Y_{\overline{s}}, \mathbb{Z}_l)(1)$ and $H^2_{\text{log-et}}(Y_{\overline{s}}, \mathbb{Q}_l)(1)$ (cf. the proof of (9.9) below).

In the p-adic case, we have only to give the same argument as that of [BO2, §3] as follows.

Let $\iota_{Y/W} \colon \widetilde{Y}_{\mathrm{zar}} \longrightarrow \widetilde{(Y/W)}^{\log}_{\mathrm{crys}}$ be a morphism of topoi defined by $\iota^*_{Y/W}(F) = F_{(Y,Y,0)}$ for a sheaf F in $\widetilde{(Y/W)}^{\log}_{\mathrm{crys}}$ as in [BO1, 5.19]. Set $\mathcal{J}_{Y/W} := \mathrm{Ker}(\mathcal{O}_{Y/W} \longrightarrow \iota_{Y/W*}(\mathcal{O}_Y))$. Then we have the following exact sequence

$$0 \longrightarrow 1 + \mathcal{J}_{Y/W} \longrightarrow \mathcal{O}^*_{Y/W} \longrightarrow \iota_{Y/W*}(\mathcal{O}^*_Y) \longrightarrow 0.$$

By using the boundary morphism of the exact sequence above, we have the following composite morphism

$$\begin{aligned} c_{1,p} \colon H^1(Y, \mathcal{O}^*_Y) &\longrightarrow H^2((Y/W)^{\log}_{\mathrm{crys}}, 1 + \mathcal{J}_{Y/W}) \\ &\xrightarrow{\log} H^2((Y/W)^{\log}_{\mathrm{crys}}, \mathcal{J}_{Y/W}) \longrightarrow H^2((Y/W)^{\log}_{\mathrm{crys}}, \mathcal{O}_{Y/W}). \end{aligned}$$

Set $\lambda_p = c_{1,p}(\mathcal{L})$ of $\mathcal{L}$ in $H^2_{\text{log-crys}}(Y/W)$ and $H^2_{\text{log-crys}}(Y/W) \otimes_W K_0$. More generally, in the same way, we can define the class $c_{1,p}(\mathcal{L})_S$ of $\mathcal{L}$ in $H^2_{\text{log-crys}}(Y/S)$ and $H^2_{\text{log-crys}}(Y/S) \otimes_W K_0$, where S is any p-adic log PD-thickening of s, that is, an exact closed immersion defined by a PD-thickening of s into a p-adic noetherian formal log scheme.

Assume that $\mathcal{L}$ is ample. Then K. Kato has suggested the following:

CONJECTURE 9.5 [Log hard Lefschetz conjecture (K. Kato)]. Let X be a projective SNCL variety over s. Then the induced morphism

(9.5.1; $\star$) $$\lambda^j_\star = \lambda^j \cup \colon H^{d-j}_{\log,\star}(X) \longrightarrow H^{d+j}_{\log,\star}(X)(j) \quad (\star = l, p) \quad (j \in \mathbb{N})$$

by $\mathcal{L}^j$ is an isomorphism.

Note that $\lambda_\star^j$ is an isomorphism if and only if it is surjective and if and only if it is injective by the log Poincaré duality ([Nak1, (0.1)], [Ts2, (5.6)] or [Hy, (3.7)]+[HK, (4.19)] whose proof has been completed in [Nakk3, (7.19)]). Recently Kajiwara has proved the log l-adic hard Lefschetz conjecture for the first log l-adic cohomology ([Ka]). If X/s is the special fiber of a projective strict semistable family $\mathcal{X}$ over a complete discrete valuation ring V and if $\mathcal{L}$ is the restriction of a relative ample invertible sheaf on $\mathcal{X}/V$, then (9.5.1; l) holds because the isomorphism in [FK], [Nak2, (4.2)] is compatible with the l-adic (log) Chern class of an invertible sheaf and because the usual hard Lefschetz theorem holds by [D4, (4.1.1)]. Moreover, if V is of mixed characteristics, (9.5.1; p) holds by [O, Theorem 2, Theorem 3] and proofs in [BO2, §3]. Though the isomorphism in [HK, (5.1)] is very delicate (cf. [HK, (4.10)]), (9.5.1; p) in the p-adic case above also follows from [HK, (5.1)], by proofs in [BO2, §3] and by the following, which gives the compatibility of the p-adic Chern classes of invertible sheaves with Hyodo-Kato's isomorphism:

LEMMA 9.6. *Let s be a log point such that $\overset{\circ}{s}$ is the spectrum of a perfect field κ of characteristic $p > 0$. Let Y be a proper log smooth scheme of Cartier type over s. Let $\widehat{W\langle t\rangle}$ be the p-adic completion of the divided power polynomial $W\langle t\rangle$ over W. Endow $\widehat{W\langle t\rangle}$ with log structure given by a morphism $\mathbb{N} \ni 1 \longmapsto t \in \widehat{W\langle t\rangle}$. Let $\mathcal{L}$ be an invertible sheaf on $\overset{\circ}{Y}$. Then, under the isomorphism* [HK, (4.13)] (see (9.7) below)

$$\widehat{W\langle t\rangle} \otimes_W (H^2_{\text{log-crys}}(Y/W) \otimes_W K_0) \overset{\sim}{\longrightarrow} H^2_{\text{log-crys}}(Y/\widehat{W\langle t\rangle}) \otimes_W K_0, \tag{9.6.1}$$

$1 \otimes c_{1,p}(\mathcal{L})$ corresponds to the log cohomology class $c_{1,p}(\mathcal{L})_{\widehat{W\langle t\rangle}}$ with respect to $Y/\widehat{W\langle t\rangle}$.

PROOF. By the definitions of $c_{1,p}(\ast)_{\widehat{W\langle t\rangle}}$ and $c_{1,p}(\ast)$, the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Pic} Y & = & \operatorname{Pic} Y \\ {\scriptstyle c_{1,p}}\downarrow & & \downarrow{\scriptstyle c_{1,p}(\ast)_{\widehat{W\langle t\rangle}}} \\ H^2_{\text{log-crys}}(Y/W) & \overset{0 \leftarrow\!\shortmid\, t}{\longleftarrow} & H^2_{\text{log-crys}}(Y/\widehat{W\langle t\rangle}). \end{array}$$

