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Generic 2-Coverings of Finite Groups of Lie Type (*).

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Dedicated to G. Zappa for his 90^{th} birthday

ABSTRACT - In [10] it was shown by R.H. Dye that in a symplectic group $G := Sp_{2\ell}(2^f) = \mathrm{Iso}\,(V, \langle .,. \rangle)$ defined over a finite field of characteristic 2 every element in G stabilizes a quadratic form of maximal or non-maximal Witt index inducing the bilinear form $\langle .,. \rangle$. Thus G is the union of the two G-conjugacy classes of subgroups isomorphic to $O^+_{2\ell}(2^f)$ and $O^-_{2\ell}(2^f)$ embedded naturally. In this paper we classify all finite groups of Lie type (G, F) with this generic 2-covering property (Thm. A). In particular, we will show that there exists also an interesting example in characteristic 3, i.e., in the finite group of Lie type $G := F_4(3^f)$ every element in G is conjuagte to an element of the subgroup $B_4(3^f) \leq F_4(3^f)$ or of the subgroup $3.{}^3D_4(3^f) \leq F_4(3^f)$.

1. Introduction.

Let G be a finite group and let $H, K \lneq G$ be proper subgroups of G. The pair of subgroups $\{H, K\}$ is called a 2-*covering* of G, if every element in G is conjugate to an element in H or K, i.e.,

$$G = \bigcup_{g \in G} H^g \cup \bigcup_{g \in G} K^g. \tag{1.1}$$

If G is a Frobenius group with complement H and kernel K then $\{H, K\}$ is obviously a 2-covering of G. Another example of a finite group with a 2-

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covering is the group $G := SL_2(q)$, $q = p^f$. It can be 2-covered by the pair $\{H, K\}$, where H is the normalizer of a p-Sylow subgroup, and K is the normalizer of a maximal non-split torus. A more sophisticated class of examples is the following: Let $\mathbb{F}_{2^f}$ be a finite field of characteristic 2 and order 2^f, let V be an $\mathbb{F}_{2^f}$-vector space of dimension 2ℓ and let $\langle .,. \rangle: V \otimes V \to \mathbb{F}_{2^f}$ be a non-degenerate symmetric bilinear form on V. In particular, $G := \mathrm{Iso}\,(V, \langle .,. \rangle) = Sp_{2\ell}(2^f)$ is the symplectic group of degree 2ℓ defined over the finite field $\mathbb{F}_{2^f}$. Let $Q: V \to V$ be a quadratic form of maximal Witt index inducing $\langle .,. \rangle$, and let $Q': V \to V$ be a quadratic form of non-maximal Witt index inducing $\langle .,. \rangle$. Then $M_1 := \mathrm{Iso}\,(V, Q) \simeq O^+_{2\ell}(2^f)$ and $M_2 := \mathrm{Iso}\,(V, Q') \simeq O^-_{2\ell}(2^f)$ are maximal subgroups of G, and it was shown by R.H. Dye ([10]) that every element of G is contained in a G-conjugate of M_1 or M_2, i.e., $\{M_1, M_2\}$ is a 2-covering of $G = Sp_{2\ell}(2^f)$.

The study of covering properties of finite groups was originally motivated by number theoretic questions. In [15], W. Jehne discovered that Kronecker equivalence of number fields is related to the existence of a certain type of covering of a finite group (cf. [17, §1C]). Therefore, one calls the proper subgroup U of the finite group G a *covering subgroup of* G, if $G = \bigcup_{a \in \mathrm{Aut}\,(G)} U^a$. The study of finite groups with a covering subgroup was initiated by R. Brandl (cf. [2]), who also conjectured that a finite group with solvable covering subgroup must itself be solvable. Finally, J. Saxl proved this conjecture in showing that a non-abelian finite simple group does not have a covering subgroup (cf. [19]).

The examples mentioned above show that in contrast to the non-existence of covering subgroups there are non-abelian finite simple groups possessing 2-coverings. Special classes of finite simple groups with this covering property have been investigated in [3] and [4]. For finite groups of Lie type there exists a more particular type of covering – a generic 2-covering – which allows to analyze the class of groups with this covering property systematically. The main purpose of this paper will be to classify all generic 2-coverings of finite groups of Lie type.

Let G be a simple algebraic group defined over the algebraically closed field $\mathbb{F}$. A closed subgroup H of G is called *of maximal rank*, if it contains a maximal torus T of G. By a *finite group of Lie type* (1) (G, F) we understand a simple algebraic group G defined over the algebraic closure $\bar{\mathbb{F}}_p$ of a finite field of characteristic p together with a Frobenius morphism $F: G \to G$.

(1) The finite group G_F of elements fixed under F is usually called the finite group of Lie type.

For such a group a pair of proper closed F-stable subgroups $\{H,K\}$ will be called a *generic 2-covering of* (G,F), if

(i) every F-stable maximal torus T is G_F-conjugate to a subgroup of H or K,

(ii) $\{H_F,K_F\}$ is a 2-covering of G_F,

where $\_{F}$ denotes the subgroup of elements fixed under F. In [25, §1] it was shown that there is a canonical one-to-one correspondence between G_F-conjugacy classes of F-stable reductive subgroups H of maximal rank of G and W-orbits of certain cosets of Weyl subgroups of W, where W denotes the Weyl group of G. We refer to the W-orbit of this coset as the *type of* H. This correspondence extends the well-known correspondence between G_F-conjugacy classes of F-stable maximal tori, and F-conjugacy classes in the Weyl group of G (cf. [8, Prop. 3.3.3]). The main purpose of this article is to establish the following theorem (Thm. 4.5, § 5).

THEOREM A. *Let* (G,F) *be a finite simple group of Lie type, and let* $\{H,K\}$ *be a generic* 2*-covering of* (G,F)*.*

(a) *Assume that for one of the subgroups - say* H *- the connected component* H° *is not reductive. Then* (G,F) *is of type* A_1, A_2, A_3 *or* ${}^2A_3$, H *is maximal parabolic, and* K° *is reductive. Moreover, the types of* $H/R_u(H)$ *and* K° *are as listed in Table 1.1. Every pair of* F*-stable subgroups* $\{H,K\}$, H *parabolic,* $K:=N_G(K^\circ)$, *satisfying the conditions of Table 1.1 defines a generic* 2*-covering of* (G,F)*.*

TABLE 1.1. – Parabolic-type generic 2-coverings

type (G,F)		*type* $H/R_u(H)$	*type* K°
$A_\ell,\ \ell\geq 1$	A_1	1^W	2^W
	A_2	$W(A_1)^W$	3^W
	A_3	$W(A_2)^W$	$2.W(A_1^2)^W$
${}^2A_3$		$2.W(A_1^2)^W$	$2.W(A_2)^W$

(b) *Assume that* H° *and* K° *are reductive subgroups. Then the types of* (G,F)*,* H *and* K *are as listed in Table 1.2 modulo interchanging the role of* H *and* K*. Moreover, the characteristic of the algebraically closed field* $\mathbb{F}$ *must be as indicated in the* 5^{th}*-column. Every pair of* F*-stable subgroups*

$\{H,K\}$, $H := N_G(H^\circ)$, $K := N_G(K^\circ)$, *satisfying the conditions of Table 1.2, defines a generic* 2*-covering of* (G,F).

TABLE 1.2. – Reductive-type generic 2-coverings

type (G,F)		*type* H°	*type* K°	*conditions*
$A_\ell,\ \ell\geq 1$	A_1	1^W	2^W	$p=2$
	A_2	$W(A_1)^W$	3^W	$p=3$
${}^2A_\ell,\ \ell\geq 2$	${}^2A_2$	$2.W(A_1)^W$	6^W	$p=3$
$C_\ell,\ \ell\geq 2$	C_2	$W(A_1^2)^W$	$2.W(A_1^2)^W$	$p=2$
	C_3	$W(C_2\times A_1)^W$	$3.W(A_1^3)^W$	$p=3$
	C_ℓ	$W(\tilde{D}_\ell)^W$	$2.W(\tilde{D}_\ell)^W$	$p=2$
F_4		$W(B_4)^W$	$3.W(D_4)^W$	$p=3$
G_2		$W(A_2)^W$	$2.W(A_2)^W$	$p=2$

Here the connected component of an algebraic group X is denoted by X°. In Table 1.2 we used the convention that a rootsystem with a $\tilde{}$ consists of short roots. Moreover, for a subrootsystem Ψ of the rootsystem Φ of G and $n\in\mathbb{N}$, $n.W(\Psi)$ will stand for a coset $n_\Psi.W(\Psi)$, $n_\Psi\in N_W(W(\Psi))$, with $n_\Psi.W(\Psi)\in N_W(W(\Psi))/W(\Psi)$ of order n. While this notations is not inambiguous, it will suffice for our needs, and from the context it will always be clear which particular coset is meant to be considered.

It should be remarked that Theorem A is independent of the isogeny class of G. The proof of Theorem A is organized in several steps. Using the results in [25, § 1] one concludes that a pair of reductive subgroups $\{H,K\}$ satisfying property (i) corresponds to a certain type of covering of the Weyl group (coset) which will be called an *admissible* 2*-coset covering* (cf. Thm. 2.2). All such coverings can be classified (cf. Thm. 3.4, Table 3.1).

In [20] closed subgroups of a simple algebraic group of positive dimension containing a regular unipotent element were classified. Fortunately, there are few examples of these groups which occur also in Table 3.1. Apart from 3 exceptions this exclusion principle will lead precisely to the examples listed in Table 1.1 and 1.2 (cf. Thm. 4.5).

In order to obtain a full classification of all generic 2-coverings, we have to show that a pair of subgroups $\{H,K\}$ with the properties described in

Table 1.1 and 1.2 is indeed a generic 2-covering. This will be done in section 5, 6, 7 by a case-by-case analysis. Unfortunately, for some examples we could not find a geometric argument, and hence in those cases the only way in proving the 2-covering property is by brute force (cf. § 6, § 7).

In a subsequent paper [5] we will determine all 2-coverings of classical groups of Lie type using the classification of finite simple group. It turns out that there are also 2-coverings of classical groups of Lie type which are non-generic.

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2. F-stable reductive subgroups of maximal rank

2.1 – *Weyl groups and admissible cosets*

By $W(\Phi) \leq \mathrm{Iso}(\mathbb{E}, (.,.))$ we denote the *Weyl group* of the finite crystallographic rootsystem $\Phi \subset \mathbb{E}$ in the euclidean space $(\mathbb{E}, (.,.))$. Let $\Delta \subset \Phi$ be a basis of Φ, and let $\gamma \in \mathrm{Aut}(W(\Phi))$ be a graph automorphism of $W(\Phi)$ leaving Δ invariant. In particular,

$$\widetilde{W} := \langle \gamma \rangle . W(\Phi) \leq Gl_{\mathbb{R}}(\mathbb{E}). \tag{2.1}$$

The coset $\gamma.W(\Phi)$ carries a natural right $W(\Phi)$-action given by

$$(\gamma.w)^x := \gamma.(\gamma(x^{-1})wx), \quad x, w \in W, \tag{2.2}$$

where $\gamma(x) = \gamma^{-1}x\gamma$. We call the coset $\gamma.W(\Phi) \subseteq \widetilde{W}$ an *admissible* $W(\Phi)$-*coset*.

Let (G, F) be a finite group of Lie type, i.e., G is a simple algebraic group defined over $\bar{\mathbb{F}}_p$, the algebraic closure the finite field $\mathbb{F}_p$, and $F: G \to G$ is a Frobenius morphism. Let $B \leq G$ be an F-stable Borel subgroup of the algebraic group G containing the F-stable maximal torus T. Then $W := N_G(T)/T$ is the Weyl group of a finite irreducible crystallographic rootsystem Φ, i.e., $W = W(\Phi)$. Let $\Delta \subset \Phi$ be a basis of Φ corresponding to the Borel subgroup B. The Frobenius morphism $F: G \to G$ is acting on Φ leaving Δ invariant, and thus corresponds to a graph automorphism $\gamma \in \mathrm{Aut}(W)$. On the contrary, every pair (γ, Φ), where Φ is a finite irreducible crystallographic rootsystem and γ is a graph automorphism of $W(\Phi)$, defines a finite group of Lie type $(G, F_0\gamma)$ for every standard Frobenius morphism F_0 (cf. [8, § 1.1.18]).

We use the same symbols which are used in the classification of finite groups of Lie type for the definition of the *type* of an admissible coset $\gamma.W(\Phi)$, i.e., we say that $\gamma.W(\Phi)$ is of type ${}^2A_\ell$, if Φ is a rootsystem of type A_ℓ and if γ is the graph automorphism of order 2, etc.

2.2 – *Subrootsystems and saturated subrootsystems*

Every subrootsystem $\Psi \subseteq \Phi$ of the finite crystallographic root system Φ is again crystallographic. Moreover, there exists an easy algorithm described in [6, p. 8] to determine all maximal subrootsystems of Φ.

For two roots $\alpha, \beta \in \Phi$ one defines the *root cone* through α and β by

$$C(\alpha,\beta) := \{\, k_1.\alpha + k_2.\beta \mid k_1, k_2 \in \mathbb{N}\} \cap \Phi. \tag{2.3}$$

A subrootsystem Ψ is called *saturated*, if for all α, $\beta \in \Psi$, $C(\alpha,\beta) \subseteq \Psi$. Obviously, if in the finite irreducible crystallographic root system Φ all roots are of the same length, every subrootsystem $\Psi \subseteq \Phi$ is saturated (cf. [12, § 9.4]). But for example, the subrootsystem $\tilde{A}_1^\ell$ consisting of all short roots in $\Phi(B_\ell)$ is not saturated. It is an easy exercise to show that a subrootsystem Ψ is saturated, if and only if for every pair of roots $\alpha, \beta \in \Psi$, Ψ contains also the α-string through β.

Let $\mathfrak{L} = \mathfrak{L}(\Phi)$ be the simple finite-dimensional $\mathbb{C}$-Lie algebra with rootsystem Φ and Cartan decomposition

$$\mathfrak{L} = \mathfrak{C} \oplus \coprod_{\alpha \in \Phi} \mathfrak{L}_\alpha. \tag{2.4}$$

Then $\mathfrak{L}(\Psi) := \mathfrak{C} \oplus \coprod_{\alpha \in \Psi} \mathfrak{L}_\alpha$ is a Lie-subalgebra, if and only if $\Psi \subseteq \Phi$ is saturated (cf. [7, § 4.2], [12, Prop. 8.4(e)]).

Let G be the simple simply-connected algebraic group defined over $\mathbb{C}$ with rootsystem Φ, and let $T \leq G$ be a maximal torus. From Chevalley's commutator formula (cf. [7, Thm. 5.2.2]) one concludes that $H := \langle T, U_\alpha \mid \alpha \in \Psi \rangle$ is a reductive subgroup with rootsystem of type Ψ, if and only if $\Psi \subseteq \Phi$ is saturated.