Let ϕ be a Frobenius of $\widehat{W\langle t\rangle} \otimes_W K_0$ defined by $\phi(t) = t^p$ and $\phi(a) = \sigma(a)$, where σ is the Frobenius of K_0. Let $\{\gamma_i\}_{i=1}^n$ be a basis of $H^2_{\text{log-crys}}(Y/W) \otimes_W K_0$. Hence, by [loc. cit., (4.13.1)], $c_{1,p}(\mathcal{L})_{\widehat{W\langle t\rangle}}$ corresponds to $1 \otimes c_{1,p}(\mathcal{L}) + \sum_{i=1}^{n} f_i(t) \otimes \gamma_i$ on the left hand side of (9.6.1), where $f_i(t) \in \widehat{W\langle t\rangle} \otimes_W K_0$ such that $f_i(0) = 0$. Because the isomorphism in {[HK, (4.13)]} is compatible with the Frobenius ([loc. cit., (4.13.2)]) and because the Frobenius actions on the log cohomology classes of an invertible sheaf are the multiplication by p, we obtain

$$(\phi(f_1(t)), \ldots, \phi(f_n(t)))(a_{ij}) = (pf_1(t), \ldots, pf_n(t)), \tag{9.6.2}$$

where (a_{ij}) is an element of $\mathrm{GL}_n(K_0)$. We claim that $f_i(t) = 0$ $(\forall i)$. Indeed, if it did not hold, then, by renumbering of the indexes i's of $f_i(t)$, we may assume that there exists $j \leq n$ such that $f_1(t) = \cdots = f_{j-1}(t) = 0$ and $f_j(t) \neq 0, \ldots, f_n(t) \neq 0$. We can also assume that the degree of the leading term of $f_k(t)$ is less that or equal to that of $f_{k+1}(t)$ $(j \leq \forall k \leq n-1)$. By (9.6.2), we obtain

$$\sum_{k=j}^{n} \phi(f_k(t)) a_{kj} = pf_j(t). \tag{9.6.3}$$

For a field L and a nonzero element $f = \sum_{i=0}^{\infty} a_i t^i \in L[[t]]$, let $\mu(f)$ denote $\min\{i \in \mathbb{N} \mid a_i \neq 0\}$. By (9.6.3), $\mu\left(\sum_{k=j}^{n} \phi(f_k(t)) a_{k,j}\right) = \mu(pf_j(t)) = \mu(f_j(t))$. On the other hand, $\mu\left(\sum_{k=j}^{n} \phi(f_k(t)) a_{k,j}\right) \geq p(\mu(f_j(t))$ since $f_j(0) = 0$. Since $\mu(f_j(t)) \neq 0$, we would obtain the contradiction. Therefore we obtain $f_i(t) = 0$ $(\forall i)$ and hence (9.6). □

REMARK 9.7. The sheaf [HK, (4.6.1)] is wrong and the proofs of [loc. cit., (4.6)] and [loc. cit., (4.8)] are mistaken. In [Nakk3, §7] I have corrected all of them. See [Nakk3, §7] for the details. Hence we can use [HK, (4.13)].

We also define the log Chern class of an invertible sheaf when the base field is $\mathbb{C}$ as follows.

Let Y be a log smooth scheme over the log point $(\mathrm{Spec}\,\mathbb{C}, \mathbb{N} \oplus \mathbb{C}^*)$. Let $\mathcal{L}$ be an invertible sheaf on $\overset{\circ}{Y}$. Assume that $\overset{\circ}{Y}$ is reduced. Then there exists the exponential exact sequence

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathcal{O}_Y \overset{\exp}{\longrightarrow} \mathcal{O}_Y^* \longrightarrow 0 \tag{9.7.1}$$

on $\overset{\circ}{Y}$. The exact sequence (9.7.1) induces a morphism

$$\mathcal{O}_Y^* \longrightarrow \mathbb{Z}(1)[1] \tag{9.7.2}$$

in the derived category of the complexes of the abelian sheaves on $\overset{\circ}{Y}$, and hence a morphism

$$H^1(Y, \mathcal{O}_Y^*) \longrightarrow H^2(Y, \mathbb{Z})(1). \tag{9.7.2; H}$$

Composing (9.7.2; H) with a natural morphism $H^2(Y, \mathbb{Z})(1) \longrightarrow$ $\longrightarrow H^2(Y_\infty, \mathbb{Z})(1)$, we have a morphism

$$c_{1,\infty} \colon H^1(Y, \mathcal{O}_Y^*) \longrightarrow H^2(Y_\infty, \mathbb{Z})(1). \tag{9.7.3}$$

Composing (9.7.2) with a natural inclusion $\mathbb{Z}(1) \xrightarrow{\subset} \mathbb{Q}(1)$, we have a morphism

$$\mathcal{O}_Y^* \longrightarrow \mathbb{Q}(1)[1] \tag{9.7.4}$$

and hence a morphism

$$H^1(Y, \mathcal{O}_Y^*) \longrightarrow H^2(Y, \mathbb{Q})(1). \tag{9.7.4; H}$$

As above, we have a morphism

$$c_{1,\infty} \colon H^1(Y, \mathcal{O}_Y^*) \longrightarrow H^2(Y_\infty, \mathbb{Q})(1). \tag{9.7.5}$$

Set $\lambda_\infty := c_{1,\infty}(\mathcal{L}) \in H^2(Y_\infty, \mathbb{Z})(1)$ or $H^2(Y_\infty, \mathbb{Q})(1)$.

DEFINITION 9.8. Let $\lambda_\star := c_{1,\star}(\mathcal{L}) \in \underline{H}^2_{\log,\star}(X)$ (resp. $\in H^2_{\log,\star}(X)$) ($\star = l, p, \infty$) be the log cohomology class of an invertible sheaf $\mathcal{L}$ on $\overset{\circ}{X}$. We say that $\lambda_\star$ is *compatible with the weight spectral sequence* (2.0.7; $\star$) ($\star = l, p, \infty$) (resp. (2.0.8; $\star$)) if the induced morphism of the left cup product of $\lambda_\star$ on the E_1-terms of (2.0.7; $\star$) (resp. (2.0.8; $\star$)) is equal to the induced morphism of the restriction of $\mathcal{L}$ to various $X^{(k)}$'s ($k \in \mathbb{Z}_{>0}$).

PROPOSITION 9.9. *Let X be a* (*not necessarily proper*) *SNCL algebraic* (*or analytic*) *variety over a log point s. Let $\lambda_\star := c_{1,\star}(\mathcal{L})$* ($\star = l, \infty$) *be the log cohomology class of an invertible sheaf $\mathcal{L}$ on $\overset{\circ}{X}$. Then $\lambda_\star$ is compatible with* (2.0.7; $\star$)*, and hence with* (2.0.8; $\star$) ($\star = l, \infty$).

PROOF. Let the notations be as in §5. We give only the proof for the l-adic case; the proof for the ∞-adic case is almost the same.

Let Y be a SNCL variety over a separably closed field κ_{sep}. Let the notations be as in the l-adic case in §5. Fix a positive integer n. Let

$$0 \longrightarrow \mathbb{Z}/l^n(1) \longrightarrow \mathbb{G}_m \xrightarrow{l^n} \mathbb{G}_m \longrightarrow 0$$

be the Kummer sequence in $\widetilde{\overset{\circ}{Y}}_{\mathrm{et}}$ and in $\widetilde{Y}^{\log}_{\mathrm{et}}$. Let $\mathcal{L}$ be an invertible sheaf on $\overset{\circ}{Y}$.

Then $\mathcal{L}$ defines elements $\overset{\circ}{c}_{1,l}(\mathcal{L}) \in H^2(\overset{\circ}{Y}_{\rm et}, \mathbb{Z}/l^n) = {\rm Hom}_{{\rm D}^+(\overset{\circ}{Y}_{\rm et},\mathbb{Z}/l^n)}(\mathbb{Z}/l^n\{-1\}, \mathbb{Z}/l^n[1](1))$ and $c_{1,l}(\mathcal{L}) \in H^2(Y^{\log}_{\rm et}, \mathbb{Z}/l^n) = {\rm Hom}_{{\rm D}^+(Y^{\log}_{\rm et},\mathbb{Z}/l^n)}(\mathbb{Z}/l^n\{-1\}, \mathbb{Z}/l^n[1](1))$. By the functoriality, $c_{1,l}(\mathcal{L})$ is the image of $\overset{\circ}{c}_{1,l}(\mathcal{L})$ by a natural morphism $H^2(\overset{\circ}{Y}_{\rm et}, \mathbb{Z}/l^n) \longrightarrow H^2(Y^{\log}_{\rm et}, \mathbb{Z}/l^n)$.

By the functoriality we have an element of ${\rm Hom}_{{\rm D}^+(Y_{1/l^m},\mathbb{Z}/l^n)}(\mathbb{Z}/l^n\{-1\}, \mathbb{Z}/l^n[1](1))$ ($m \in \mathbb{Z}_{>0}$) again. By applying the direct image of a natural morphism $Y_{1/l^m} \longrightarrow Y$ to this element and taking the inductive limit with respect to m, we have a morphism

$$c_{1,l}(\mathcal{L})\colon K^\bullet\{-1\} \longrightarrow K^\bullet(1)[1] \tag{9.9.1}$$

and

$$c_{1,l}(\mathcal{L})\colon L^\bullet\{-1\} \longrightarrow L^\bullet(1)[1]. \tag{9.9.2}$$

Let $A^\bullet_{Y,l,n}$ be the single complex of the double complex (5.0.3). The morphism (9.9.2) induces a morphism $A^\bullet_{Y,l,n}\{-1\} \longrightarrow A^\bullet_{Y,l,n}(1)[1]$ and a morphism

$$({\rm gr}^P_k A^\bullet_{Y,l,n})\{-1\} \longrightarrow ({\rm gr}^P_k A^\bullet_{Y,l,n})(1)[1]. \tag{9.9.3}$$

By the proof of [Nak3, (1.4)], (9.9.3) is equal to the following morphism

$$\bigoplus_{j\geq \max\{-k,0\}} R^{2j+k+1}\varepsilon_{Y*}(\mathbb{Z}/l^n)(j+1)[-2j-k]\{-1\} \longrightarrow \bigoplus_{j\geq \max\{-k,0\}} R^{2j+k+1}\varepsilon_{Y*}(\mathbb{Z}/l^n)(j+1)[-2j-k](1)[1], \tag{9.9.4}$$

and by [loc. cit., the proof of (1.8.3)], (9.9.4) is equal to

$$\bigoplus_{j\geq \max\{-k,0\}} (\mathbb{Z}/l^n)_{\overset{\circ}{Y}(2j+k+1)}(-j-k)[-2j-k]\{-1\} \longrightarrow \bigoplus_{j\geq \max\{-k,0\}} (\mathbb{Z}/l^n)_{\overset{\circ}{Y}(2j+k+1)}(-j-k)[-2j-k](1)[1]. \tag{9.9.5}$$

Here note that the multiplications of the source and the target of (9.9.5) by $(-1)^{2j+k+1}$ does not change the morphism (9.9.5).

By the construction of the morphism (9.9.3), the morphism (9.9.5) is induced by the left cup product of $\overset{\circ}{c}_{1,l}(\mathcal{L}) \in H^2(\overset{\circ}{Y}_{\rm et}, \mathbb{Z}/l^n)$. Hence, by the functoriality of the usual Chern class applied to the closed immersions from the irreducible components of $\overset{\circ}{Y}$ to $\overset{\circ}{Y}$, we have the compatibility. □

PROPOSITION 9.10. *If* (9.5.1;l) *is an isomorphism for* $j=1$, *then* $N_l\colon E^{-12}_{2,l} \longrightarrow E^{10}_{2,l}(-1)$ *and* $N_l\colon E^{-1,2d}_{2,l} \longrightarrow E^{1,2d-2}_{2,l}(-1)$ *are isomorphisms.*

Consequently, if (9.5.1; l) *is an isomorphism for* $j = 1$, *then the monodromy filtrations and the weight filtrations on* $H^1_{\log,l}(X)$ *and* $H^{2d-1}_{\log,l}(X)$ *coincide.*

PROOF. We give two proofs.

First proof: Because $\lambda_l^{d-1}\colon H^1_{\log,l}(X) \longrightarrow H^{2d-1}_{\log,l}(X)(d-1)$ is a surjection by the assumption and because (2.0.8; l) degenerates at E_2 ([Nak3, (2.1)]), we have only to know, by (9.1) (2), that the morphism λ_l^{d-1} induces a morphism $E_{2,l}^{-12} \longrightarrow E_{2,l}^{-1,2d}(d-1)$. This is clear by (9.9).

Second proof: As above, we have only to prove that λ_l^{d-1} induces a morphism $E_{2,l}^{-12} \longrightarrow E_{2,l}^{-1,2d}(d-1)$. This is clear if the base field κ is finite. Indeed, let $H^i_{\log,l}(X)_{\mathrm{wt}\leq j}$ ($i, j \in \mathbb{N}$) be the subobject of $H^1_{\log,l}(X)$ of Frobenius weight $\leq j$. Since λ_l comes from an invertible sheaf and the Frobenius weight of λ_l is 2, λ_l induces a morphism $\lambda_l^{d-1}\colon H^1_{\log,l}(X)_{\mathrm{wt}\leq 1} \longrightarrow$ $\longrightarrow H^{2d-1}_{\log,l}(X)_{\mathrm{wt}\leq 2d-1}(d-1)$. Thus we see that λ_l^{d-1} induces a morphism

$$E_{2,l}^{-12} = H^1_{\log,l}(X)/H^1_{\log,l}(X)_{\mathrm{wt}\leq 1} \longrightarrow H^{2d-1}_{\log,l}(X)/H^{2d-1}_{\log,l}(X)_{\mathrm{wt}\leq 2d-1}(d-1) = E_{2,l}^{-1,2d-1}(d-1).$$

We can reduce the general l-adic case to this case by a specialization argument in [Nak3]. □

COROLLARY 9.11. *Let* X *be a projective SNCL surface over* s. *Then* (9.5.1; l) for $j = 1$ implies (2.0.9; l) for X. □

PROOF. (9.11) follows from (9.10) and (6.8) (1).

REMARK 9.12. (1) By (9.10) and Kajiwara's result mentioned after (9.5), the monodromy filtrations and the weight filtrations on $H^1_{\log,l}(X)$ and $H^{2d-1}_{\log,l}(X)$ coincide for a projective SNCL variety X/s.

(2) The obvious analogues of (9.10) and (9.11) also hold in the p-adic case. One can prove these by using the convergence of the weight filtrations on the log crystalline cohomologies of a family of SNCL varieties ([Nakk4]), and by using the specialization argument (cf. [Nakk3]); if $\mathcal{L}$ is ample, then one can also prove (9.9) by using the theory of log de Rham Witt complexes $W_n\Lambda^{\bullet}_{X}(\log D)$ for an effective Cartier divisor D on $\overset{\circ}{X}$ which meets X transversally in the sense of the algebraic analogue of (10.1) below, and to use the p-adic analogue of (10.1.5) below. In a future paper we would like to discuss this theory in detail.

(3) (6.15) also follows from (6.5), Kajiwara's result and (9.11).