Let G be the simple simply-connected algebraic group defined over $\bar{\mathbb{F}}_p$ with rootsystem Φ, and let $T \leq G$ be a maximal torus. We call the subrootsystem $\Psi \subseteq \Phi$ *p-saturated*, if $H := \langle T, U_\alpha \mid \alpha \in \Psi \rangle$ is a reductive subgroup with rootsystem of type Ψ. One has the following:

PROPOSITION 2.1. *Let Φ be a finite irreducible crystallographic rootsystem and let $\Psi \subseteq \Phi$ be a subrootsystem.*

(a) *If $\Psi \subseteq \Phi_{long}$ consists only of long roots, Ψ is saturated.*

(b) *If Ψ is saturated, it is p-saturated for all primes p.*

(c) *Assume Φ has two root lengths r_l, r_s and that $p \neq r_l/r_s$. Then, if Ψ is p-saturated, it is also saturated.*

(d) *If Φ has two root lengths r_l, r_s and $p = r_l/r_s$, every subrootsystem is p-saturated.*

Proof. Let $\alpha, \beta \in \Psi$ be linearly independent roots, and let $\Gamma \subseteq \Phi$ be the subrootsystem generated by $C(\alpha, \beta)$. If Γ is of type A_2 or $A_1 \times A_1$, Γ coincides with the rootsystem spanned by α and β. The same is true if Γ is of type B_2 or G_2 and α and β are long roots (cf. [7, p. 46, fig. 1]). This yields (a). (b) is a direct consequence of Chevalley's commutator formula. Assume that Ψ is p-saturated. The rootsystem Γ coincides with the rootsystem spanned by α and β unless Γ is of type B_2 or G_2 and α and β are short roots. If $p = r_l/r_s$, the constants $C_{i,j,\alpha,\beta,}$ (cf. [7, p. 62, p. 77]) vanish whenever $i\alpha + j\beta$ is a long root. This yields (d). However, if $p \neq r_l/r_s$, they do not vanish. Thus, if Ψ is p-saturated, $C(\alpha, \beta) \subseteq \Psi$ must hold which implies (c). $\square$

2.3 – *γ-Subrootsystems*

Let γ be a graph automorphism of the finite irreducible crystallographic rootsystem Φ and let $\Psi \subseteq \Phi$ be a subrootsystem of Φ. Then Ψ is called a *γ-subrootsystem*, if there exists an element $w \in W(\Phi)$ such that ${}^{\gamma}\Psi = {}^{w}\Psi$.

Let G a simple algebraic group with rootsystem of type Φ defined over $\bar{\mathbb{F}}_p$, and let $F\colon G \to G$ be a Frobenius morphism acting on Φ through γ. Let $T \leq G$ be a maximal torus and let $\Psi \subseteq \Phi$ be a subrootsystem of Φ. In [25, Prop. 2] it was shown that G contains an F-stable reductive group $H = \langle T, U_\alpha \mid \alpha \in \Psi \rangle^g$, $g \in G$, if and only if Ψ is p-saturated and if Ψ is a γ-subrootsystem. Additionally, for all twisted types the γ-subrootsystems were determined.

2.4 – *Admissible 2-coset coverings for Weyl groups*

Let Φ be a finite irreducible crystallographic rootsystem, let $W = W(\Phi)$ be its Weyl group, and let $\gamma\colon W \to W$ be as in § 2.1. Let G_F be a finite group of Lie type, where G is a simple algebraic group defined over $\bar{\mathbb{F}}_p$ with rootsystem Φ and assume that the Frobenius morphism $F\colon G \to G$

is acting on Φ through γ. Let B be an F-stable Borel subgroup of G containg the maximal torus T and let U_a, $a \in \Phi$, denote the root groups for T in B and B^-.

For a p-saturated γ-subrootsystem $\Psi \subseteq \Phi$ and $n_\Psi \in N_{\widetilde{W}}(W(\Psi)) \cap \cap\, \gamma.W(\Phi)$, the W-orbit $(n_\Psi.W(\Psi))^W$ of the coset $n_\Psi.W(\Psi)$ corresponds to a G_F-conjugacy class of finite reductive groups H_F, where $H:= \langle T, U_a \mid a \in \Psi \rangle^g$ is F-stable, $F(g)g^{-1} \in N_G(T)$, $\pi(F(g)g^{-1}) = n_\Psi$, $\pi: N_G(T) \to W$ the canonical projection. This follows from [25, Prop. 4] bearing in mind that left translation with γ, $\gamma.\_: W \to \gamma.W$, is a bijection of right W-sets, where W carries the right W-action given by $w^x = \gamma(x^{-1})wx$, $x, w \in W$. In order to analyze the problem further we make the following definition:

DEFINITION 2.1. Let $\gamma.W$ be an admissible W-coset of the finite Weyl group $W = W(\Phi)$. Let Ψ, $\Xi \subset \Phi$ be proper γ-subrootsystems of Φ, and let n_Ψ, $n_\Xi \in \gamma.W$ be elements with the following properties:

(i) $n_\Psi \in N_{\widetilde{W}}(W(\Psi))$, $n_\Xi \in N_{\widetilde{W}}(W(\Xi))$,
(ii)

$$(\gamma.W)^W = (n_\Psi.W(\Psi))^W \cup (n_\Xi.W(\Xi))^W. \tag{2.5}$$

Then we call $(n_\Psi.W(\Psi), n_\Xi.W(\Xi))$ an *admissible 2-coset covering* of $\gamma.W$.

From the definition and [25, § 1] one concludes the following:

THEOREM 2.2. *Let (G, F) be a finite group of Lie type with rootsystem Φ whose F-action is given by $\gamma \in$ Aut(Φ). Put $W := W(\Phi)$. Let H and K be F-stable reductive groups of maximal rank with G_F-conjugacy class corresponding to the W-conjugacy class $(n_\Psi.W(\Psi))^W$ and $(n_\Xi.W(\Xi))^W$, respectively, n_Ψ, $n_\Xi \in \gamma.W$ (cf.* [25, Prop. 4]*).*

(a) *$(n_\Psi.W(\Psi), n_\Xi.W(\Xi))$ is an admissible 2-coset covering of $\gamma.W$, if and only if every F-stable maximal torus T is G_F-conjugate to a torus in H or K.*

(b) *If $(n_\Psi.W(\Psi), n_\Xi.W(\Xi))$ is an admissible 2-coset covering of $\gamma.W$, every semisimple element $s \in G_F$ is G_F-conjugate to an element in H_F or K_F.*

PROOF. For (a) see [25, Prop. 4, Thm. 5]. Since every semisimple element $s \in G_F$ is contained in a maximal F-stable torus, (b) is a direct consequence of (a). $\square$

3. Admissible 2-coset coverings for Weyl groups

3.1 – *The Coxeter class*

Let Φ be a finite irreducible crystallographic rootsystem and let $W = W(\Phi)$ denote its Weyl group. There exists a very particular W-conjugacy class w^W, the class of *Coxeter elements* (cf. [7, § 10.4]). Elements of this class have the following well-known property:

PROPOSITION 3.1. *Let Φ be a finite irreducible crystallographic rootsystem and let $W = W(\Phi)$ denote its Weyl group. Let w be a Coxeter element of W and let $\Psi \subseteq \Phi$ be a subrootsystem of Φ, such that $w \in W(\Psi)$. Then $\Psi = \Phi$.*

PROOF. We denote by $\mathbb{E}_{\mathbb{C}} := \mathbb{E} \otimes \mathbb{C}$ the complexification of $\mathbb{E}$ and denote by $(.,.)_{\mathbb{C}}$ the standard W-invariant hermitian form induced by $(.,.)$. Let $\Psi \subseteq \Phi$ be a subrootsystem such that $w \in W(\Psi)$.

The Coxeter element $w \in W$ is an element of order $k := |\Phi|/\ell$, where $\ell = \mathrm{rk}\,(\Phi)$ is the rank of Φ. It is also regular, i.e., there exists a primitive k^{th}-root of unity $\xi \in \mathbb{C}^*$ such that the eigenspace $V_\xi(w)$ of w on $\mathbb{E}_{\mathbb{C}}$ for the eigenvalue ξ is one-dimensional, and all eigenvectors $v \in V_\xi(w) \setminus \{0\}$ are regular, i.e., they are not invariant under any non-trivial element of W (cf. [23, Thm. 4.2]).

Let $\Psi := \bigcup_{i=1}^{r} \Psi_i$ be the decomposition of Ψ in irreducibility components, and let $V_i := \mathrm{span}_{\mathbb{C}}\,(\Psi_i)$, $V_0 := (V_1 \oplus \cdots \oplus V_r)^\perp$, where $\perp$ denotes the orthogonal complement with respect to the hermitian form $(.,.)_{\mathbb{C}}$. The subspaces V_i, $i = 0,..,r$, are $W(\Psi)$-invariant subspaces, and hence in particular w-invariant. Since w does not have the eigenvalue 1 (cf. [7, Prop. 10.5.6]), $V_0 = 0$ and Ψ must be of rank $\ell = \mathrm{rk}\,(\Phi)$.

Assume $r \geq 2$. As $V_\xi(w)$ coincides with the ξ-Fitting component of w on $\mathbb{E}_{\mathbb{C}}$, it must be contained in one of the V_i, $i \in \{1,..,r\}$. Hence $W(\Psi_j)$, $j \neq i$, fixes $V_\xi(w)$, a contradiction. Hence Ψ must be an irreducible subrootsystem of maximal rank.

The element $w \in W(\Psi)$ is also regular in $W(\Psi)$, and hence its order $o(w)$ is a regular number of Ψ. In particular, $o(w) \leq |\Psi|/\ell$ (cf. [23, § 5]). This yields $\Psi = \Phi$. □

3.2 – *Conjugacy classes in $W(A_\ell)$*

The Weyl group $W = W(A_\ell)$ of type A_ℓ is isomorphic to $S_{\ell+1}$. The conjugacy classes of W can be parametrizised by the set of partitions of

$\ell+1$. The mapping is given by assigning each element $w \in W$ its cycle type of the action on the set $\{1,..,\ell+1\}$. The Coxeter elements are the elements of cycle type $(\ell+1)$. We will make use of the following property:

PROPOSITION 3.2. (a) *Let $U \leq S_n$, $n \geq 2$, be a subgroup containing elements ρ, σ, τ of cycle type $(2,1,..,1)$, (n) and $(1,n-1)$. Then $U = S_n$.*

(b) *Let $U \leq S_n$, $n \geq 4$, be a subgroup containing elements ρ, σ, τ of cycle type $(2,1,..,1)$, $(1,n-1)$ and $(2,n-2)$. Then $U = S_n$.*

PROOF. (a) The subgroup U is 2-transitive on $\{1,..,n\}$, and thus in particular primitive. Thus the claim follows from Jordan's theorem (cf. [14, Satz 4.5(b)]).

(b) From the hypothesis $n \geq 4$, one concludes that U is transitive, and the cycle type of σ yields that U is 2-transitive. The claim then follows by the same argument as used in (a). $\square$

3.3 – *Conjugacy classes in $W(B_\ell)$*

Let $W = W(B_\ell)$ be the Weyl group associated to the rootsystem of type B_ℓ. In this case W is acting faithfully on the set $\mathfrak{E}_\ell := \{\pm e_i \mid 1 \leq i \leq \ell\}$ of positive and negative vectors of an orthonormal basis $e_1,..,e_\ell$ of the euclidean space $(\mathbb{E},(.,.))$. Assigning every element $w \in W$ the cycle type of this action yields a parametrization of W-conjugacy classes by partitions $(\_\,|\,\_)$ with positive and negative parts. The W-conjugacy class containing the unit element $1_W \in W$ corresponds to the $\pm$-partition $(1,..,1|)$; the Coxeter class corresponds to the $\pm$-partition $(|\ell)$.

The Weyl group $W(B_\ell)$ contains the element $w_0 \in W(B_\ell)$ acting as $-id_{\mathbb{E}}$ on $\mathbb{E}$. Multiplication with w_0 yields an automorphism $w_0.\_ : W^W \to W^W$ of order 2. It corresponds to the automorphism of the set of $\pm$-partitions of ℓ fixing blocks of even length and changing the sign of blocks of odd length. The conjugacy class consisting of the element w_0 corresponds to the the $\pm$-partition $(|1,..,1)$.

The subrootsystem $\Phi_{long} \subset \Phi$ consisting of long roots is of type D_ℓ. A conjugacy class is contained in the Weyl subgroup $W(\Phi_{long})$, if and only if the number of negative blocks is even. The conjugacy class of long reflections, i.e., reflections corresponding to long roots, corresponds to the $\pm$-partition $(2,1,..,1|)$. The conjugacy class of short reflections, i.e., reflections corresponding to short roots, corresponds to the $\pm$-partition $(1,..,1|1)$.

There exists a unique W-orbit of subrootsystems Ψ of type $A_{\ell-1}$ consisting of long roots. The W-conjugacy classes of elements being contained in a W-conjugate of $W(\Psi)$ correspond to the $\pm$-partitions with trivial negative part $(\_\,|)$.

Let $E \lhd W$ denote the normal subgroup generated by all reflections corresponding to short roots. The canonical projection $\pi\colon W \to W/E \simeq S_\ell$ is mapping reflections to reflections, and thus in particular Weyl subgroups to Weyl subgroups. Omitting the "$|$" in the notation describes its effect on the conjugacy classes. Let $\Psi \subseteq \Phi$ be a subrootsystem such that $\pi(W(\Psi)) = W/E$. Then Ψ is of type $A_{\ell-1}$, D_ℓ or B_ℓ. This is a consequence of the following more general property:

PROPOSITION 3.3. *Let Φ be a crystallographic finite rootsystem with at most two root lengths. Let $\tau\colon W(\Phi) \to S_m$, $m \geq 2$, be a surjective morphism mapping long reflections to long reflections. Then Φ contains a subrootsystem of type A_{m-1} consisting of long roots.*

PROOF. Let $R(S_m)$ denote the set of reflections in S_m, and let R_l denote the set of long reflections in $W(\Phi)$. Since $R(S_m)$ is a single S_m-conjugacy class, the hypothesis implies that τ induces a surjective map $\tau_R\colon R_l \to$ $\to R(S_m)$. Let $\Sigma' = \{\, \sigma_i' \mid 1 \leq i \leq m-1 \,\} \subset R(S_m)$ be a simple generation system of S_m, and let $\Sigma = \{\sigma_i \mid 1 \leq i \leq m-1\,\}$ be a preimage of Σ' under τ_R. For any pair (σ_i, σ_j) of long reflections the order of its product satisfies $o(\sigma_i\sigma_j) \in \{1, 2, 3\}$, and either two of these numbers are coprime. This implies that the Coxeter matrices of Σ and Σ' coincide. This yields the claim. $\square$

3.4 – *Conjugacy classes in $W(D_\ell)$*

Conjugacy classes in the Weyl group $W = W(D_\ell)$ can be best understood to think of $W(D_\ell)$ as the Weyl subgroup of $W(B_\ell)$ being generated by long reflections. For $w \in W(D_\ell)$ let $t(w)$ denote the $\pm$-partition associated to w being considered as an element in $W(B_\ell)$. We call $t(w)$ the *type* of w. Obviously, $W(D_\ell)$-conjugate elements have the same type, and a $\pm$-partition is the type of an element $w \in W(D_\ell)$, if and only if the number of negative blocks is even. If ℓ is odd, one has $W(B_\ell) = \langle w_0 \rangle W(D_\ell)$. Hence for every $\pm$-partition λ with an even number of negative blocks, $t^{-1}(\{\lambda\})$ consists of a single $W(D_\ell)$-conjugacy class. When ℓ is even, $t^{-1}(\{\lambda\})$ consists of either 2 or 1 $W(D_\ell)$-conjugacy classes.

Restriction of the mapping $\pi\colon W(B_\ell) \to S_\ell$ of section 3.3 yields a surjective mapping $\pi'\colon W(D_\ell) \to S_\ell$ with kernel $ker(\pi') = W(A_1^{\ell-1})$. Note that the only proper rootsubsystems Ψ of Φ for which $\pi'(W(\Psi)) = S_\ell$ are of type $A_{\ell-1}$ (cf. Prop. 3.3). There is a unique $W(D_\ell)$-orbit of such subrootsystems in case that ℓ is odd, and there are two such $W(D_\ell)$-orbits for ℓ even.