We conclude the l-adic case in this section by considering a relative case:

Proposition 9.13. *Let Y be a smooth variety over a field κ of characteristic $p \geq 0$ and let D be a smooth divisor of Y. Let $f\colon X \longrightarrow Y$ be a projective morphism with strict semistable reduction along D. Endow X (resp. Y) with the log structure associated to the closed subscheme $f^{-1}(D)$ (resp. D). If the log hard Lefschetz conjecture* (9.5.1; l) *holds, then $Rf_*(\mathbb{Q}_l) \simeq \oplus R^i f_*(\mathbb{Q}_l)\{-i\}$ and the Leray spectral sequence*

$$H^j_{\text{log-et}}(Y_{\bar{s}}, R^i f_*(\mathbb{Q}_l)) \Longrightarrow H^{i+j}_{\text{log-et}}(X_{\bar{s}}, \mathbb{Q}_l)$$

degenerates at E_2.

Proof. (9.13) follows from the log proper base change theorem of Nakayama ([Nak1, (5.1)]) and [D1, (1.5)]. □

Let us turn to the case where the characteristic of the base field is 0. Let $s = (\operatorname{Spec} \mathbb{C}, \mathcal{M}_s) := (\operatorname{Spec} \mathbb{C}, \mathbb{N} \oplus \mathbb{C}^*)$ be a log point. Let X/s be a proper SNCL analytic variety.

Theorem 9.14 [Log hard Lefschetz theorem over $\mathbb{C}$]. *Let X/s be a projective SNCL variety. Let $\lambda_\infty := c_{1,\infty}(\mathcal{L})$ be the log cohomology class of an ample invertible sheaf $\mathcal{L}$ on X. Then the left cup product of λ^j_∞ ($j \geq 0$)*

$$\lambda^j_\infty\colon H^{d-j}(X_\infty, \mathbb{Q}) \longrightarrow H^{d+j}(X_\infty, \mathbb{Q})(j) \tag{9.14.1}$$

is an isomorphism of mixed Hodge structures.

Proof. (9.14) follows from (9.9) and [SaM, (4.2.2)]. □

The Hodge symmetry holds for a proper smooth variety over $\mathbb{C}$ ([D1, (5.3)]). However the log Hodge symmetry for a proper SNCL variety over $\mathbb{C}$ does not hold in general by (7.6) (5). On the other hand, this holds for a projective SNCL variety over $\mathbb{C}$:

Corollary 9.15. *Let X/s be a projective SNCL variety of pure dimension d. Then the log Hodge symmetry for X holds*: $\dim_{\mathbb{C}} H^j(X, \Lambda^i_{X/\mathbb{C}}) = \dim_{\mathbb{C}} H^i(X, \Lambda^j_{X/\mathbb{C}})$ ($i, j \in \mathbb{N}$)

Proof. By (9.14), we have an isomorphism

$$\lambda^j_\infty\colon H^{d-j}_{\text{log-dR}}(X/\mathbb{C}) \xrightarrow{\ \sim\ } H^{d+j}_{\text{log-dR}}(X/\mathbb{C})(j).$$

Because λ^j_∞ is a morphism of mixed Hodge structures, λ^j_∞ is strictly compatible with the log Hodge filtration by [D2, (2.3.5)]. Moreover, the log Hodge-de Rham spectral sequence

$$E_1^{ij} = H^j(X, \Lambda^i_{X/\mathbb{C}}) \Longrightarrow H^{i+j}_{\text{log-dR}}(X/\mathbb{C}) \tag{9.15.1}$$

degenerates at E_1 by [FN, (3.12)] and by mixed Hodge theory [D3, (8.1.9) (v)]. (As mentioned in §2, the $\mathbb{Q}$-structure of the Steenbrink complex in [St2] alone is incomplete, though we can deduce the degeneration at E_1 of (9.15.1) from his incomplete $\mathbb{Q}$-structure; we can also prove the degeneration at E_1 of (9.15.1) by the method of Deligne-Illusie [DI2, (2.7)] and by [Kk1, (4.12) (3)].) Hence λ^j_∞ induces an isomorphism $H^{d-j-i}(X, \Lambda^i_{X/\mathbb{C}}) \xrightarrow{\sim} H^{d-i}(X, \Lambda^{i+j}_{X/\mathbb{C}})$. By the log Serre duality of Tsuji ([Ts2, (2.21)]), $H^{d-i}(X, \Lambda^{i+j}_{X/\mathbb{C}})$ and $H^{d-(d-i)}(X, \Lambda^{d-i-j}_{X/\mathbb{C}})$ are dual. Hence $\dim_{\mathbb{C}} H^{d-j-i}(X, \Lambda^i_{X/\mathbb{C}}) = \dim_{\mathbb{C}} H^i(X, \Lambda^{d-j-i}_{X/\mathbb{C}})$. □

We should remark the following:

PROPOSITION 9.16. *The following hold*:

(1) *Let X/s be a proper SNCL analytic variety. Assume that each irreducible component of $\overset{\circ}{X}$ is Kähler or algebraic. If X is the special fiber of a proper analytic strict semistable family over a unit disk such that the generic fibers are Kähler or algebraic, then the log Hodge symmetry for X holds*: $\dim_{\mathbb{C}} H^j(X, \Lambda^i_{X/\mathbb{C}}) = \dim_{\mathbb{C}} H^i(X, \Lambda^j_{X/\mathbb{C}})$ $(i, j \in \mathbb{N})$.

(2) *Let X be the special fiber of an algebraic proper strict semistable family over a discrete valuation ring with residue field of characteristic* 0. *Then the log Hodge symmetry holds.*

PROOF. (1): Let Δ be a unit disk and Δ^* the punctured disk. Let $\mathcal{X}$ be a proper analytic strict semistable family over Δ such that the special fiber is X. Let t be an element of Δ^* and let $\mathcal{X}_t$ be the generic fiber of $\mathcal{X}$ over t. Since $\Omega^i_{\mathcal{X}/\Delta}(\log X)$ $(i \in \mathbb{N})$ is a locally free $\mathcal{O}_{\mathcal{X}}$-module, we have $\dim_{\mathbb{C}} H^j(X, \Lambda^i_{X/\mathbb{C}}) \geq$ $\geq \dim_{\mathbb{C}} H^j(\mathcal{X}_t, \Omega^i_{\mathcal{X}_t/\mathbb{C}})$ by the upper semicontinuity theorem of Grauert ([Gra, Satz 3, p. 290] for the analytic case and [EGA III-2, (7.7.5)] for the algebraic case). As in the proof of (9.15), the log Hodge de Rham spectral sequence of X degenerates at E_1 by the assumptions since the singular cohomologies of the intersections of the irreducible components of X have pure Hodge structures (see [D1, (5.3)] for the algebraic case). Furthermore, the classical Hodge de Rham spectral sequence of $\mathcal{X}_t$ degenerates at E_1 by the assumptions (see [loc. cit.] or [DI2, (2.7)] for the algebraic case). Moreover, we have $\dim_{\mathbb{C}} H^h(X, \Lambda^\bullet_{X/\mathbb{C}}) = \dim_{\mathbb{C}} H^h(\mathcal{X}_t, \Omega^\bullet_{\mathcal{X}_t/\mathbb{C}})$ $(h \in \mathbb{N})$ by the proof of [St1, (2.18)].

Hence we have $\dim_{\mathbb{C}} H^j(X, \Lambda^i_{X/\mathbb{C}}) = \dim_{\mathbb{C}} H^j(\mathcal{X}_t, \Omega^i_{\mathcal{X}_t/\mathbb{C}})$. Thus the log Hodge symmetry reduces to the classical Hodge symmetry.

(2): By the Lefschetz principle, we may assume that the residue field is $\mathbb{C}$. By using Neron's blow up ([A, (4.6)], [SGA 7-I, I (0.5)]), we may assume that the semistable family comes from a semistable family over a smooth henselian $\mathbb{Q}[t]_{(t)}$-algebra R which is essentially of finite type over $\mathbb{Q}[t]_{(t)}$. Here we say that a family over R is semistable if it is locally defined by an equation $x_0 \cdots x_r - t$ ($r \in \mathbb{N}$) with independent variables $x_0, \ldots, x_r$ over R (I have learnt this notion from K. Fujiwara.). Extending a scalar extension $\mathbb{Q} \subset \mathbb{C}$, we may assume that Spec(R) is a smooth curve over $\mathbb{C}$. Now (2) follows from (1) and GAGA. □

PROBLEM. How does the log Hodge symmetry for a proper SNCL variety X/s fail? For example, is the difference $\dim_{\mathbb{C}} H^j(X, \Lambda^i_{X/\mathbb{C}}) - \dim_{\mathbb{C}} H^i(X, \Lambda^j_{X/\mathbb{C}})$ bounded for all proper SNCL varieties X over s of a fixed dimension $d \geq 2$? (If $d = 1$, then the equality $\dim_{\mathbb{C}} H^1(X, \mathcal{O}_X) = \dim_{\mathbb{C}} H^0(X, \Lambda^1_{X/\mathbb{C}})$ follows from the log Serre duality of Tsuji [Ts2, (2.21)].)

In order that the Problem make sense, the condition "the fixed dimension" above is necessary:

PROPOSITION 9.17. *Let X/s be the proper SNCL surface in* (6.5)*, where $\kappa = \mathbb{C}$. Let n be a positive integer and let X^n be the n-times product of X. Then* $\dim_{\mathbb{C}} H^0(X^n, \Lambda^1_{X^n/\mathbb{C}}) = ng$ *and* $\dim_{\mathbb{C}} H^1(X^n, \mathcal{O}_{X^n}) = n(g+1)$.

PROOF. The log Hodge de Rham spectral sequence

$$E_1^{ij} = H^j(X^n, \Lambda^i_{X^n/\mathbb{C}}) \Longrightarrow H^{i+j}_{\text{log-dR}}(X^n/\mathbb{C})$$

degenerates at E_1. By (7.6) (5), we obtain $\dim_{\mathbb{C}} H^0(X, \Lambda^1_{X/\mathbb{C}}) = g$ and $\dim_{\mathbb{C}} H^1(X, \mathcal{O}_X) = g + 1$. By the log Künneth formula (cf. [B, V Théorème 4.2.1] and [Kk1, (6.12)] for the (log) crystalline case),

$$H^1_{\text{log-dR}}(X^n/\mathbb{C}) = (H^1_{\text{log-dR}}(X/\mathbb{C}))^{\oplus n}. \tag{9.17.1}$$

Taking the log Hodge filtration on (9.17.1), we obtain (9.17) (cf. [B, p. 379]). □

10. First log Chern classes over $\mathbb{C}$

In this section we establish a relationship between the first log Chern class ((9.7.5)) and El-Zein's Chern class ([E]) of an ample invertible sheaf on a

SNCL analytic variety X over $s := (\operatorname{Spec}\mathbb{C}, \mathcal{M}_s) := (\operatorname{Spec}\mathbb{C}, \mathbb{N}\oplus\mathbb{C}^*)$. By abuse of notation, we sometimes omit the symbol $\circ$ in the notation $\overset{\circ}{X}$ below.

Let $c_{1,\mathrm{ldR}}$ be the following composite morphism

$$c_{1,\mathrm{ldR}} \colon \mathcal{O}_X^* \overset{d\log}{\longrightarrow} \Lambda^1_{X/\mathbb{C}} \overset{\subset}{\longrightarrow} \Lambda^\bullet_{X/\mathbb{C}}\{1\} \tag{10.0.1}$$

of complexes. Then $c_{1,\mathrm{ldR}}$ induces a morphism of cohomologies, which is denoted by the same symbol $c_{1,\mathrm{ldR}}$:

$$c_{1,\mathrm{ldR}} \colon H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \Lambda^\bullet_{X/\mathbb{C}}). \tag{10.0.1; H}$$

We also have a morphism

$$c_{1,\mathrm{ldRSZ}} \colon \mathcal{O}_X^* \overset{c_{1,\mathrm{ldR}}}{\longrightarrow} \Lambda^\bullet_{X/\mathbb{C}}\{1\} \underset{\simeq}{\overset{(d\log t\wedge *)\{1\}}{\longrightarrow}} A^\bullet_{X/\mathbb{C}}\{1\} \tag{10.0.2}$$

of complexes. In particular, we have a morphism

$$c_{1,\mathrm{ldRSZ}} \colon H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, A^\bullet_{X/\mathbb{C}}) \tag{10.0.2; H}$$

of cohomologies.

On the other hand, as in [E, II (3.15)], we can define a class $c_{1,\mathbb{C},\mathrm{EZ}}(\mathcal{L})$ of an ample invertible sheaf $\mathcal{L}$ on $\overset{\circ}{X}$ in $H^2(X, A^\bullet_{X/\mathbb{C}})$ by using the Steenbrink complex $A^\bullet_{X/\mathbb{C}}$. For the completeness of this paper, we recall El-Zein's construction. By using the argument in [loc. cit], we shall define $c_{1,\mathbb{C},\mathrm{EZ}}(\mathcal{L})$ for a more general invertible sheaf than an ample one.

DEFINITION 10.1. Let x be a point of $\overset{\circ}{X}$. Let D be an effective Cartier divisor on $\overset{\circ}{X}$. We say that D *meets X transversally* at x if there exist independent variables $z_0, \ldots, z_d$ such that there exists an isomorphism $\mathcal{O}_{X,x} \simeq \mathbb{C}\{z_0, \ldots, z_d\}/(z_0 \cdots z_r)$ and $D|_{(X,x)}$ is defined by $z_{r+1} = 0$. We say that D *meets X transversally* if D meets X transversally at any point x of $\overset{\circ}{X}$.

Note that the scheme $\operatorname{Supp} D$ with the restriction of the log structure of X is a SNCL variety over s. By abuse of notation, we denote $\operatorname{Supp} D$ only by D in the following.