3.5 – *The classification of admissible 2-coset coverings for Weyl groups*

The information we have collected in the previous subsection can now be used to classify all admissible 2-coset coverings:

THEOREM 3.4. *Let Φ be a finite irreducible crystallographic rootsystem, let $W = W(\Phi)$ be its Weyl group and let $\gamma.W$ be an admissible 2-coset (cf. 2.1). Assume that $(n_\Psi.W(\Psi), n_\Xi.W(\Xi))$ is an admissible 2-coset covering of $\gamma.W$. Then the types of $(n_\Psi.W(\Psi), n_\Xi.W(\Xi))$ - or $(n_\Xi.W(\Xi), n_\Psi.W(\Psi))$ - are as listed in Table 3.1. Moreover, every tuple $(n_\Xi.W(\Xi), n_\Psi.W(\Psi))$ given as in Table 3.1 is an admissible 2-coset covering of $\gamma.W$.*

TABLE 3.1. – Admissible 2-coset coverings

type $\gamma.W$		$n_\Psi.W(\Psi)$	$n_\Xi.W(\Xi)$
$A_\ell,\ \ell \geq 1$	A_1	1	2
	A_2	$W(A_1)$	3
	A_3	$W(A_2)$	$2.W(A_1^2)$
${}^2A_\ell,\ \ell \geq 2$	${}^2A_2$	$2.W(A_1)$	6
	${}^2A_3$	$2.W(A_2)$	$2.W(A_1^2)$
$B_\ell,\ \ell \geq 2$	B_2	$W(\tilde{A}_1^2)$	$2.W(\tilde{A}_1^2)$
	B_3	$W(B_2 \times \tilde{A}_1)$	$3.W(\tilde{A}_1^3)$
	B_4	$W(B_3 \times \tilde{A}_1)$	$2.W(B_2^2)$
	B_ℓ	$W(D_\ell)$	$2.W(D_\ell)$
F_4		$W(B_4)$	$3.W(D_4)$
		$W(C_4)$	$3.W(\tilde{D}_4)$
G_2		$W(A_2)$	$2.W(A_2)$
		$W(\tilde{A}_2)$	$2.W(\tilde{A}_2)$

The case Φ of type C_ℓ leads to the same examples as for Φ of type B_ℓ with short and long roots exchanged. Therefore, we have not listed these cases in Table 3.1 explicitly.

PROOF. Assume that $(n_\Psi.W(\Psi), n_\Xi.W(\Xi))$ is an admissible 2-coset covering of $\gamma.W$. We denote by $\bar{n}_\Psi$ and $\bar{n}_\Xi$ the canonical homomorphic images of n_Ψ and n_Ξ in $N_W(W(\Psi))/W(\Psi)$ and $N_W(W(\Xi))/W(\Xi)$, respectively. We may assume that $\gamma \in n_\Psi.W(\Psi)$. Thus in case that $\gamma = 1$, $n_\Xi.W(\Xi)$ contains a Coxeter element and thus $\bar{n}_\Xi \neq 1$ (cf. Prop. 3.1). The proof proceeds by a case-by-case analysis depending on the type of $\gamma.W$.

3.5.1 — $\gamma.W$ of type A_ℓ

For $\ell \leq 3$ an easy exercise shows that the cases listed in Table 3.1 are the only examples one has in this case. Since for $\ell \geq 4$ no proper Weyl subgroup of W is normal in W, $(W(\Psi), N_W(W(\Xi))$ is a 2-covering of W. From [3, Thm. 4.2] one concludes that this implies $\ell \leq 5$. It remains to analyze the cases $\ell = 4, 5$ explicitly.

Assume $\ell = 4$, i.e. $W = S_5$. For any proper non-trivial subrootsystem Ξ of Φ, $N_W(W(\Xi))/W(\Xi)$ is a 2-group. Hence $\Xi = \emptyset$ must be trivial, and n_Ξ is an element of order 5. The maximal subrootsystems of Φ are of type A_3 and $A_2 \times A_1$, and thus $W(\Psi)$ does not contain elements of cycle type $(2, 3)$ in the first case and does not contain elements of cycle type $(1, 4)$ in the latter, respectively. Hence admissible 2-coset coverings do not exist for $\ell = 4$.

Assume $\ell = 5$, i.e., $W = S_6$. There is one W-conjugacy class of Weyl subgroups $W(\Psi)^W$ containing a 5-cycle corresponding to the subrootsystems of type A_4. The group $W(\Psi)$ does not contain elements of cycle type $(2, 2, 2)$, $(3, 3)$ and (6). The only proper non-trivial subrootsystems Ξ for which $N_W(W(\Xi))$ contains elements of cycle type (6) are of type A_1^3 and A_2^2. However, for Ξ of type A_1^3 and $\bar{n}_\Xi$ of order 3, $n_\Xi.W(\Xi)$ does not contain elements of cycle type $(2, 2, 2)$; for Ξ of type A_2^2 and $\bar{n}_\Xi$ of order 2, $n_\Xi.W(\Xi)$ does not contain elements of cycle type $(3, 3)$. This implies that $n_\Xi.W(\Xi)$ must contain elements of cycle type $(1, 5)$. As it already contains a Coxeter element, one has $\Xi \neq \emptyset$. This implies that $N_W(W(\Xi)) = S_6$ (cf. Prop. 3.2(a)), and thus $W(\Xi) = W$, a contradiction. Hence admissible 2-coset coverings do not exist for $\ell = 5$.

3.5.2 — $\gamma.W$ of type ${}^2A_\ell$

In this case γ is an inner automorphism, and $\widetilde{W} = \langle \gamma \rangle.W = \langle -id_{\mathbb{E}} \rangle.W$, where $-id_{\mathbb{E}}$ denotes the endomorphism acting as -1 on $\mathbb{E}$. Thus if

$(n_\Psi.W(\Psi), n_\Xi.W(\Xi))$ is an admissible 2-coset covering of $\gamma.W$, the pair of admissible cosets $((-id_{\mathbb{E}})n_\Psi.W(\Psi), (-id_{\mathbb{E}})n_\Xi.W(\Xi))$ is an admissible 2-coset covering of W. Thus the claim can be deduced from 3.5.1.

3.5.3 — $\gamma.W$ of type B_ℓ (or C_ℓ)

By 3.5.1 and the previous discussion (cf. § 3.3), we have to consider three cases:

(1) $\pi(W(\Psi)) = W/E$,

(2) $\pi(n_\Xi.W(\Xi)) = W/E$,

(3) $\ell \leq 4$ and $(\pi(W(\Psi)), \pi(n_\Xi.W(\Xi))$ is an admissible 2-coset covering of S_ℓ.

Note that $\pi(n_\Xi.W(\Xi))$ is a group, if and only if $n_\Xi.W(\Xi) \cap E \neq \emptyset$. Hence $\pi(n_\Xi.W(\Xi)) = W/E$ implies $\pi(W(\Xi)) = W/E$.

Cases (1) and (2): The only proper subrootsystems Ψ of Φ satisfying (1) are of type D_ℓ and $A_{\ell-1}$ (cf. Prop. 3.3). A similar argument yields that (2) implies that Ξ is either of type D_ℓ or $A_{\ell-1}$. Hence in order to prove the claim it suffices to show that Ψ of type D_ℓ implies Ξ of type D_ℓ, and that Ξ of type D_ℓ implies Ψ of type D_ℓ.

Assume that $\Psi = \Phi_{long}$ is of type D_ℓ. Then $n_\Xi.W(\Xi)$ contains a Coxeter element and elements of the W-conjugacy classes $(2, 1, .., 1|1)$ and $(1|\ell - 1)$. Hence $\pi(n_\Xi.W(\Xi)) = W/E$ (cf. Prop. 3.2(a)). As $\pi(n_\Xi.W(\Xi)) = \pi(W(\Xi))$, the rootsystem Ξ must be of type $A_{\ell-1}$ or D_ℓ. Assume Ξ is of type $A_{\ell-1}$. As $w_0 \in N_W(W(\Xi))$ and $|N_W(W(A_{\ell-1}))/W(A_{\ell-1})| = 2$, one has $n_\Xi.W(\Xi) = w_0.W(\Xi)$. However, in this case $w_0.W(\Xi)$ does not contain short reflections corresponding to the $\pm$-partition $(1, .., 1|1)$ (cf. § 3.3). Thus $\Psi = \Phi_{long}$ implies $\Xi = \Phi_{long}$.

Assume that $\Xi = \Phi_{long}$ and that n_Ξ is inducing the graph automorphism of order 2. Then $n_\Xi.W(\Xi)$ does not contain any W-conjugacy class being contained in $W(\Phi_{long})$. In particular, $W(\Psi)$ contains elements of type $(2, 1, .., 1|1, 1), (1, \ell - 1|), (\ell|)$. Thus $\pi(W(\Psi)) = W/E$ (cf. Prop. 3.2(a)), and Ψ is either of type $A_{\ell-1}$ or D_ℓ. However, since $W(\Psi)$ contains elements of type $(2, 1, .., 1|1, 1)$ the first case is impossible. This yields the claim.

Case (3): The 3 examples for Φ of type A_ℓ, $\ell = 1, 2, 3$ yield the 3 additional cases for Φ of type B_ℓ, $\ell = 2, 3, 4$. A straightforward calculation shows that these are the only exceptions.

3.5.4 — $\gamma.W$ of type D_ℓ

By § 3.5.1 one has to consider three cases:

(1) $\pi(W(\Psi)) \simeq W/E'$,

(2) $\pi(n_\Xi.W(\Xi)) \simeq W/E'$,
(3) $\ell = 4$, $\pi(W(\Psi)) \simeq S_3$, $\pi(n_\Xi.W(\Xi)) = 2.W(A_1^2) \subseteq S_4$.

Case (1): The hypothesis implies that $W(\Psi)$ is of type $A_{\ell-1}$ (cf. Prop. 3.3). Hence $n_\Xi.W(\Xi)$ must contain an element of type $(1,..,1|1,1)$, and thus $\pi(n_\Xi.W(\Xi))$ is a group. Furthermore, $n_\Xi.W(\Xi)$ must contain elements of type $(|1,\ell-1)$, $(|2,\ell-2)$ and $(2,1..,1|1,1)$, and this implies that $\pi(n_\Xi.W(\Xi)) = W/E'$ (cf. Prop. 3.2(b)). So also $W(\Xi)$ must be of type $A_{\ell-1}$. But this yields that ℓ must be odd and that $n_\Xi.W(\Xi) = w_0.W(\Xi)$. However, in this case neither $W(\Psi)$ nor $n_\Xi.W(\Xi)$ contains elements of type $(1,..,1|1,1)$.

Case (2): The hypothesis implies that $\pi(W(\Xi)) = W/E'$, and thus Ξ is of type $A_{\ell-1}$, ℓ must be even $n_\Xi.W(\Xi) = w_0.W(\Xi)$. So $W(\Psi)$ must contain elements of type $(2,1,..,1|)$, $(|2,\ell-2)$ and $(1,\ell-1|)$. This yields that $\pi(W(\Psi)) = W/E'$ (cf. Prop. 3.2(b)) and thus Ψ must be of type $A_{\ell-1}$. As in the previous case neither $W(\Psi)$ nor $n_\Xi.W(\Xi)$ contain elements of type $(1,..,1|1,1)$. So an admissible 2-coset covering cannot exist in this case.

Case (3) can be excluded, since in this case $N_W(W(\Xi))$ is a 2-group, but must also contain a Coxeter element which is of order 6, a contradiction.

3.5.5 — $\gamma.W$ of type ${}^2D_\ell$

If ℓ is odd, $\gamma.W(\Phi) = w_0.W(\Phi) \subset W(B_\ell)$, and with the same argument as used in § 3.5.2 one concludes that there are no admissible 2-coset coverings in this case. So ℓ must be even. We may assume that $n_\Psi = \gamma$. Note that γ is of type $(1,..,1|1)$ considered as an element in $W(B_\ell)$. Hence $\pi(W(\Psi))$ is a group. By § 3.5.1, we have to consider 2 cases:

(1) $\pi(W(\Psi)) \simeq W/E'$,
(2) $\pi(n_\Xi.W(\Xi)) \simeq W/E'$,
(3) $\ell = 4$, $\pi(W(\Psi)) \simeq S_3$, $\pi(n_\Xi.W(\Xi)) = 2.W(A_1^2) \subseteq S_4$.

Case (1): The hypothesis implies that Ψ is of type $A_{\ell-1}$. However, for ℓ even there does not exist a γ-subrootsystem of type $A_{\ell-1}$ (cf. [25, § 2, (C)]).

Case (2): The hypothesis implies that $\pi(W(\Xi)) \simeq S_\ell$, and thus Ξ must be of type $A_{\ell-1}$. The argument which was used in case (1) yields that this is impossible.

Case (3): By hypothesis, $\pi(n_\Xi.W(\Xi))$ is not a group. Thus $n_\Xi.W(\Xi) \cap \cap E = \emptyset$. The hypothesis $\pi(\gamma.W(\Psi)) \simeq S_3$ yields that Ψ is of type A_2 and that γ is centralizing $W(\Psi)$. So the elements of $\gamma.W(\Xi)$ are of type $(1,1,1|1)$, $(1,2|1)$, $(3|1)$. Hence neither $\gamma.W(\Psi)$ nor $n_\Xi.W(\Xi)$ contains elements of type $(1|1,1,1)$, and thus an admissible 2-coset covering cannot exist in this case.

3.5.6 — $\gamma.W$ of type $^3D_4$

If one identifies the Weyl group W with the Weyl subgroup $W(D_4) \leq \leq W(F_4)$ associated to the subrootsystem of long roots in the rootsystem of type F_4, $\gamma.W = 3.W(D_4) \subset W(F_4)$ is a $W(D_4)$-coset in the Weyl group $W(F_4)$ of type F_4. Since $W(F_4)$ has an admissible 2-covering $(W(B_4), 3.W(D_4))$ (cf. § 3.5.11), one concludes that $3.W(D_4)$ contains Coxeter elements of the Weyl group $W(F_4)$ which are of order 12 and which are the twisted Coxeter elements in $3.W(D_4)$ (cf. [23, § 7]). Let $w \in \gamma.W$ be such an element being contained in $n_{\Xi}.W(\Xi)$. From [25, Prop. 7] one concludes that $\Xi = \emptyset$ must be trivial.

The proper γ-invariant subrootsystems of the rootsystem Φ are of type A_1^4, $A_1^3, A_2, A_1, \emptyset$ (cf. [25, § 2, (B)]). The coset $n_{\Psi}.W(\Psi)$ must intersect the two W-conjugacy classes of elements of order 3 being contained in $3.W(D_4)$ non-trivially. Hence $W(\Psi)$ cannot be a 2-group which leaves only the case Ψ of type A_2. In particular, one may assume that $n_{\Psi} = \gamma$ and that γ is centralizing $W(\Psi)$. It is straightforward, that $n_{\Psi}.W(\Psi)$ is a subset of a Weyl subgroup $W(A_2 \times \tilde{A}_2) \leq \leq W(F_4)$ of type $A_2 \times \tilde{A}_2$ in $W(F_4)$. Hence $n_{\Psi}.W(\Psi)$ has trivial intersection with the $W(F_4)$-conjugacy class containing $\gamma.(-id_{\mathbb{E}})$, a contradiction.

3.5.7 — $\gamma.W$ of type E_6

We may assume that $\Psi \subset \Phi$ is a maximal subrootsystem of Φ. The maximal subrootsystems of the rootsystem of type E_6 are of type D_5, $A_5 \times A_1$ and A_2^3. From Table 3.2 one concludes that $W(\Psi)$ has trivial intersection with a W-conjugacy class which is not the Coxeter class. This implies that Ξ is not empty.

Since proper Weyl subgroups of $W = W(\Phi)$ are not normal in W, $N_W(W(\Xi)) \neq W$. Hence it suffices to show that $W(\Phi)$ cannot be covered by the W-conjugates of $W(\Psi)$, Ψ a maximal subrootsystem of Φ, and a maximal subgroup $M < W$ containing $N_W(W(\Xi))$. The maximal subgroups $M < W$ of $W \simeq O_6^-(2)$ are known (cf. [9, p. 26]), and from their permutation characters χ_M corresponding to the left $\mathbb{C}[W]$ module $\mathrm{Ind}_M^W(\mathbb{C})$ one can determine the W-conjugacy classes they do not contain.

In Table 3.2 we have listed some W-conjugacy classes which have trivial intersection with some of the maximal subgroups, and also the decomposition of the permutation character χ_M in terms of the characters occuring in the character table in [9]. We have used the symbol $\times$ to express that the conjugacy class has non-trivial intersection with the maximal subgroup and the symbol $\varnothing$ in case the intersection is trivial. By $sgn\colon W \to \{\pm 1\}$ we denote the sign character.

TABLE 3.2. – Maximal subgroups and conjugacy classes of $O_6^-(2)$

maximal subgroup	χ_M	$9A$	$12A$	$8A$	$10A$	$12C$
$M_1 = W(D_5)$	$\chi_1 + \chi_4 + \chi_9$	$\varnothing$	$\varnothing$	$\times$	$\varnothing$	$\times$
$M_2 = W(A_5 \times A_1)$	$\chi_1 + \chi_8 + \chi_9$	$\varnothing$	$\varnothing$	$\varnothing$	$\times$	$\varnothing$
$M_3 = 3_+^{1+2} : 2S_4$	$\chi_1 + sgn \cdot \chi_7 + \chi_{10}$	$\times$	$\times$	$\times$	$\varnothing$	$\varnothing$
$M_4 = N_W(W(A_2^3))$	$\chi_1 + \chi_8 + \chi_{10}$	$\times$	$\varnothing$	$\varnothing$	$\varnothing$	$\times$
$M_5 = C_W(2A)$	$\chi_1 + \chi_9 + \chi_{10}$	$\varnothing$	$\times$	$\times$	$\varnothing$	$\varnothing$

Since M has to contain Coxeter elements (cf. Prop. 3.1) which correspond to the conjugacy class $12A$, one concludes from Table 3.2 that M has to be conjugate to either M_3 or M_5. Since M_1 and M_4 have trivial intersection with the conjugacy class $10A$, Ψ has to be of type $A_5 \times A_1$. Hence M cannot be conjugate to M_5, as this group has trivial intersection with the conjugacy class of type $9A$. But M cannot be of type M_3 either, since this group has trivial intersection with the conjugacy class of type $12C$.

3.5.8 — $\gamma.W$ of type ${}^2E_6$.

Since $\gamma.W = (-id_{\mathbb{E}}).W$, the same argument which was used in § 3.5.2 and the non-existence of admissible 2-coset covers for $W = W(E_6)$ (cf. § 3.5.7) show that admissible 2-coset covers do not exist in this case.

3.5.9 — $\gamma.W$ of type E_7

We may assume that Ψ is a maximal subrootsystem of Φ. Thus Ψ is of type E_6, A_7, $D_6 \times A_1$ or $A_5 \times A_2$. Let $\bar{\ } : W \to Sp_6(2)$ denote the canonical projection on the symplectic group of degree 6 over the field with 2 elements. Some information on maximal subgroups and conjugacy classes of the group $Sp_6(2)$ has been collected in Table 3.3 (cf. [9, p. 46]).

From Table 3.3 one concludes that there exists a W-conjugacy class which is not the Coxeter class having trivial intersection with $W(\Psi)$. Hence Ξ is non-trivial. Since proper Weyl subgroups of W are not normal, $N_W(W(\Xi))$ is contained in a maximal subgroup M of W containing $w_0 = -id_{\mathbb{E}}$. The same technique which was used in § 3.5.7 and the information listed in Table 3.3 show that either Ψ is of type E_6 and M is W-conjugate to $\langle w_0 \rangle.W(A_7)$, or Ψ is of type A_7 and M is W-conjugate to

TABLE 3.3. – Maximal subgroups and conjugacy classes of $Sp_6(2)$

maximal subgroup	$3B$	$7A$	$8A$	$9A$	$15A$
$\overline{M}_1 = \overline{W}(E_6)$	$\times$	$\varnothing$	$\times$	$\times$	$\varnothing$
$\overline{M}_2 = \overline{W}(A_7)$	$\varnothing$	$\times$	$\varnothing$	$\varnothing$	$\times$
$\overline{M}_3 = \overline{W}(D_6 \times A_1)$	$\varnothing$	$\varnothing$	$\times$	$\times$	$\varnothing$
$\overline{M}_4 = U_3(3) : 2$	$\times$	$\varnothing$	$\times$	$\varnothing$	$\varnothing$
$\overline{M}_5 = 2^6 : L_3(2)$	$\times$	$\times$	$\times$	$\varnothing$	$\varnothing$
$\overline{M}_6 = 2.[2^6] : (S_3 \times S_3)$	$\times$	$\varnothing$	$\times$	$\varnothing$	$\varnothing$
$\overline{M}_7 = \overline{W}(A_5 \times A_2)$	$\times$	$\varnothing$	$\varnothing$	$\varnothing$	$\times$
$\overline{M}_8 = S_2(8)$	$\times$	$\times$	$\varnothing$	$\times$	$\varnothing$

$\langle w_0 \rangle.W(E_6)$. Since w_0 is not contained in Weyl subgroups of type A_7 or E_6, $w_0 \in n_{\Xi}.W(\Xi)$, and we may assume that Ξ is of type A_7 or E_6, respectively. However, in the first case, neither $W(E_6)$ nor $w_0.W(A_7)$ contain Coxeter elements which are of order 18; in the second case neither $W(A_7)$ nor $w_0.W(E_6)$ contain elements of order 9. Hence admissible 2-coset coverings cannot exist.

3.5.10 — $\gamma.W$ of type E_8

Let $\chi\colon W \to \{\pm 1\}$ denote the sign character of the reflection group $W = W(E_8)$, and let $\bar{\ }\colon W \to O_8^+(2)$ denote the canonical projection on the orthogonal group of maximal Witt index of degree 8 over the field with 2 elements. As in the previous case we may assume that Ψ is a maximal subrootsystem of Φ. Hence Ψ is of type $E_7 \times A_1$, D_8, A_8, $E_6 \times A_2$ or A_4^2 (cf. [6]), and thus $\overline{W}(\Psi)$ is $\overline{W}$-conjugate to a subgroup of one of the maximal subgroups $\overline{M}_1$, $\overline{M}_4$, $\overline{M}_7$, $\overline{M}_{10}$, $\overline{M}_{15}$ of $\overline{W} = O_8^+(2)$ (cf. Table 3.4). We have chosen the enumeration of the $\overline{W}$-conjugacy classes of maximal subgroups in such a way that it coincides with its appearance in the list of maximal subgroups in [9, p. 85].

For each of the maximal subgroups $\overline{M}_1$, $\overline{M}_4$, $\overline{M}_7$, $\overline{M}_{10}$, $\overline{M}_{15}$, there exists one $\overline{W}$-conjugacy class which has trivial intersection with the maximal subgroup and which is not the homomorphic image of the Coxeter class (cf. Table 3.4). As $\chi(w) = 1$ for a Coxeter element w, this implies that Ξ is not empty.

Since W does not contain proper normal Weyl subgroups, $N_W(W(\Xi)) \neq W$ and $N_W(W(\Xi))$ is contained in a maximal subgroup $M < W$. Let $\overline{M}$ denote its homomorphic image in $\overline{W}$. From the previous remark one concludes that $\chi(M) = \{\pm 1\}$. Hence $\overline{M}$ is $\overline{W}$-conjugate to one of the maximal subgroups $\overline{M}_1$, $\overline{M}_4$, $\overline{M}_7$, $\overline{M}_{10}$, $\overline{M}_{13} = C_{\overline{W}}(2A)$, $\overline{M}_{14}$, $\overline{M}_{15}$. As $\overline{M}_{13}$ and $\overline{M}_{14}$ do not have elements of order 5, they cannot contain the homomorphic image of a Coxeter element whose order is divisible by 15. Hence $\overline{M}$ is $\overline{W}$-conjugate to one of the maximal subgroups listed in Table 3.4.

Assume that Ψ is of type $E_7 \times A_1$. Then $\overline{W}(\Psi)$ has trivial intersection with the $\overline{W}$-conjugacy classes $9B$ and $15B$. The first fact implies that $\overline{M}$ cannot be $\overline{W}$-conjugate to $\overline{M}_{15}$, while the latter implies that $\overline{M}$ cannot be $\overline{W}$-conjugate to $\overline{M}_1$, $\overline{M}_4$, $\overline{M}_7$ or $\overline{M}_{10}$ (cf. Table 3.4). The cases Ψ of type D_8, A_8, $E_6 \times A_2$ and A_4^2 can be ruled out by a similar argument and the information listed in Table 3.4.

TABLE 3.4. – Maximal subgroups and conjugacy classes of $O_8^+(2)$

maximal subgroup	$7A$	$9A$	$9B$	$15B$
$\overline{M}_1 = \overline{W}(E_7 \times A_1)$	$\times$	$\times$	$\varnothing$	$\varnothing$
$\overline{M}_4 = \overline{W}(D_8)$	$\times$	$\varnothing$	$\varnothing$	$\varnothing$
$\overline{M}_7 = \overline{W}(A_8)$	$\times$	$\varnothing$	$\times$	$\varnothing$
$\overline{M}_{10} = \overline{W}(E_6 \times A_2)$	$\varnothing$	$\times$	$\times$	$\varnothing$
$\overline{M}_{15} = N_{\overline{W}}(\overline{W}(A_4^2))$	$\varnothing$	$\varnothing$	$\varnothing$	$\times$

3.5.11 — $\gamma.W$ of type F_4

The maximal subrootsystems of Φ are of type B_4, C_4, $C_3 \times A_1$, $B_3 \times \tilde{A}_1$, $\tilde{A}_3 \times A_1$, $A_3 \times \tilde{A}_1$ and $A_2 \times \tilde{A}_2$. Note that since there exists a surjective morphism $\beta\colon W(F_4) \to S_3^2$ with $ker(\beta)$ a 2-group, $W = W(F_4)$ has precisely 3-conjugacy classes σ_i^W, $i = 1, 2, 3$, of elements of order 3. The graph automorphism of F_4 permutes 2 of these conjugacy classes and leaves σ_3^W fixed. We may assume that σ_1^W has non-trivial intersection with Weyl subgroups of type C_4, and σ_2^W has non-trivial intersection with Weyl subgroups of type B_4. Since no maximal Weyl subgroup contains Coxeter elements of the Weyl subgroup of type B_4 - which have order 8 - and also elements from the W-conjugacy class σ_3^W, Ξ cannot be trivial.

Since $n_\Xi.W(\Xi)$ contains Coxeter elements, one concludes from [25, Prop. 7] that $N_W(W(\Xi)) = N_W(W(D_4))$ or $N_W(W(\Xi)) = N_W(W(\tilde{D}_4))$. Hence Ξ is either of type D_4 or $\tilde{D}_4$.

Assume Ξ is of type D_4. Since $(W(B_4), 3.W(D_4))$ is an admissible 2-coset covering, $W(\Psi)$ must contain elements from the W-conjugacy class σ_2^W and also Coxeter elements from the Weyl group $W(B_4)$. Thus from the knowledge of maximal Weyl subgroups of W one concludes that $W(\Psi) \leq W(B_4)$ is a subgroup of a Weyl group of type B_4. Proposition 3.1 then implies that $W(\Psi)$ is of type B_4. Applying the graph automorphism yields the case Ξ of type $\tilde{D}_4$.

3.5.12 — $\gamma.W$ of type G_2

The subrootsystems of Φ are of type A_2, $\tilde{A}_2$, $A_1 \times \tilde{A}_1$, A_1, $\tilde{A}_1$ or $\emptyset$. The W-conjugacy classes are 1, $w_0 = -id_{\mathbb{E}}$, ρ_l^W, ρ_s^W, σ^W, τ^W, where ρ_l is a long root, ρ_s a short root, σ of order 3 and τ of order 6. Straightforward arguments show that for $\Psi = A_2$ or $\tilde{A}_2$, $n_\Xi.W(\Xi)$ must be as described in Table 3.1, and that there does not exist an admissible 2-coset covering for Ψ of type $A_1 \times \tilde{A}_1$.

3.5.13 — $\gamma.W$ of type ${}^2B_2$

Non-trivial γ-invariant proper subrootsystems do not exist in this case (cf. [25, § 2, (D)]). Hence $\Psi = \Xi = \emptyset$, and this yields that admissible 2-coset coverings do not exist in this case.

3.5.14 — $\gamma.W$ of type ${}^2G_2$

Every non-trivial proper γ-subrootsystem is of type $\tilde{A}_1 \times A_1$ (cf. [25, § 2, (D)]). Note that $\widetilde{W} \simeq D(24)$ is isomorphic to a dihedral group of order 24. In particular, $\gamma.W$ contains two W-conjugacy classes of elements of order 12 and one W-conjugacy class of elements of order 4. As $N_{\widetilde{W}}(W(\tilde{A}_1 \times A_1))$ is a 2-group of order 8, an admissible 2-coset covering $(n_\Psi.W(\Psi), n_\Xi.W(\Xi))$ must satisfy $\Psi = \Phi = \emptyset$ and n_Ψ and n_Φ must be of order 12. However, for this choice neither $n_\Psi.W(\Psi)$ nor $n_\Xi.W(\Xi)$ contains elements of order 4, and thus admissible 2-coset coverings cannot exist in this case.

3.5.15 — $\gamma.W$ of type ${}^2F_4$

Every non-trivial proper γ-subrootsystem is of type B_2^2, $\tilde{A}_2 \times A_2$ or $\tilde{A}_1 \times A_1$. (cf. [25, § 2, (D)]). Let ρ_i, $i = 1, 2, 3, 4$, be the simple reflection associated to a basis Δ of Φ ordered in the natural way. Then $\sigma := \gamma\rho_1\rho_2 \in \gamma.W$ is an element of order 24. We may assume that $n_\Xi.W(\Xi)$ contains an element W-conjugate to σ. Since $N_W(W(\tilde{A}_1 \times A_1))$ and

$N_W(W(B_2^2))$ are 2-groups and as $N_W(W(\tilde{A}_2 \times A_2)) = (- id_{\mathbb{E}}).W(\tilde{A}_2 \times A_2)$ (cf. [6, Tab. 8]), Ξ must be trivial.

Moreover, $\gamma.W$ contains elements of order 6 and 8. Hence the same argument yields that Ψ cannot be of type $\tilde{A}_1 \times A_1$, B_2^2 or $\tilde{A}_2 \times A_2$. But as $\gamma.W$ has more than two W-orbits, also $\Psi = \emptyset$ is impossible showing that admissible 2-coset coverings do not exist in this case. □

4. The proof of Theorem A - part I

In this section we will prove that a generic 2-covering $\{H, K\}$ of the finite group of Lie type (G, F) must satisfy the conditions of Theorem A (cf. Thm. 4.5). In the subsequent 3 sections it will be proved that every pair of subgroup $\{H, K\}$ as described in Theorem A is indeed a generic 2-covering. We start with the following property - the easy proof is left to the reader.