Assume that an effective Cartier divisor D on X meets X transversally. Consider the following exact sequence

$$0 \longrightarrow \Lambda^\bullet_{X/\mathbb{C}}(1) \longrightarrow \Lambda^\bullet_{X/\mathbb{C}}(\log D)(1) \overset{(2\pi\sqrt{-1})^{-1}\mathrm{Res}}{\longrightarrow} \Lambda^\bullet_{D/\mathbb{C}}\{-1\} \longrightarrow 0. \tag{10.1.1}$$

(Though we can formulate the logarithmic differential forms $\Lambda^\bullet_{X/\mathbb{C}}(\log D)$ in terms of the log structure in the theory of Fontaine-Illusie-Kato ([Kk1]), we suppress the formulation here.) The boundary morphism induces a morphism

$$\delta_D\colon H^0(D,\Lambda^\bullet_{D/\mathbb{C}})\longrightarrow H^2(X,\Lambda^\bullet_{X/\mathbb{C}})(1). \tag{10.1.2}$$

Let $X=\bigcup_{i\in I}X_i$ be an open covering of X such that there exists a local equation $t_i=0$ $(i\in I)$ of $D\cap \overset{\circ}{X}_i$. Then $(2\pi\sqrt{-1})^{-1}\mathrm{Res}(d\log t_i\otimes 2\pi\sqrt{-1})=1$. Following the convention on the sign of torsors in [SGA $4\frac{1}{2}$, Cycle 1.1] and using a standard argument, we have the following formula in $H^2(X,\Lambda^\bullet_{X/\mathbb{C}})$:

$$\begin{aligned} c_{1,\mathrm{ldR}}(\mathcal{O}_X(D))\otimes 2\pi\sqrt{-1} &= [\{d\log(t_j^{-1}(t_i^{-1})^{-1})\}]\otimes 2\pi\sqrt{-1} \\ &= [\{d\log t_i - d\log t_j\}]\otimes 2\pi\sqrt{-1} \\ &= -\delta_D(1). \end{aligned} \tag{10.1.3}$$

Following [E, I (3.3.1)] and [SZ, (5.5)], consider a sheaf

$$A^{ij}_{X/\mathbb{C}}(\log D):=\Omega^{i+j+1}_{X/\mathbb{C}}(\log\mathcal{M}_X)(\log D)/P^{\mathcal{M}_X}_j\Omega^{i+j+1}_{X/\mathbb{C}}(\log\mathcal{M}_X)(\log D)\quad (i,j\in\mathbb{N})$$

on X, where $P^{\mathcal{M}_X}_j$ is the weight filtration with respect to the log structure $\mathcal{M}_X$ of X ([St2, p. 113]). Then we obtain the following double complex $A^{\bullet\bullet}_{X/\mathbb{C}}(\log D)$ with boundary morphisms

$$\begin{array}{ccc} A^{i,j+1}_{X/\mathbb{C}}(\log D) & & \\ {\scriptstyle (-1)^i d\log t\wedge}\uparrow & & \\ A^{ij}_{X/\mathbb{C}}(\log D) & \xrightarrow{(-1)^{j+1}d} & A^{i+1,j}_{X/\mathbb{C}}(\log D) \end{array} \tag{10.1.4}$$

and the associated single complex $A^\bullet_{X/\mathbb{C}}(\log D):=s(A^{\bullet\bullet}_{X/\mathbb{C}}(\log D))$. Here t is a global section of $\mathcal{M}_s$ whose image in $\Gamma(s,\mathcal{M}_s/\mathcal{O}_s^*)$ is the generator. Then we have the following exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^\bullet_{X/\mathbb{C}}(1) & \longrightarrow & A^\bullet_{X/\mathbb{C}}(\log D)(1) & \xrightarrow{(2\pi\sqrt{-1})^{-1}\mathrm{Res}} & A^\bullet_{D/\mathbb{C}}\{-1\} & \longrightarrow & 0 \\ & & {\scriptstyle d\log t\wedge}\uparrow & & {\scriptstyle d\log t\wedge}\uparrow & & {\scriptstyle d\log t\wedge}\uparrow & & \\ 0 & \longrightarrow & \Lambda^\bullet_{X/\mathbb{C}}(1) & \longrightarrow & \Lambda^\bullet_{X/\mathbb{C}}(\log D)(1) & \xrightarrow{(2\pi\sqrt{-1})^{-1}\mathrm{Res}} & \Lambda^\bullet_{D/\mathbb{C}}\{-1\} & \longrightarrow & 0. \end{array} \tag{10.1.5}$$

Thus we obtain a morphism

$$\partial_D\colon H^0(D,A^\bullet_{D/\mathbb{C}})\longrightarrow H^2(X,A^\bullet_{X/\mathbb{C}})(1). \tag{10.1.6}$$

El Zein's Chern class $c_{1,\mathbb{C},\mathrm{EZ}}(\mathcal{O}_X(D))$ of $\mathcal{O}_X(D)$ is, by definition, $-\partial_D(1)$. By

(10.0.2; H), (10.1.3) and (10.1.5), we have the following formula

$$c_{1,\mathrm{ldRSZ}}(\mathcal{O}_X(D)) \otimes 2\pi\sqrt{-1} = c_{1,\mathbb{C},\mathrm{EZ}}(\mathcal{O}_X(D)). \tag{10.1.7}$$

Let $\mathcal{L}$ be an invertible sheaf on $\overset{\circ}{X}$ such that $\mathcal{L} \simeq \mathcal{O}_X(D)$, where D is an effective Cartier divisor on $\overset{\circ}{X}$ which meets X transversally. Set

$$c_{1,\mathbb{C},\mathrm{EZ}}(\mathcal{L}) := c_{1,\mathbb{C},\mathrm{EZ}}(\mathcal{O}_X(D)). \tag{10.1.8}$$

Then, by (10.1.7), $c_{1,\mathbb{C},\mathrm{EZ}}(\mathcal{L})$ is independent of the choice of D.

Let $\{X_m\}_{m=1}^n$ be the irreducible components of $\overset{\circ}{X}$. Let $\mathcal{L}$ be a very ample invertible sheaf on $\overset{\circ}{X}$; the SNC analytic variety $\overset{\circ}{X}$ is the associated analytic space of a projective scheme $\overset{\circ}{X}{}^{\mathrm{pr}}$ over $\mathbb{C}$. Let $\overset{\circ}{X}{}^{\mathrm{pr}}_m$ be an irreducible component of $\overset{\circ}{X}{}^{\mathrm{pr}}$ such that $(\overset{\circ}{X}{}^{\mathrm{pr}}_m)_{\mathrm{an}} = X_m$. Let $\iota \colon \overset{\circ}{X}{}^{\mathrm{pr}} \overset{\subset}{\longrightarrow} \mathbb{P}^n_{\mathbb{C}}$ be the closed immersion defined by the very ample invertible sheaf on $\overset{\circ}{X}{}^{\mathrm{pr}}$ whose analytification is $\mathcal{L}$. Then, by using Bertini's theorem ([Ha2, II (8.18)]), we can find a hyperplane H' such that H' meets any intersection $\overset{\circ}{X}{}^{\mathrm{pr}}_{m_1} \cap \cdots \cap \overset{\circ}{X}{}^{\mathrm{pr}}_{m_r}$ ($r \in \mathbb{Z}_{>0}$, $1 \leq m_1 < \cdots < m_r \leq n$) transversally in $\mathbb{P}^n_{\mathbb{C}}$. Set $H := H' \cap \overset{\circ}{X}$. Then H_{an} meets $\overset{\circ}{X}$ transversally in the sense of (10.1).

More generally, let $\mathcal{L}$ be an ample invertible sheaf on $\overset{\circ}{X}$. Let l be a positive integer such that $\mathcal{L}^{\otimes l}$ is very ample. Let H be as in the last paragraph with respect to $\mathcal{L}^{\otimes l}$. Then we define $c_{1,\mathbb{C},\mathrm{EZ}}(\mathcal{L})$ in $H^2(X, A^{\bullet}_{X/\mathbb{C}})(1)$ by the following formula:

$$c_{1,\mathbb{C},\mathrm{EZ}}(\mathcal{L}) := l^{-1} c_{1,\mathbb{C},\mathrm{EZ}}(\mathcal{O}_X(H)). \tag{10.1.9}$$

By using the formula (10.1.7), it is easy to check that this definition is independent of the choice of l. Moreover, by (10.1.7) again, we have the following formula:

$$c_{1,\mathrm{ldRSZ}}(\mathcal{L}) \otimes 2\pi\sqrt{-1} = c_{1,\mathbb{C},\mathrm{EZ}}(\mathcal{L}). \tag{10.1.10}$$

Summing up, we obtain the following:

Proposition 10.2. *The following hold*:

(1) *Let $\mathcal{L}$ be an invertible sheaf on $\overset{\circ}{X}$ such that there exists an effective Cartier divisor D on $\overset{\circ}{X}$ such that $\mathcal{L} \simeq \mathcal{O}_X(D)$, and such that D meets X transversally. Then $c_{1,\mathrm{ldRSZ}}(\mathcal{L}) \otimes 2\pi\sqrt{-1} = c_{1,\mathbb{C},\mathrm{EZ}}(\mathcal{L})$.*

(2) *Let $\mathcal{L}$ be an ample invertible sheaf on $\overset{\circ}{X}$. Then $c_{1,\mathrm{ldRSZ}}(\mathcal{L}) \otimes 2\pi\sqrt{-1} =$ $= c_{1,\mathbb{C},\mathrm{EZ}}(\mathcal{L})$.*

Next, we consider the "$\mathbb{Q}$-structure" of $c_{1,\mathrm{ldRSZ}}$.

By (9.7.4) we have a morphism

$$\mathcal{O}_X^* \longrightarrow \mathbb{C}(1)[1] \tag{10.2.1}$$

and hence a morphism

$$H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{C})(1). \tag{10.2.1;H}$$

Let $\mathcal{L} = \mathcal{O}_X(D)$ be an effective invertible sheaf on $\overset{\circ}{X}$. We follow the convention on the sign of torsors in [SGA $4\frac{1}{2}$, Cycle 1.1]. Let $\{X_i\}_i$ be an open covering of X such that D is defined by an analytic equation $t_i = 0$. Set $X_{ij} := X_i \cap X_j$ and $X_{ijk} := X_i \cap X_j \cap X_k$. We may assume that a branch $\log(t_{ij}) \in \Gamma(Y_{ij}, \mathcal{O}_{Y_{ij}})$ of t_{ij} is defined. The line bundle $\mathcal{O}_X(D)$ defines a cocycle $\{t_{ij}\} \in \check{H}^1(X, \mathbb{G}_m)$ defined by $t_{ij} := t_i/t_j$. The 2-cocycle defined by the image of $\{t_{ij}\}$ by the boundary morphism of the exponential sequence $0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathcal{O}_X \overset{\exp}{\longrightarrow} \mathcal{O}_X^* \longrightarrow 0$ is $u_{ijk} = \log t_{jk} - \log t_{ik} + \log t_{ij} \in \mathbb{Z}(1)_{X_{ijk}}$. Consider the Čech double complex replacing $K^U(\Omega_Y^\bullet)$ in (4.1.2) by $\Lambda_X^\bullet$. By the same proof as that of (4.1), we have $[\{d\log t_{ij}\}] = [\{u_{ijk}\}]$ in $\check{H}^2(X, \Lambda_{X/\mathbb{C}}^\bullet)$. Hence we have the following commutative diagram

$$\begin{array}{ccc} H^1(X, \mathcal{O}_X^*) & \longrightarrow & H^2(X, \mathbb{Z})(1) \\ \| & & \downarrow \\ H^1(X, \mathcal{O}_X^*) & \xrightarrow{c_{1,\mathrm{ldR}}} & H^2(X, \Lambda_{X/\mathbb{C}}^\bullet)(1). \end{array} \tag{10.2.2}$$

Moreover, by (10.1.2;H) and by the functoriality, we have a morphism

$$c_{1,\mathbb{C}}\colon H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X_\infty, \mathbb{C})(1). \tag{10.2.3}$$

Then the morphism $c_{1,\infty}$ (resp. $c_{1,\mathbb{C}}$) is induced by the left (resp. right) vertical morphism in the following commutative diagram

$$\begin{array}{ccc} J_{\mathbb{Q}}^\bullet(1)[1] & \longrightarrow & J_{\mathbb{C}}^\bullet(1)[1] \\ \uparrow & & \uparrow \\ \mathbb{Q}_X(1)[1] & \longrightarrow & \mathbb{C}_X(1)[1] \\ \uparrow & & \uparrow \\ \mathcal{O}_X^* & = & \mathcal{O}_X^*. \end{array} \tag{10.2.4}$$

Therefore, by (9.7.4) and by a composite morphism $\mathbb{Q}_X \longrightarrow J_{\mathbb{Q}}^\bullet \longrightarrow A^\bullet(J_{\mathbb{Q}}^*)$, we have a morphism

$$c_{1,\mathbb{Q},\mathrm{SZ}}\colon H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, A^\bullet(J_{\mathbb{Q}}^*))(1). \tag{10.2.5}$$

Analogously, by a composite morphism $\mathbb{C}_X \longrightarrow J^\bullet_{\mathbb{C}} \longrightarrow A^\bullet(J^*_{\mathbb{C}})$, we have a morphism

(10.2.6) $$c_{1,\mathbb{C},\mathrm{SZ}}\colon H^1(X,\mathcal{O}^*_X) \longrightarrow H^2(X, A^\bullet(J^*_{\mathbb{C}}))(1).$$

THEOREM 10.3. (1) *Under the identification* $H^2(X, A^\bullet(J^*_{\mathbb{Q}}))$ *with* $H^2(X_\infty, \mathbb{Q})$ *by the isomorphism* $\mu_{\mathbb{Q}}\colon H^2(X_\infty,\mathbb{Q}) = H^2(X, J^\bullet_{\mathbb{Q}}) \xrightarrow{\sim} H^2(X, A^\bullet(J^*_{\mathbb{Q}}))$, $c_{1,\mathbb{Q},\mathrm{SZ}} = c_{1,\infty}$.

(2) *The composite of* $c_{1,\mathbb{Q},\mathrm{SZ}}$ *with a natural morphism* $H^2(X, A^\bullet(J^*_{\mathbb{Q}}))(1) \longrightarrow$ $\longrightarrow H^2(X, A^\bullet(J^*_{\mathbb{C}}))(1)$ *is equal to* $c_{1,\mathbb{C},\mathrm{SZ}}$.

(3) *By the identification* (3.10.1; H), $c_{1,\mathbb{C},\mathrm{SZ}} = c_{1,\mathrm{ldRSZ}}$.

PROOF. (1): (1) immediately follows from the definition of $c_{1,\mathbb{Q},\mathrm{SZ}}$.