PROPOSITION 4.1. *Let $(\tilde{G}, F)$ be a finite group of Lie type, and let $\iota: \tilde{G} \to G$ be a finite F-invariant central isogeny. Let $\{\tilde{H}, \tilde{K}\}$ be a pair of F-stable closed subgroups of maximal rank, and let H and K denote their canonical images under ι. Then, if $\{H, K\}$ is a generic 2-covering of (G, F), $\{\tilde{H}, \tilde{K}\}$ is a generic 2-covering of $(\tilde{G}, F)$. In particular, one has the following:*

(a) *Let (G, F) be a finite group of Lie type of adjoint type. Assume that the pair of F-stable subgroups $\{H, K\}$ of maximal rank is a generic 2-covering. Then for every group of Lie type G isogeneous to $\tilde{G}$ and $\tilde{H}$ and $\tilde{K}$ defined as above, $\{\tilde{H}, \tilde{K}\}$ is a generic 2-covering of $(\tilde{G}, F)$.*

(b) *Let $(\tilde{G}, F)$ be a simply-connected finite group of Lie type, and let $\tilde{H}$, $\tilde{K}$ be F-stable subgroups of maximal rank. Assume that $\{\tilde{H}_F, \tilde{K}_F\}$ is not a 2-covering of $\tilde{G}_F$. Then $\{H_F, K_F\}$ - with H and K as defined above - is not a 2-covering of G_F for all groups of Lie type (G, F) isogeneous to $(\tilde{G}, F)$.*

4.1 – *Three exceptions*

It seems remarkable to us that almost all the admissible 2-coset coverings listed in Table 3.1 give rise to a generic 2-covering - sometimes with a further characteristic restriction. However, there are three cases in Table 3.1 which will not correspond to a generic 2-covering:

(1) (G, F) of type C_4, H reductive of type $W(C_3) \times W(C_1)$, K° reductive of type $2.W(C_2^2)$,

(2) (G, F) of type A_3, H reductive of type $W(A_2)$, K° reductive of type $2.W(A_1^2)$,

(3) (G, F) of type ${}^2A_3$, H reductive of type $2.W(A_2)$, K° reductive of type $2.W(A_1^2)$.

These cases will now be analyzed by a case-by-case analysis. If the characteristic of the field is different from 2, neither H_F nor K_F will contain regular unipotent elements (cf. [20]). Therefore, we may restrict our consideration to the case when the field of definition is of characteristic 2.

Case (1): Let (G, F) be the simply-connected finite group of Lie type of type C_4 defined over the algebraically closed field $\bar{\mathbb{F}}_2$ of characteristic 2, and let H and K° be F-stable reductive subgroups of maximal rank of type $W(A_1 \times C_3)^W$ and $2.W(C_2^2)^W$, respectively. We put $K := N_G(K^\circ)$. In particular,

$$G_F \simeq Sp_8(q), \quad H_F \simeq Sp_2(q) \times Sp_6(q), \quad K_F \simeq 2.Sp_4(q^2) \tag{4.1}$$

for some 2-power $q = 2^f$. One has:

PROPOSITION 4.2. *The pair of subgroups $\{H_F, K_F\}$ is not a 2-covering of G_F.*

PROOF. Assume that $\{H_F, K_F\}$ is a 2-covering of (G, F). Let $x := x_s x_u \in G_F$ be an element where x_s is a semisimple element generating a Coxeter torus in $Sp_4(q)$ and where x_u is a regular unipotent element in $Sp_4(q)$. Hence $x \in Sp_4(q) \times Sp_4(q)$, and $Sp_4(q) \times Sp_4(q) \leq \leq Sp_8(q)$ is stabilizing the 4-dimensional summands V_4 and V_4' of a orthogonal decomposition of the natural module $V = V_8$ of $Sp_8(q)$. From the action of x on V, it is obvious that x cannot be contained in any G_F-conjugate of H_F. Hence it must be contained in a G_F-conjugate of K_F, and thus we may assume that $x \in K_F$. Since x_u is acting as a Jordan block of size 4 on V_4', x_u cannot be contained in K_F°. Moreover, one concludes that $x_s \in C_{K_F^\circ}(x_u) \simeq Sp_4(q)$ is acting diagonally on V. But this yields a contradiction, since we assumed that x_s is leaving a 4-dimensional subspace of V fixed. $\square$

Case (2): Let G be the reductive group GL_4 defined over the algebraically closed field $\bar{\mathbb{F}}_2$ of characteristic 2, and let H and K° be F-stable reductive subgroups of maximal rank of type $W(A_2)^W$ and $2.W(A_1^2)^W$, respectively. We put also $K := N_G(K^\circ)$. In particular,

$$G_F \simeq GL_4(q), \quad H_F \simeq GL_3(q) \times GL_1(q), \quad K_F \simeq 2.GL_2(q^2) \tag{4.2}$$

for some 2-power $q = 2^f$. Let V_4 denote the natural $\mathbb{F}_q[GL_4(q)]$-module, and let $V := V_2 \oplus V_2'$ be a direct decomposition in subspaces of $\mathbb{F}_q$-dimension 2. Then using the element $x := x_s x_u$, where x_s is a semisimple element acting as a Singer cycle on V_2 and fixing V_2', and where x_u is a non-trivial unipotent element fixing V_2 and acting non-trivial on V_2', the same argument which was used in the proof of Proposition 4.2 shows that $\{H_F, K_F\}$ is not a 2-covering of G_F. Moreover, since in this case $GL_4(q) = SL_4(q) \times (q-1)$ this yields that $\{SL_4(q) \cap H_F, SL_4(q) \cap K_F\}$ is not a 2-covering of $SL_4(q)$. Hence one has:

PROPOSITION 4.3. *Let (G, F) be a simply-connected group of Lie type of type A_3 defined over the algebraically closed field $\bar{\mathbb{F}}_2$ of characteristic 2. Let H be an F-stable reductive subgroup of maximal rank of type $W(A_2)^W$, and let K° be an F-stable reductive subgroup of maximal rank of type $2.W(A_1^2)^W$. Then for $K := N_G(K^\circ)$, $\{H, K\}$ is not a generic 2-covering of (G, F).*

Case (3): Let G be the reductive group GL_4 defined over the algebraically closed field $\bar{\mathbb{F}}_2$ of characteristic 2, and let F be a Frobenius morphism which is the composition of a standard Frobenius morphism with a graph automorphism. Let H and K° be F-stable reductive subgroups of type $W(A_2)^W$ and $2.W(A_1^2)^W$, respectively. There are obviously two W-conjugacy classes of type $2.W(A_1^2)^W$. The one we are considering is the one corresponding to the G_F-conjugacy class of Levi complements of the maximal parabolic subgroup of type $A_1 \times A_1$. Then for $K := N_G(K^\circ)$ one has:

$$G_F \simeq GU_4(q), \quad H_F \simeq GU_3(q) \times GU_1(q), \quad K_F \simeq 2.GL_2(q^2)^\sharp \tag{4.3}$$

for some 2-power $q = 2^f$, where $GL_2(q^2)^\sharp$ denotes the group of 2-by-2 matrices X over $\mathbb{F}_{q^2}$ whose norm of the determinant is equal to 1, i.e., $N_{\mathbb{F}_q^2/\mathbb{F}_q}(det(X)) = 1$.

Let V_4 be the natural $\mathbb{F}_{q^2}[GU_4(q)]$-module equipped with the non-degenerate hermitian form $(.,.)$. Let $V_4 = V_2 \oplus V_2'$ be an orthogonal decomposition into non-degenerate 2-dimensional $\mathbb{F}_{q^2}$-subspaces V_2 and V_2'. By choosing an element $x = x_s x_u$, where x_u is acting as a non-trivial unipotent element on V_2' and fixing V_2 pointwise, and where x_s is acting as an element of order $q-1$ on V_2 and fixing V_2' pointwise, the same argument as presented for Case (1) shows that $\{H_F, K_F\}$ is not a 2-covering of $G_F = GU_4(q)$. Thus using Proposition 4.1 and applying the same technique which was used to reduce to the simply-connected case (cf. Case (2)) one obtains the following:

PROPOSITION 4.4. *Let (G, F) be a simply-connected group of Lie type of type ${}^2A_3$ defined over the algebraically closed field $\bar{\mathbb{F}}_2$ of characteristic 2. Let H be an F-stable reductive subgroup of maximal rank of type $2.W(A_2)^W$, and let K° be an F-stable reductive subgroup of maximal rank of type $2.W(A_1^2)^W$ corresponding to the Levi complement of the F-stable parabolic subgroup of type A_1^2. Then for $K := N_G(K^\circ)$, $\{H, K\}$ is not a generic 2-covering of (G, F).*

4.2 – *Generic 2-coverings*

From Theorem 2.2 and 3.4 and the main result of [20] one concludes the following:

THEOREM 4.5. *Let $\{H, K\}$ be a generic 2-covering of the finite group of Lie type (G, F). Then H and K are of maximal rank.*

(a) *Assume that for one subgroup say H, $R_u(H) \neq 1$. Then (G, F) is of type A_1, A_2, A_3 or ${}^2A_3$, H is maximal parabolic and K° is reductive. Moreover, the types of the Levi complement $H/R_u(H)$ of H and of K° is as listed in Table 1.1.*

(b) *Assume that H° and K° are reductive. Then the types of (G, F), H° and K° and the characteristic of $\bar{\mathbb{F}}$ is as listed in Table 1.2.*

PROOF. By definition, one of the subgroups - say H - must contain a maximal torus, and thus it is of maximal rank. Then either K is also of maximal rank, or H must contain an F-stable maximal torus of each G_F-conjugacy class. Assume the latter case holds. If H° is not reductive, by enlarging the group H° we may assume that H° is an F-stable parabolic subgroup of G (cf. [13, Thm. 30.4]). In particular, an F-stable Levi complement L must contain an F-stable maximal torus of each G_F-conjugacy class. Thus in either case G must contain an F-stable reductive subgroup R containing an F-stable maximal torus of each G_F-conjugacy class. From Proposition 3.1 and [25, Prop. 4] one concludes that this implies that (G, F) must be of twisted type. Moreover, the arguments used in § 3.5 show that (G, F) cannot be of type ${}^2A_\ell$, ${}^2D_\ell$, ℓ odd, ${}^2E_6$ either. For none of the remaining types there exist an admissible 2-coset covering for the Weyl group coset (cf. Thm. 3.4). Hence such an F-stable reductive subgroup cannot exist, and thus K is of maximal rank, too.

If H° is not reductive, we enlarge it to an F-stable parabolic subgroup H°. Let $L \leq H^\circ$ be an F-stable Levi complement of H°, and let $n_\Psi.W(\Psi)^W$ denote its type. If H° is reductive, we denote by $n_\Psi.W(\Psi)^W$ the type of H°.

Second we do the same for K° and denote by $n_\Xi.W(\Xi)^W$ the type of either K° - in case K° is reductive - or of an F-stable Levi complement of an F-stable parabolic containing K°.

Thus, the hypothesis and Theorem 2.2 imply, that $(n_\Psi.W(\Psi), n_\Xi.W(\Xi))$ is an admissible 2-coset covering of the Weyl group coset $\gamma.W$ (cf. § 2.1). Hence by Theorem 3.4, $\gamma.W$ and the admissible 2-coset covering $(n_\Psi.W(\Psi), n_\Xi.W(\Xi))$ must occur in the list of Table 3.1.

The only cosets corresponding to Levi complements occur for the groups of Lie type A_1, A_2, A_3 or ${}^2A_3$. From this one concludes that if H° is not reductive, (G, F) must be of type A_1, A_2, A_3 or ${}^2A_3$ and K° must be reductive. In particular, a direct verification shows that the groups $\{H, K\}$ listed in Table 1.1 are the only possibilities in this case. This yields (a).

For (b) we may assume that H° and K° are reductive of maximal rank of one of the types listed in Table 3.1. From the covering property (ii) (cf. § 1) one concludes that either H or K must contain a regular unipotent element of G. Hence from [20] one concludes that $\{H, K\}$ is one of the groups listed in Table 1.2, or (G, F) is of type A_3, ${}^2A_3$ or C_4, $p = 2$ and the type of H° and K° must be as indicated in Table 3.1. In the previous section (cf. §4.1) we showed that in neither of these cases $\{H_F, K_F\}$ is a 2-covering for G_F. This completes the proof. □

5. Some 2-coverings for finite groups of Lie type

In this section we shall give an easy geometric argument showing the generic 2-covering property for the cases listed in Table 1.1 and for some of the cases listed in Table 1.2. The two remaining case will be handled in the two subsequent sections.

5.1 – *Parabolic-type generic 2-coverings*

For groups G of type A_ℓ, $\ell = 1, 2, 3$, the covering proverty of the pair of subgroups $\{H, K\}$ as listed in Table 1.1 was shown in [4, Prop. 4.2]. Therefore, it remains to show the generic 2-covering property for G of type

${}^2A_3$, and H and K given as described in Table 1.1. This property can be deduced from the following elementary geometric fact:

PROPOSITION 5.1. *Let $\mathbb{F}_{q^2}$ be a finite field with q^2 elements, where q is a prime power. Let $(V, \langle .,. \rangle)$ be a 4-dimensional non-degenerate hermitian $\mathbb{F}_{q^2}$-space, and let $GU_4(q) := \mathrm{Iso}\,(V, \langle .,. \rangle)$ denotes its isometry group. Then for an element $g \in GU_4(q)$ one (or both) of the following holds:*

(i) *There exists a 1-dimensional non-singular g-invariant $\mathbb{F}_{q^2}$-space V_1.*

(ii) *There exists a 2-dimensional singular g-invariant $\mathbb{F}_{q^2}$-space V_2.*

PROOF. Let $\mathfrak{N}(g)$ be the set of all non-trivial g-invariant non-singular subspaces of g. Note that if $W \in \mathfrak{N}(g)$ is directly indecomposable, $\dim_{\mathbb{F}_{q^2}}(W)$ must be odd. Furthermore, for $W \in \mathfrak{N}(g)$, $W^{\perp} \in \mathfrak{N}(g)$. This shows that either $\mathfrak{N}(g) = \emptyset$, or there exists a 1-dimensional g-invariant subspace $W \in \mathfrak{N}(g)$.

Assume that $g \in GU_4(q)$ is an element for which neither (i) nor (ii) holds, and let $W \leq V$ be a minimal non-trivial g-invariant subspace. Hence from the previous remark one concludes that W must be singular. By hypothesis, $\dim_{\mathbb{F}_{q^2}}(W) = 1$. In particular, $W \leq W^{\perp} = rad(W)$ has codimension 2 in its radical.

Assume that there exists a 3-dimensional g-invariant subspace W' different from $W^{\perp}$. By hypothesis, $rad(W') \neq W$ and W' is not non-singular. Hence either $rad(W') \leq W^{\perp}$ and $rad(W') \oplus W$ is a 2-dimensional g-invariant singular subspace, or $W^{\perp} \oplus rad(W') = V$ and thus $U := W' \cap W^{\perp}$ is g-invariant and has dimension 2. Since U cannot be non-singular, it contains either a g-invariant 1-dimensional subspace U_0 or it is a 2-dimensional singular subspace. Hence by assumption, U_0 is a 1-dimensional singular subspace, and $U_0 \oplus rad(W')$ is a singular 2-dimensional subspace, a contradiction. This shows that $W^{\perp}$ is a unique 3-dimensional g-invariant subspace, and also that W is the unique 1-dimensional g-invariant subspace.

The element g acts as unitary transformation on the induced hermitian space $W^{\perp}/W$, and thus has a 1-dimensional invariant subspace $\bar{R} \leq$ $\leq W^{\perp}/W$. Let $R \leq W^{\perp}$ be its canonical preimage, i.e., R is a g-invariant 2-dimensional subspace containing W satisfying $\bar{R} = R/W$. By hypothesis, R cannot be singular and thus $\bar{R}$ is a 1-dimensional non-singular space. In particular, $R \not\leq R^{\perp}$ and this implies $W = R \cap R^{\perp}$.

Let $x_1 \in W$ be a vector spanning W, and let $y_1 \in V$ be such that x_1 and y_1 span a hyperbolic plane. Let $x_2 \in R$ such that $R = = \mathrm{span}_{\mathbb{F}_{q^2}}(x_1, x_2)$, and $y_2 \in R^\perp$ such that $R^\perp = \mathrm{span}_{\mathbb{F}_{q^2}}(x_1, y_2)$. Then with respect to the ordered basis (y_1, y_2, x_2, x_1) the element g is represented by an upper triangular matrix over $\mathbb{F}_{q^2}$. The entries on the diagonal of this matrix are μ^{-1}, λ', λ, μ for some $\mu \in \mathbb{F}_q = = \{\, a \in \mathbb{F}_{q^2} \mid a^q = a \,\}$, and $\lambda, \lambda' \in \mathbb{F}_{q^2}$, $\lambda^{q+1} = 1$. From the Jordan normal form one concludes that either there exists another 1-dimensional g-invariant subspace different from W, or $\lambda = \mu$. Similarly, the non-existence of a 3-dimensional g-invariant subspace implies that $\mu^{-1} = \lambda'$. Thus the semi-simple part of g has to be a central element of order 1 or 2. In particular, g satisfies (ii), a contradiction, and this yields the claim. □

5.2 – *Reductive-type generic* 2*-coverings for groups of type* A_ℓ *or* ${}^2A_\ell$

Case (1): G of type A_1, $p = 2$. In case (G, F) is of type A_1, it is well-known that $\{H_F, K_F\}$, where H is the normalizer of an F-stable maximally split torus, and K is the normalizer of an F-stable maximally non-split torus, is a 2-covering for $G = GL_2(q)$, $q = 2^f$ (cf. [14]).

Case (2): G of type A_3, $p = 3$. Let $G_F := Gl_3(q), q = 3^f$, and let $V = V_3$ be its natural 3-dimensional $\mathbb{F}_q[G]$-module. We assume that $H_F := GL_2(q)\times$ $\times GL_1(q)$, $K_F := 3.T_F$, T_F a Singer torus. From Jordan's normal form theorem one concludes that a non-trivial element g has either no proper g-invariant subspace - in which case it is G_F-conjugate to K_F - or it has a 1-dimensional and 2-dimensional g-invariant subspace, say V_1 and V_2. If there exist V_1 and V_2 with $V_1 \cap V_2 = 0$, g is G_F-conjugate to H_F. Thus we may assume $V_1 \leq V_2$. Hence the same argument which was used in the proof of Proposition 5.1 shows that in this case g is either G_F-conjugate to $GL_2(q) \times GL_1(q)$, or g has Jordan decomposition $g = x_s x_u$ with x_s central and x_u regular unipotent. In particular, g is G_F-conjugate into K_F, and this yields the claim in this case.

Case (3): G of type ${}^2A_3$, $p = 3$. For $G_F := GU_3(q)$, $q = 3^f$, the same argument as was used in Case (2) shows the 2-covering property for the subgroups $\{H_F, K_F\}$.

5.3 – *Reductive type generic* 2*-coverings in characteristic* 2

Case (4): G of type C_ℓ, $p = 2$. Let (G, F) be a finite group of Lie type of type C_ℓ defined over the algebraically closed field $\bar{\mathbb{F}}_2$ of characteristic 2 and let $F: G \to G$ be a standard Frobenius morphism. Let H° be an F-stable reductive subgroup of maximal rank of type $W(\tilde{D}_\ell)^W$, and let K° be an F-stable reductive subgroup of maximal rank of type $2.W(\tilde{D}_\ell)^W$. Then by Theorem 2.2, 3.4 and Dye's theorem, $\{H, K\}$, where $H := N_G(H^\circ)$, $K := N_G(K^\circ)$, is a generic 2-covering of (G, F).

Case (5): G of type C_2, $p = 2$. Let $\ell = 2$ and (G, F) as in Case (4). Let H° be an F-stable reductive subgroup of maximal rank of type $W(\tilde{A}_1^2)^W$, and let K° be an F-stable reductive subgroup of maximal rank of type $2.W(\tilde{A}_1^2)^W$. By Theorem 2.2 and 3.4, every maximal F-stable torus of (G, F) is G_F-conjugate to a torus in H° or K°. Let $H := N_G(H^\circ)$, $K := N_G(K^\circ)$. Then $\boldsymbol{a}(H_F)$ and $\boldsymbol{a}(K_F)$ are G_F-conjugate to the subgroups constructed in Case (4), where $\boldsymbol{a}: G_F \to G_F$ denotes the non-trivial graph automorphism. Hence $\{H, K\}$ is a generic 2-covering of (G, F).

Case (6): G of type G_2, $p = 2$. Let $\ell = 3$, and let (G, F) be the finite group of Lie type of type C_3, and let $X \leq G$ be an F-stable subgroup of G of type G_2. Let $Y^\circ \leq X$ be an F-stable reductive subgroup of maximal rank of type $W(A_2)^W$, and let $Z^\circ \leq X$ be an F-stable reductive subgroup of maximal rank of type $2.W(A_2)^W$. We put also $Y := N_X(Y^\circ)$, $Z := N_X(Z^\circ)$. Then Y is contained in an F-stable subgroup H as constructed in Case (4), and Z is contained in an F-stable subgroup K as constructed in Case (4). Moreover, since X cannot be contained in H, and as Y is maximal in X, one has $Y = X \cap H$. In a similar fashion one shows that $Z = X \cap K$.

By Theorem 2.2 and 3.4, every F-stable maximal torus of X is X_F-conjugate to a maximal torus in Y° or Z°. Moreover, a straightforward order argument shows that

$$G_F = H_F.X_F, \quad G_F = K_F.X_F. \tag{5.1}$$

Hence the following proposition shows that $\{Y, Z\}$ is a generic 2-covering for (X, F):

PROPOSITION 5.2. *Let A be a finite group with 2-covering $\{B, C\}$, and let $R \leq A$ be a subgroup satisfying $A = B.R = C.R$. Then $\{B \cap R, C \cap R\}$ is a 2-covering for R.*

PROOF. By hypothesis, $A = \bigcup_{r \in R} B^r \cup \bigcup_{r \in R} C^r$. Hence since $B^r \cap R = (B \cap R)^r$, this yields

$$R \subseteq \bigcup_{r \in R} (B \cap R)^r \cup \bigcup_{r \in R} (C \cap R)^r. \tag{5.2}$$

□

6. Groups of type C_3, $p = 3$

Throughout this section we assume that (G, F) is a finite group of Lie type of type C_3 of adjoint type defined over the algebraically closed field $\bar{\mathbb{F}}_3$, i.e., $G = PCSp_6$. Furthermore, we assume that $H = H^\circ$ is an F-stable reductive subgroup of maximal rank of type $W(C_2 \times A_1)^W$, and that K° is an F-stable reductive subgroup of maximal rank of type $3.W(A_1^3)^W$. Moreover, $K := N_G(K^\circ)$. We denote by $W(\Psi)$ and $n_\Xi.W(\Xi)$ the Weyl group coset corresponding to the G_F-orbit containing H and K°, respectively.

In order to show that $\{H, K\}$ is a generic 2-covering of G, it suffices to show by Theorem 3.4(b) that every element $x = x_s x_u \in G_F$ with non-trivial unipotent part x_u is G_F-conjugate to an element in H_F or K_F.

PROPOSITION 6.1. *Let (G, F), H and K be given as described above. Let $x = x_s x_u \in G_F$ be an element satisfying:*

(i) *the semisimple part x_s of x is not contained in the center of G_F,*
(ii) *$C_G(x_s)^\circ$ is not of type $W(\tilde{A}_2)^W$.*

Then x is G_F-conjugate to an element of H_F or K_F°.

PROOF. Since 3 is not a bad prime for G, every unipotent element of $C_G(x_s)$ is contained in $C_G(x_s)^\circ$ (cf. [24]). Thus $x \in C_G(x_s)^\circ_F$. If the rootsystem of $C_G(x_s)^\circ$ is of type $\emptyset$, x_s is a regular semisimple element and thus $x_u = 1$. Hence in this case Theorem 3.4 yields the claim. Thus it suffices to show that every F-stable reductive subgroup C of maximal rank of type $n.W(\Pi)^W$ - with Π not of type $\emptyset$, $\tilde{A}_2$, C_3 - is G_F-conjugate to a subgroup in H or K°. By [25, Prop. 4], it suffices to show that $n.W(\Pi)$ is W-conjugate to a subset in $W(\Psi)$ or $n_\Xi.W(\Xi)$.

In Table 6.1 we have listed all 3-saturated subrootsytem of the rootsytem Φ of type C_3. In the second column we have indicated whether such a subrootsystem is W-conjugate to a subrootsytem in $W(\Psi)$. If this is the case, we assume $\Pi \subseteq \Psi$ and denote by $\bar{N}_W(W(\Pi)) := N_W(W(\Pi))/W(\Pi)$ and $\bar{N}_{W(\Psi)}(W(\Pi)) := N_{W(\Psi)}(W(\Pi))/W(\Pi)$ the respective outer normalizers. We used the following notation: 2^n denotes an elementary abelian 2-group of order 2^n, S_n denotes the symmetric group of degree n, $D(n)$ denotes the dihedral group of order $2n$.

Every 3-saturated subrootsystem Π of Φ of type different from $\tilde{A}_2$ and C_3 is contained in a subrootsystem of type $C_2 \times A_1$. Thus we may assume that Π is contained in Ψ. If $\bar{N}_{W(\Psi)}(W(\Pi))$ contains an element of

TABLE 6.1. – p-Saturated subrootsystems of $\Phi = \Phi(C_3)$, p odd.

type Π	$\subseteq_W \Psi$	$\bar{N}_W(W(\Pi))$	$\bar{N}_{W(\Psi)}(W(\Pi))$
C_3	$-$	1	$-$
$C_2 \times A_1$	$+$	1	1
A_1^3	$+$	S_3	2
C_2	$+$	2	2
$\tilde{A}_2$	$-$	2	$-$
A_1^2	$+$	2^2	2^2
$A_1 \times \tilde{A}_1$	$+$	2	2
A_1	$+$	$D(4)$	$D(4)$
$\tilde{A}_1$	$+$	2^2	2^2
$\emptyset$	$+$	$S_3 \ltimes 2^3$	$D(4) \times 2$

every conjugacy class of $\bar{N}_W(W(\Pi))$, then every F-stable reductive subgroup of maximal rank of type Π is contained in a G_F-conjugate of H. Hence by Table 6.1, the claim follows for elements x for which $C_G(x_s)^\circ$ is not of type A_1^3.

Let x be an element such that $C := C_G(x_s)^\circ$ has rootsystem Π of type $\tilde{A}_1^3$. There are 3 G_F-conjugacy classes of F-stable subgroups C with rootsystem of type A_3^1 depending whether for the corresponding W-orbit $n_\Pi.W(\Pi)^W$ the element $n_\Pi.W(\Pi) \in \bar{N}_W(W(\Pi))$ has order 1, 2 or 3. In the first 2 cases C is G_F-conjugate to a subgroup of H; while in the third case C is G_F-conjugate to K° (cf. [25, Prop. 4]). This yields the claim. □

6.1 – *G and G_F-conjugacy classes*

In order to relate G_F-conjugacy classes in G_F to G-conjugacy classes we use the following lemma:

LEMMA 6.2. *Let X be a connected affine algebraic group defined over the algebraic closure $\bar{\mathbb{F}}_p$ of a finite field $\mathbb{F}_p$, and let $F: X \to X$ be a Frobenius morphism.*

(a) *Let $x \in X$ be an element of X. Then the X-conjugacy class x^X contains an F-fixed point, if and only if $F(x) \in x^X$.*

(b) *Assume x is an F-fixed point, i.e., $F(x) = x$. Then the mapping*

$$\rho\colon \{\, x^g \mid g \in X,\ F(x^g) = x^g \,\} \longrightarrow C_X(x)/C_X(x)^\circ =: \bar{C}_X(x), \qquad (6.1)$$
$$\rho((x^g)^{X_F})\colon = F(g)g^{-1}.C_X(x)^\circ,$$

is surjective and yields a bijection $\bar{\rho}$ between X_F-conjugacy classes of F-fixed points in x^X and F-conjugacy classes of $\bar{C}_X(x)$. In particular, if $C_X(x)$ is connected and if x is an F-fixed point, there exists a unique X_F-conjugacy class of F-fixed points in x^X.

(c) *Let $Y \leq X$ be an F-stable closed connected subgroup and let $x \in Y$ be an F-fixed point. Then if $y\colon = x^h$, $h \in Y$, is an F-fixed point one has*

$$\tau(\rho_Y(y)) = \rho_X(y), \qquad (6.2)$$

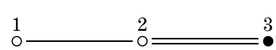
where $\tau\colon \bar{C}_Y(x) \to \bar{C}_X(x)$ is the canonical map.

PROOF. (a) and (b) are direct consequence of the Lang-Steinberg theorem. (c) is a direct consequence of (b). □

6.2 – *Semisimple elements with connected centralizer of type $W(A_2)^W$ and $2.W(A_2)^W$*

In view of Proposition 6.1 we need to take a closer look at semisimple elements $x_s \in G_F$ whose connected centralizer $C_G(x_s)^\circ$ is of type $W(\tilde{A}_2)^W$ or $2.W(\tilde{A}_2)^W$.

(6.3)



Let $\{a_1, a_2, a_3\}$ denote a basis of the rootsystem Φ with numbering given as in (6.3), i.e., a_1, a_2 are short roots and a_3 is a long root. The subrootsystem Δ spanned by a_1 and a_2 is of type $\tilde{A}_2$. Let $T \leq G$ be an F-stable maximal torus of G being contained in an F-stable Borel subgroup of G, and let

$$X\colon = \mathrm{Hom}\,(T, \bar{\mathbb{F}}_3^*), \quad Y\colon = \mathrm{Hom}\,(\bar{\mathbb{F}}_3^*, T) \qquad (6.4)$$

denote the character group and co-character group, respectively. We put

$$C\colon = \langle\, T, U_a \mid a \in \Delta \,\rangle. \qquad (6.5)$$

Note that C is a Levi complement of the maximal parabolic subgroup P_3, in particular $C \simeq GL_3(\bar{\mathbb{F}}_3)/\{\pm 1\}$ and thus C has precisely 3 unipotent conjugacy classes: regular unipotent elements, root elements, and the trivial element. Obviously, the centralizer in C for either of these unipotent elements is connected.

Let $\check{\varepsilon} \in \Phi^\vee$ be the sum of all short positive coroots. Then with respect to the canonical pairing, $\Delta \subset \Phi$ is orthogonal to $\check{\varepsilon} \in \Phi^\vee$. In terms of affine group schemes the kernel of $\check{\varepsilon}$ is non-trivial, but coincides with the constant group scheme of order 3. However, on $\bar{\mathbb{F}}_3$-rational points $\check{\varepsilon}\colon \bar{\mathbb{F}}_3^* \to T$ is injective. Moreover, $(\bar{\mathbb{F}}_3^*)_F$ is a cyclic group of order $q-1 = 3^f - 1$ and the induced map on F-fixed points $\check{\varepsilon}_F\colon (\bar{\mathbb{F}}_3^*)_F \to T_F$ is also injective. Thus

$$im(\check{\varepsilon}) = Z(C), \;\; im(\check{\varepsilon}_F) = Z(C_F). \tag{6.6}$$

Moreover, every semisimple element $x_s \in G_F$ for which $C_G(x_s)^\circ$ is of type $W(\Delta)^W$ is G_F-conjugate to a non-trivial element in $im(\check{\varepsilon}_F)$.