(2): (2) follows from the following two obvious commutative diagrams:

(10.3.1) $$\begin{array}{ccc} \mathcal{O}^*_X & \longrightarrow & \mathbb{Q}_X(1)[1] \\ \| & & \downarrow \\ \mathcal{O}^*_X & \longrightarrow & \mathbb{C}_X(1)[1], \end{array}$$

(10.3.2) $$\begin{array}{ccccc} \mathbb{Q}_X(1) & \longrightarrow & J^\bullet_{\mathbb{Q}} & \longrightarrow & A^\bullet(J^*_{\mathbb{Q}}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{C}_X(1) & \longrightarrow & J^\bullet_{\mathbb{C}} & \longrightarrow & A^\bullet(J^*_{\mathbb{C}}). \end{array}$$

(3): (3) follows from the definition (10.0.2) and the commutativity of (10.2.4), (10.2.2), and (3.10.1; H). □

11. Appendix.

In this Appendix we determine the log $\star$-adic ($\star = l, p, \infty$) cohomologies of $X = E_{q,C}(\mathcal{L})$ in §6 with $\deg \mathcal{L} = 0$.

THEOREM 11.1. *Assume that* $\deg \mathcal{L} = 0$. *Then the following hold*:

(1) *The log $\star$-adic* ($\star = l, p, \infty$) *cohomologies of X are the following*:

(a) $\underline{H}^0_{\log,\star}(X) = \underline{\mathbf{1}}_\star$.

(b) $P_0\underline{H}^1_{\log,\star}(X) = \underline{\mathbf{1}}_\star$, $\mathrm{gr}^P_1 \underline{H}^1_{\log,\star}(X) = \underline{H}^1_\star(C)$, $\mathrm{gr}^P_2 \underline{H}^1_{\log,\star}(X) = \underline{\mathbf{1}}_\star(-1)$.

(c) $P_1\underline{H}^2_{\log,\star}(X) = \underline{H}^1_\star(C)$, $\mathrm{gr}^P_2 \underline{H}^2_{\log,\star}(X) = \underline{\mathbf{1}}_\star(-1)^{\oplus 2}$, $\mathrm{gr}^P_3 \underline{H}^2_{\log,\star}(X) = \underline{H}^1_\star(C)(-1)$.

(d) $P_2\underline{H}^3_{\log,\star}(X) = \underline{\mathbf{1}}_\star(-1)$, $\mathrm{gr}^P_3 \underline{H}^3_{\log,\star}(X) = \underline{H}^1_\star(C)(-1)$, $\mathrm{gr}^P_4 \underline{H}^3_{\log,\star}(X) = \underline{\mathbf{1}}_\star(-2)$.

(e) $\underline{H}^4_{\log,\star}(X) = \underline{\mathbf{1}}_\star(-2)$.

(2) *The monodromy filtration and the weight filtration on* $H^h_{\log,\star}(X)$ $(h = 0, 1, 2, 3, 4)$ *coincide.*

PROOF. We may assume that κ is algebraically closed. Here we give the proof only for the l-adic case.

(1): Let $E^{\bullet\bullet}_{r,l}$ $(r \in \mathbb{Z}_{\geq 1} \cup \{\infty\})$ be the E_r-term of the weight spectral sequence $(2.0.7; l)$. By the same proof as that of (7.1), we obtain $E^{01}_{2,l} = H^1_{\mathrm{et}}(C, \mathbb{Z}_l) = E^{11}_{2,l}$, $E^{-13}_{2,l} = H^1_{\mathrm{et}}(C, \mathbb{Z}_l)(-1) = E^{03}_{2,l}$. Let $\{X_i\}_{i \in \mathbb{Z}/e}$ be the irreducible components of X such that $C_i := X_i \cap X_{i+1} \neq \phi$ and let $\pi_i : X_i \longrightarrow C$ be the projection. As in (6.5.1),

$$\pi_i^*(\mathcal{L}) = \mathcal{O}_{X_i}(\pm(C_i - C_{i-1})).$$

This element in $H^2(X_i, \mathbb{Z}_l)$ is the zero since the first chern class of $\mathcal{L}$ in $H^2(C, \mathbb{Z}_l)$ is so by the assumption. Hence, as in the proof of (6.5), we see that $E^{-12}_{2,l} = \mathbb{Z}_l(-1)$. By using the variant of (7.2) (Set $\alpha = 0$ in (7.2), though this α is a zero-divisor.), we see that $E^{12}_{2,l} = \mathbb{Z}_l(-1)$ and $E^{02}_{2,l} = \mathbb{Z}_l(-1)^{\oplus 2}$ by (7.2.4) and (7.2.7). Since $E^{11}_{2,l}$ and $E^{12}_{2,l}$ are free $\mathbb{Z}_l$-modules, we have $E^{-12}_{2,l} = E^{-12}_{\infty,l}$, $E^{11}_{2,l} = E^{11}_{\infty,l}$, $E^{-13}_{2,l} = E^{-13}_{\infty,l}$ and $E^{12}_{2,l} = E^{12}_{\infty,l}$ by the E_2-degeneration of $(2.0.8; l)$ ([Nak3, (2.1)]). (In the p-adic case, we use the E_2-degeneration of $(2.0.8; p)$ ([Nakk3, (3.6)]).) Thus we obtain (1).

(2): Let $E^{\bullet\bullet}_{r,l}$ $(r \in \mathbb{Z}_{\geq 1} \cup \{\infty\})$ be the E_r-term of the weight spectral sequence $(2.0.8; l)$. As in (8.4), we have only to prove that $v_l : E^{-12}_{2,l} \longrightarrow E^{10}_{2,l}(-1)$ is an isomorphism. Since $E^{-12}_{2,l}$ and $E^{10}_{2,l}$ are 1-dimensional, we have only to prove that v_l is not the zero morphism. By (1), $E^{-12}_{2,l} = \{(a, a, \ldots, a) | a \in \mathbb{Q}_l(-1)\} \subset H^0_{\mathrm{et}}(X^{(2)}, \mathbb{Q}_l)(-1)$. On the other hand, the boundary morphism

$$d^{00}_1 : E^{00}_{1,l} = H^0_{\mathrm{et}}(X^{(1)}, \mathbb{Q}_l) \longrightarrow E^{10}_{1,l} = H^0_{\mathrm{et}}(X^{(2)}, \mathbb{Q}_l)$$

is given by $(a_1, a_2, \ldots, a_e) \longmapsto (a_2 - a_1, a_3 - a_2, \ldots, a_1 - a_e)$. Since $v_l : E^{-12}_{1,l} \longrightarrow E^{10}_{1,l}(-1)$ is the identity by the same argument as that of [Nakk3, (11.7)] for the p-adic case, the image of a non-zero element of $E^{-12}_{2,l}$ in $E^{10}_{2,l}$ is not the zero. $\square$

To the reader, we leave the statements and the proofs of the analogous formulae of (7.4), (7.5), (7.6), (7.10), (7.11) and (7.12) for X with $\deg \mathcal{L} = 0$. (For the proof of the analogues of (7.10) and (7.11), we use [Ue, (6.1) 2)] instead of [loc. cit., (6.1) 3)] used in the proofs of (7.10) and (7.11).)

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