Let $(-1) \in W$ be the element acting as $-id_X$ on X, and let $e \in G$ be an element such that $F(e)e^{-1} \in N_G(T)$, $\pi(F(e)e^{-1}) = (-1)$, where $\pi\colon N_G(T) \to W$ denotes the canonical map. Then C^e is an F-stable reductive subgroup of maximal rank of G. Let

$$\check{\varepsilon}^e\colon \bar{\mathbb{F}}_3^* \longrightarrow T^e \tag{6.7}$$

denote the composition of $\check{\varepsilon}$ with $\_^e$. Then, if we let F_0 act on $\bar{\mathbb{F}}_3^*$ by $F_0(x) := F(x^{-1})$, $x \in \bar{\mathbb{F}}_3^*$, one has

$$F \circ \check{\varepsilon}^e = \check{\varepsilon}^e \circ F_0. \tag{6.8}$$

Moreover, $\bar{\mathbb{F}}_{F_0}^*$ is a cyclic group of order $q+1 = 3^f + 1$ and one has

$$im(\check{\varepsilon}^e_{F_0}) = Z(C^e_F). \tag{6.9}$$

Thus every semisimple element x_s for which $C_G(x_s)^\circ$ is of type $(-1).W(\Delta)^W$ is G_F-conjugate to a non-trivial element in $im(\check{\varepsilon}^e_{F_0})$.

PROPOSITION 6.3. *Let (G, F), H and K be as defined above. Let $x = x_s \cdot x_u \in G_F$ be an element such that $C_G(x_s)^\circ$ is of type $W(A_2)^W$ or $(-1)W(A_2)^W$.*

(a) *For x_u one of the following holds: (i) x_u is trivial, or (ii) x_u is a non-trivial short root element, or (iii) x_u is a regular unipotent element in $A := [C_G(x_s)^\circ, C_G(x_s)^\circ]$.*

(b) *If x_u is a short root element, then x is G_F-conjugate to an element in H_F.*

(c) *If x_u is a regular unipotent element in A, then x is G_F-conjugate to an element in K_F.*

PROOF. The type of $C_G(x_s)^\circ$ is either $W(\Delta)^W$ or $(-1).W(\Delta)^W$. Part (a) has been proved already in the discussion above. Thus it remains to establish (b) and (c).

(b) Assume that $C_G(x_s)^\circ$ is of type $W(\Delta)^W$. Let $H' = \langle T, U_a \mid a \in \Psi' \rangle$, where $\Psi' \subset \Phi$ is the subrootsystem spanned by a_2, a_3 and the heighest long root a_*. Then Ψ' is a subrootsystem of type $C_2 \times A_1$, H' is an F-stable subgroup of G of maximal rank of type $C_2 \times A_1$ and thus G_F-conjugate to H. Hence we may assume that $x_s \in im(\check{\varepsilon}_F) \leq H'_F$ and $C_G(x_s)^\circ = C$. As $U_{a_2} \leq C \cap H'$, there exists an element $h \in C_F$ such that $x_u^h = u \in (U_{a_2})_F$. Hence $x^h \in H'_F$ and the claim follows.

Assume that $C_G(x_s)^\circ$ is of type $(-1).W(\Delta)^W$. As $(-1) \in W(\Psi')$, there exists an element $e \in H'$ such that $F(e)e^{-1} \in N_{H'}(T)$, $\pi(F(e)e^{-1}) = (-1)$. In this case we may assume that $x_s \in im(\check{\varepsilon}_F^e) \leq H'_F$. Moreover, $C_{H'}(x_s)^\circ$ is an F-stable reductive subgroup of H' of maximal rank of type $(-1).W(\{\pm .a_2\})^W$. Hence $C_{H'}(x_s)_F^\circ$ contains short root elements, and there exists $h \in C_G(x_s)_F$ such that $x_u^h \in C_{H'}(x_s)_F$. Thus $x^h \in H'_F$ and the claim follows.

(c) Assume that $C_G(x_s)^\circ$ is of type $W(\Delta)^W$. Then we may assume that $x_s \in im(\check{\varepsilon}_F)$, $C = C_G(x_s)^\circ$ and $x_u \in C_F$. In particular, $W(\Delta)$ coincides with the stabilizer of $\check{\varepsilon}$ in W. Let $\tau \in W(\Delta)$ be an element of order 3. As $|T_F|$ is coprime to 3, there exists an element $u \in N_C(T)_F$ of order 3 with $\pi(u) = \tau$. Moreover, u is regular unipotent in G (cf. [20]). Moreover, since $A = SL_3(\bar{\mathbb{F}}_3)$ has a unique conjugacy class of regular unipotent elements, and as $C_A(u)$ is connected, x_u is A_F-conjugate to u (cf. Lemma 6.2). We assume therefore that $x_u = u$.

Since C is connected, the Lang-Steinberg theorem implies the existence of an element $g \in A$ such that $F(g)g^{-1} = u$. Note that by construction $x_s^g = x_s$. In particular, x is A_F-conjugate to x^g (cf. Lemma 6.2).

Let $K_1^\circ := \langle T, U_a \mid a \in \Xi \rangle^g$. As τ is stabilzing Ξ, K_1° is an F-stable reductive subgroup of G of type $n_\Xi.W(\Xi)^W$ which is G_F-conjugate to K°. Hence we may assume that K_1° coincides with K°. By construction, $x_s = x_s^g \in K_1^\circ$, and also, $u^g \in N_G(K_1^\circ)$. This yields the claim in this case.

Assume that $C_G(x_s)^\circ$ is of type $(-1).W(\Delta)^W$. Let $e \in G_F$ and $\check{\varepsilon}^e$ be given as in (6.7). Then we may assume that $x_s \in im(\check{\varepsilon}^e)_F$ and $C_G(x_s)^\circ = C^e$. As in the previous case one concludes that there exists a regular unipotent element $u \in N_{C^e}(T^e)_F$ of order 3 which centralizes $\check{\varepsilon}^e \in Y(T^e)$, and that we may assume that $x = x_s.u$. Let $g \in A^e$ such that $F(g)g^{-1} = u$. Then $u^g \in A_F^e$ and x and x^g are A_F-conjugate. Note that

$$F(eg)g^{-1}e^{-1} = F(e)e^{-1}eue^{-1} \in N_G(T). \tag{6.10}$$

Furthermore, $\pi(F(eg)(eg)^{-1})$ is an element of order 6 and $K'' := \langle\, T, U_a \mid a \in \Xi \,\rangle^{eg}$ is an F-stable reductive subgroup of G which is G_F-conjugate to K. We therefore assume that K and K'' coincide. By construction, $x_s^g = x_s \in K''$, and $u^g \in N_G(K'')$. This completes the proof. $\square$

6.3 – G_F-*Conjugacy classes of unipotent elements*

The unipotent G-conjugacy classes as well as their centralizers have been determined already in [24, § IV]. It turns out that in this case the Bala-Carter theorem (cf. [8, Thm. 5.9.6]) remains valid, and thus the G-conjugacy classes can be parameterized by certain weighted Dynkin diagrams corresponding to Levi subgroups of G. We list the weighted Dynkin diagrams occuring in the 1st column of Table 6.2 (cf. [8, p. 174, p. 400]). Moreover, every such G-conjugacy class can be represented as a regular unipotent element of reductive subgroup of maximal rank, and the saturated subrootsystems corresponding to the respective conjugacy class are listed in the 2nd column. As a consequence, every G-conjugacy class u^G of unipotent elements contains an F-fixed point (cf. [24, III.1.19(a)]). In the 3rd column we describe the isomorphism type of

$$\bar{C}_G(u) := C_G(u)/C_G(u)^\circ \tag{6.11}$$

TABLE 6.2. – Unipotent conjugacy classes

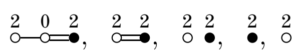
type	reg	$\lvert\bar{C}_G(u)\rvert$	$J(\tilde{u})$	u^3	H_F	K_F
$\overset{2}{\circ}-\overset{2}{\circ}=\overset{2}{\bullet}$	C_3	1	6	$\overset{2}{\circ}\ \overset{2}{\bullet}$	$-$	$+$
$\overset{2}{\circ}-\overset{0}{\circ}=\overset{2}{\bullet}$	$C_2 \times A_1$	2	$4,2$	$\overset{2}{\circ}$	$+/+$	$-/-$
$\overset{2}{\circ}=\overset{2}{\bullet}$	C_2	1	$4,1,1$	$\overset{2}{\circ}$	$+$	$-$
$\overset{2}{\circ}-\overset{2}{\circ}$	$\tilde{A}_2$	1	$3,3$	$\emptyset$	$-$	$+$
$\overset{2}{\circ}\ \overset{2}{\bullet}$	$A_1 \times \tilde{A}_1$ $\tilde{A}_1^3$	1	$2,2,2$	$\emptyset$	$+$	$+$
$\overset{2}{\circ}$	$\tilde{A}_1$ A_1^2	2	$2,2,1,1$	$\emptyset$	$+/+$	$-/-$
$\overset{2}{\bullet}$	A_1	1	$2,1,1,1,1$	$\emptyset$	$+$	$-$
$\emptyset$	$\emptyset$	1	$1,1,1,1,1,1$	$\emptyset$	$+$	$+$

for every G-conjugacy class of unipotent elements (cf. [8, p. 400], [24, IV.2]). From Lemma 6.2 one concludes that the F-fixed points of a unipotent G-conjugacy class form a single G_F-conjugacy class apart from regular unipotent elements of type $C_2 \times \tilde{A}_1$ and short root elements, where one has 2 G_F-conjugacy classes.

In the 4th column we list the size of the Jordan blocks of the unipotent element $\tilde{u} \in Sp_6(\bar{\mathbb{F}}_3)$ with $\imath(\tilde{u}) = u$ in the natural 6-dimensional representation. Here $\imath\colon Sp_6(\bar{\mathbb{F}}_3) \to PCSp_6(\bar{\mathbb{F}}_3)$ denotes the canonical map. These values can be easily obtained from the 2nd column. In the 5th column the G-conjugacy class of u^3 is given terms of weighted Dynkin diagrams. In the 6th and 7th column we list whether an F-fixed point of the unipotent conjugacy class is contained in G_F-conjugate of H_F or K_F. In case that $C_G(u)$ is not connected, there will be 2 G_F-conjugacy classes, and this is the reason why for some G-conjugacy classes there occur two signs in this column.

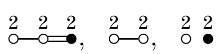
Proposition 6.4. *Let (G,F), H and K be given as before.*

(a) *If u is a unipotent element of type*

(6.12)  *or* $\emptyset$

which is an F-fixed point, then u is G_F-conjugate to an element in H_F.

(b) *If u is a unipotent element of type*

(6.13)  *or* $\emptyset$

which is an F-fixed point, then u is G_F-conjugate to an element in K_F.

Proof. Let $u_1,..,u_8$ denote representatives of the unipotent G-conjugacy classes listed in Table 6.2.

(a) Obviously, H_F contains an F-fixed point of either of the unipotent classes u_2^G, u_3^G, u_5^G, u_6^G, u_7^G, u_8^G. For $i = 2, 6$ and $u_i \in H_F$ the canonical mapping $c_i\colon C_H(u_i) \to C_G(u_i)$ induces a morphism

$$\bar{c}_i\colon C_H(u_i)/C_H(u_i)^{\circ} \longrightarrow C_G(u_i)/C_G(u_i)^{\circ}. \tag{6.14}$$

Hence by Lemma 6.2, it suffices to show that c_2 and c_6 are isomorphisms. We will show that $C_G(u_i) = im(c_i).R_u(C_G(u_i))$ for $i = 2, 6$.

Since $u_2 \in G$ is a distinguished unipotent element, $C_G(u_2)^{\circ} = R_u(C_G(u_2))$. As $Z(H) \simeq \mathbb{Z}/2\mathbb{Z}$ consists of semisimple elements, $C_G(u_2) = im(c_i).R_u(C_G(u_2))$.

Assume that $u_6 \in H_F$. By [24, § IV.2],

$$C_H(u_6)/R_u(C_H(u_6)) \simeq (\bar{\mathbb{F}}_3^* \circ A_1).2$$

(cf. [8, p. 400]), where A_1 is a simple algebraic group of type A_1 whose rootgroups are long roots, and "$\circ$" stands for direct product modulo a finite central subgroup. As this group is isomorphic to $C_G(u_6)/R_u(C_G(u_6))$, one has $C_G(u_6) = im(c_6).R_u(C_G(u_6))$ and this yields the claim.

(b) Note that $K_F^\circ \simeq PGL_2(3^{3f})$. Hence there exists a unique K_F°-conjugacy class of elements of order 3 being contained in K_F° and this conjugacy class is of type $\overset{2}{\circ}\ \overset{2}{\bullet}$. By [21, III.1, Lemma 1], K_F contains two conjugacy classes of elements of order 3 which are not contained in K_F°. Furthermore, there exist elements of order 9 (cf. [9, p. 18]), and for each element v of order 9, v^3 is a unipotent element of type $\overset{2}{\circ}\ \overset{2}{\bullet}$. Hence every element of order 9 is of type $\overset{2}{\circ}\!-\!\overset{2}{\circ}\!\Rightarrow\!\overset{2}{\bullet}$. This yields the claim. $\square$

7. Groups of type F_4, $p = 3$

Let (G, F) be a finite group of Lie type of type F_4 defined over the algebraically closed field $\bar{\mathbb{F}}_3$. Let H be an F-stable reductive group of maximal rank of type $W(B_4)^W$, and let K° be an F-stable reductive group of maximal rank of type $3.W(D_4)^W$. Note that G is simply-connected and of adjoint type. By W we denote its Weyl group. We denote by $W(\Psi)^W$ the type of H, and by $n_\Xi.W(\Xi)$ the type of K°. We also assume that n_Ξ is an element of order 3.

Let $\bar{\gamma} \in K_F$ be an element of order 3 which is not contained in K_F°. Then $\bar{\gamma}$ is a unipotent element. Its G-conjugacy class is the one of type $\tilde{A}_2$ (cf. [8, p. 400]), and its G_F-conjugacy class corresponds to the class $x_7^{G_F}$ in the list of T. Shoji (cf. [22, Table 6]). In particular, $C_G(\bar{\gamma})$ is connected.

From Theorem 2.2(b) and 3.4 one knows already that every semisimple element $s \in G_F$ is G_F-conjugate to an element in H_F or K_F°. Therefore, it suffices to consider elements with non-trivial unipotent part.

7.1 – *Elements with non-trivial semisimple and unipotent part*

In order to study elements with non-trivial semisimple and non-trivial unipotent part we determine first all W-orbits of saturated subrootsystems.

PROPOSITION 7.1. *Let $\Pi \subseteq \Phi(F_4)$ be a saturated subrootsystem of the rootsystem of type F_4. Then the type of Π is as listed in Table 7.1. Furthermore, any two saturated subrootsystems of the same type are W-conjugate.*

TABLE 7.1. – Saturated subrootsystems of $\Phi = \Phi(F_4)$

Π	$N_W(W(\Pi))$	$\Pi \subseteq \Psi$	$N_W(W(\Pi)) = N_{W(\Psi)}(W(\Pi))$
$\emptyset$	$W(F_4)$	$\times$	no
A_1	$W(A_1 \times C_3)$	$\times$	no
$\tilde{A}_1$	$W(\tilde{A}_1 \times B_3)$	$\times$	yes
A_1^2	$W(B_2^2)$	$\times$	yes
$A_1 \times \tilde{A}_1$	$W(A_1^2 \times \tilde{A}_1^2)$	$\times$	yes
A_2	$(\pm\, id_{\mathbb{E}}) \times W(A_2 \times \tilde{A}_2)$	$\times$	yes
$\tilde{A}_2$	$(\pm\, id_{\mathbb{E}}) \times W(A_2 \times \tilde{A}_2)$	$\varnothing$	—
B_2	$W(B_2^2)$	$\times$	yes
A_1^3	$W(A_1^4) : S_3$	$\times$	no
$A_1^2 \times \tilde{A}_1$	$W(B_2 \times \tilde{A}_1^2)$	$\times$	yes
$A_1 \times \tilde{A}_2$	$(\pm\, id_{\mathbb{E}}) \times W(A_1 \times \tilde{A}_2)$	$\varnothing$	—
$A_1 \times B_2$	$W(A_1^2 \times B_2)$	$\times$	yes
$\tilde{A}_1 \times A_2$	$(\pm\, id_{\mathbb{E}}) \times W(\tilde{A}_1 \times A_2)$	$\times$	yes
A_3	$W(\tilde{A}_1 \times B_3)$	$\times$	yes
B_3	$W(\tilde{A}_1 \times B_3)$	$\times$	yes
C_3	$W(A_1 \times C_3)$	$\varnothing$	—
A_1^4	$W(C_4)$	$\times$	no
$A_1^2 \times B_2$	$W(B_2^2)$	$\times$	yes
$A_1 \times C_3$	$W(A_1 \times C_3)$	$\varnothing$	—
$\tilde{A}_1 \times A_3$	$W(\tilde{A}_1 \times B_3)$	$\times$	yes
$A_2 \times \tilde{A}_2$	$(\pm\, id_{\mathbb{E}}) \times W(A_2 \times \tilde{A}_2)$	$\varnothing$	—
B_4	$W(B_4)$	$\times$	yes
D_4	$W(F_4)$	$\times$	no
F_4	$W(F_4)$	$\varnothing$	—

PROOF. From the standard algorithm established by A. Borel and J. de Siebenthal and E. B. Dynkin (cf. [1], [11]) one deduces easily that the type of a subrootsystem $\Psi \subseteq \Phi(F_4)$ must be one of the types listed in Table 7.1. Every W-orbits of subrootsystem consisting entirely of long roots or short roots is uniquely determined by its type (cf. [6, Table 4]). Since subrootsystems of type B_2, B_3, C_3 are the saturation of subrootsystems of type $\tilde{A}_1^2, \tilde{A}_1^3, A_3$, this yields the claim. □

In the 3rd column of Table 7.1 we have indicated which subrootsystems are W-conjugate to a subrootsystem of Ψ. In the 4th column we specified the subrootsystems for which there exists a containment $\Pi \subseteq \Psi$, satisfying

$$N_W(W(\Pi)) = N_{W(\Psi)}(W(\Pi)). \tag{7.1}$$

However, the reader should be warned: Ψ contains two $W(\Psi)$-orbits of subrootsystems of type A_1^2 and A_3. Only for one such $W(\Psi)$-orbit (7.1) is satisfied.

From Proposition 7.1 one deduces the following property:

PROPOSITION 7.2. *Let (G, F), H and K be as defined above. Let $x = x_s x_u \in G_F$ be an element with non-trivial semisimple part. Then x is contained in H_F or K_F, or $C_G(x_s)$ is of type $W(\tilde{A}_2)^W$, $W(A_1 \times \tilde{A}_2)^W$, $W(C_3)^W$ or $W(A_1 \times C_3)^W$.*

PROOF. Since G is simply-connected, Steinberg's theorem implies that $C := C_G(x_s)$ is an F-stable reductive group of maximal rank containing x. Let $n_\Pi.W(\Pi)^W$ denote the type of C. In particular, $\Pi \subset \Phi$ is a saturated subrootsystem and thus one of the subrootsystems listed in Table 7.1. We may assume that $\Pi \neq \emptyset, \Phi$.

Suppose Π is W-conjugate to a subrootsystem of Ψ. We may assume that $\Pi \subseteq \Psi$. In case that $N_W(W(\Pi)) \simeq N_{W(\Psi)}(W(\Pi))$, C_F is G_F-conjugate to a subgroup of H_F (cf. [25, § 1]) and the conclusion follows.

In case Π is of type A_1, A_2, A_1^3, A_1^4 or D_4, Π is also a subrootsystem of $\Phi_{long} = \Xi$. Since $W = W(\Xi) : S_3$, this shows that for every element $n \in N_W(W(\Pi))$, $n.W(\Pi)$ is either contained in one of the W-conjugates of $W(\Psi)$, or in one of the W-conjugates of the coset $3.W(\Xi)$.

As the defining characteristic equals 3, the reductive group with rootsystem of type $A_2 \times \tilde{A}_2$ is semisimple with trivial center. Hence C cannot be of type $W(A_2 \times \tilde{A}_2)^W$ leaving the four possibilities. □

From the 2-covering property for groups of type C_3 (cf. § 6) one deduces the following:

PROPOSITION 7.3. *Let (G, F), H and K be defined as above. Let $C \leq G$ be a reductive subgroup of maximal rank of type $W(A_1 \times C_3)^W$ or $W(C_3)^W$. Then every element $x \in C_F$ is G_F-conjugate to an element in H_F or K_F.*

PROOF. If C is of type C_3, $C' := C \circ C_G([C, C])$ is an F-stable reductive subgroup of maximal rank of type $A_1 \times C_3$. Hence it suffices to prove the claim for C of type $W(A_1 \times C_3)^W$.

The character group X_C has index 2 in the character group X of G. Hence $C \simeq SL_2(\bar{\mathbb{F}}_3) \circ Sp_6(\bar{\mathbb{F}}_3)$ is isomorphic to the central product of $SL_2(\bar{\mathbb{F}}_3)$ with $Sp_6(\bar{\mathbb{F}}_3)$, and thus is neither adjoint nor simply-connected. From the 2-covering property of groups of Lie type C_3 one deduces that every element in C_F is G_F-conjugate to an element in X_F, X an F-stable reductive subgroup of G of maximal rank of type $W(A_1^2 \times B_2)^W$, or Y_F, $Y = N_G(Y^\circ)$, Y° an F-stable reductive subgroup of maximal rank of type $3.W(A_1^4)^W$. In the proof of Proposition 7.2 it was shown that X_F is G_F-conjugate to a subgroup of H_F, and Y_F is a G_F-conjugate to a subgroup of K_F. This yields the claim. $\square$

From the 2-covering property for groups of type A_2 and ${}^2A_2$ (cf. § 5.2) one concludes:

PROPOSITION 7.4. *Let (G, F), H and K be defined as above. Let $C \leq G$ be a reductive subgroup of maximal rank of type $(\pm 1).W(A_2 \times \tilde{A}_2)^W$ or $(\pm 1).W(A_1 \times \tilde{A}_2)^W$. Then every element $x \in C_F$ is G_F-conjugate to an element in H_F or K_F.*

PROOF. Assume that C is of type $W(A_2 \times \tilde{A}_2)^W$. Hence $C = S_1 \times S_2$, $S_1 \simeq SL_3$ is generated by long root groups, and $S_2 \simeq SL_3$ is generated by short root groups. In particular, $C_F \simeq SL_3(q) \times SL_3(q)$. Let $x = x_1 x_2$, $x_1 \in (S_1)_F$, $x_2 \in (S_2)_F$. From the 2-covering property for the groups of type A_2 one concludes that $x \in C_F$ is G_F-conjugate to

(i) X_F, X an F-stable reductive subgroup of maximal rank of type $W(A_2 \times \tilde{A}_1)^W$, or

(ii) Y_F, Y an F-stable reductive subgroup of maximal rank of type $3.W(A_2)^W$, or

(iii) x_2 is a regular unipotent element in SL_3 generated by short roots.

So by Proposition 7.2 only the last case has to be considered. Since x_2 is of type $\tilde{A}_2$, we may assume that $x_2 = \bar{\gamma}$.

Let S_1^g be an F-stable G-conjugate of S_1 centralizing $\bar{\gamma}$. As $C_G(S_1) = S_2$ and $C_G(S_2) = S_1$, $\bar{\gamma}^{g^{-1}} \in S_2$. Hence there exists $s \in S_2$ such that $\bar{\gamma}_2^{g^{-1}s} = \bar{\gamma}_2$, and thus $S_1^g = S_1^{s^{-1}g}$. In particular, S_1^g and S_1 are conjugate by an element in $C_G(\bar{\gamma})$. Since $C_G(\bar{\gamma})$ is connected, the Lang-Steinberg theorem implies that S_1^g and S_1 are conjugate by an element in $C_G(\bar{\gamma})_F$.

As $C_{K^\circ}(\bar{\gamma}) \simeq G_2$, it contains an F-stable subgroup which is G_F-conjugate to S_1. Hence the previous remark yields the claim in this case. The other three cases follow by a similar argument and are therefore left to the reader. □

7.2 – *Regular unipotent elements*

For our purpose it is important to know whether K_F contains regular unipotent elements of G or not. In [20, Lemma 3.2] it was shown that the algebraic group K contains regular unipotent elements of G. However, the argument used there cannot be applied in our case.

From [16] (cf. [20, Lemma 3.2]) one knows that regular unipotent elements are the only elements of order 27 in G. Thus it suffices to show that K_F contains elements of order 27.

Let $\Delta = \{e_1 - e_2, e_2 - e_3, e_3 - e_4, e_3 + e_4\}$ be the standard basis of the rootsystem of $\Phi_{long} = \Xi$, and let γ be the graph automorphism with cycle decomposition $(e_2 - e_3)(e_1 - e_2, e_3 + e_4, e_3 + e_4)$. We assume that $\bar{\gamma} \in \mathrm{Aut}(K_F^\circ)$ is acting on Φ through γ.

For simplicity we use the same notation as was introduced in [22], i.e., $1 - 2$ is a short form of the root $e_1 - e_2$, etc. For $s \in \mathbb{F}_q$ and $t \in \mathbb{F}_{q^3}$ we define the root elements

$$u_1(s) := x_{2-3}(s), \qquad v_1(t) := x_{1-2}(t) \cdot x_{3+4}(t^q) \cdot x_{3-4}(t^{q^2}), \tag{7.2}$$

$$u_2(s) := x_{1+3}(s), \qquad v_2(t) := x_{1-3}(t) \cdot x_{2+4}(t^q) \cdot x_{2-4}(t^{q^2}), \tag{7.3}$$

$$u_3(s) := x_{1+2}(s), \qquad v_3(t) := x_{1+4}(t) \cdot x_{2+3}(t^q) \cdot x_{1-4}(t^{q^2}), \tag{7.4}$$

which are obviously contained in K_F° (cf. [7, § 13.6]), and which generate the 3-Sylow subgroup of K_F°. In particular, for $i = 1, 2, 3$ one has

$$u_i(s)^{\bar{\gamma}} = u_i(s), \tag{7.5}$$

$$v_i(t)^{\bar{\gamma}} = v_i(t^{q^2}). \tag{7.6}$$

If one chooses the sign of the structure constants as in [22, Tab. 1] one deduces the following commutator formula from Chevalley's commutator formula (cf. [7, Thm. 5.2.2]). Here $[x,y] := x^{-1}y^{-1}xy$ denotes the standard commutator.

PROPOSITION 7.5. *For the elements as defined in* (7.2) *one has the following formulae:*

$$
\begin{aligned}
[u_1(s), u_2(s')] &= u_3(-ss'),\\
[u_1(s), v_1(t)] &= v_2(-st)\cdot v_3(-st^{1+q})\cdot u_2(-sN(t))\cdot u_3(s^2N(t)),\\
[v_1(t), v_2(\omega)] &= v_3(-(t^q\omega + t\omega^q))\cdot u_2(-Tr(t^{1+q}\omega^{q^2}))\cdot u_3(-Tr(t\omega^{q+q^2})),\\
[v_1(t), v_3(\omega)] &= u_2(-Tr(t\omega^q)),\\
[v_2(t), v_3(\omega)] &= u_3(-Tr(t\omega^q),
\end{aligned}
\tag{7.7}
$$

where $s, s' \in \mathbb{F}_q$, $t, \omega \in \mathbb{F}_{q^3}$, $N\colon \mathbb{F}_{q^3} \to \mathbb{F}_q$ *denotes the norm map and* $Tr\colon \mathbb{F}_{q^3} \to \mathbb{F}_q$ *denotes the trace map. Furthermore, any pair of elements not occuring in* (7.7) *commutes.*

PROOF. This follows by a lengthy but straightforward calculation. □

LEMMA 7.6. *Let a, b, c be elements of a group A with the following properties:*

(i) $a^3 = b^3 = c^3 = [b,a]^3 = [c,b]^3 = [c,a]^3 = [[b,a],b]^3 = 1$,
(ii) $[c,b]$, $[c,a]$, $[[b,a],b] \in Z(A)$,
(iii) $[[[b,a],a],b] = 1$.

Then $(abc)^3 = [[b,a]a][[b,a],b]^2$.

PROOF. This follows by a straightforward calculation using the basic commutator identities (cf. [18, § 5.1.5]). □

We finally obtain the following property:

PROPOSITION 7.7. *Let* (G,F), *H and K be as above. Then* K_F *contains regular unipotent elements of G.*

PROOF. Let $g := \bar{\gamma}\cdot u_1(s)\cdot v_1(t)$. Then

$$g^3 \equiv v_1(Tr(t))\cdot v_2(s(t^q - t)^q) \bmod U_0, \tag{7.8}$$

where $U_0 := \langle\, u_2(s), u_3(s), v_3(t) \mid s \in \mathbb{F}_q,\ t \in \mathbb{F}_{q^3}\,\rangle$. Hence

$$g^3 = v_1(Tr(t))\cdot v_2(s(t^q - t)^q)\cdot v_3(\omega)\cdot u_2(s)\cdot u_3(s') \tag{7.9}$$

for suitable numbers $\omega \in \mathbb{F}_{q^3}$, $s, s' \in \mathbb{F}_q$. Since $u_2(s)$ and $u_3(s')$ centralize the subgroup generated by $v_1(t)$, $v_2(t')$, $v_3(t'')$, $t, t', t'' \in \mathbb{F}_{q^3}$ (cf. (7.7)), one has

$$g^9 = \big(v_1(Tr(t)) \cdot v_2(s(t^q - t)^q)v_3(\omega)\big)^3. \tag{7.10}$$

It is easy to check that Lemma 7.6 applies for $a := v_1(Tr(t))$, $b := v_2(s(t^q - t)^q)$ and $c := v_3(\omega)$. Moreover, for a, b, c as defined above, one has

$$\begin{aligned} [[b, a], b]] &= u_3(2s^2 Tr(t)(Tr(t^{q+1}) - Tr(t^2))), \\ [[b, a], a] &= u_2(2sTr(t)^2 Tr(t^q - t)) = 1. \end{aligned} \tag{7.11}$$

For $q \neq 3$, $Tr(t)(Tr(t^{q+1}) - Tr(t^2))$ is a polynomial in t of degree $3q^2$ and thus there exists an element $t \in \mathbb{F}_{q^3}$ with $Tr(t)(Tr(t^{q+1}) - Tr(t^2)) \neq 0$. Hence the claim follows. For $q = 3$, the polynomial $Tr(t)(Tr(t^{q+1}) - Tr(t^2))$ as a function on $\mathbb{F}_{27}$ is equivalent to a polynomial of degree 21, and hence the claim follows also in this case. □

7.3 – G_F-*conjugacy classes of unipotent elements*

From the information obtained in [22] one concludes the following:

PROPOSITION 7.8. *Let (G, F), H and K be as above. Then every unipotent element in G_F is G_F-conjugate to an element in H_F or K_F.*

PROOF. The unipotent G_F-conjugacy classes in the finite group of Lie type G_F of type F_4 defined over a finite field of characteristic 3 were determined by T. Shoji (cf. [22]). There are 28 unipotent G_F-conjugacy classes $x_i^{G_F}$, $i = 0, \ldots, 27$, where x_i is the representative given in [22, Tab. 6].

From the definition of the elements x_i it is obvious that unipotent elements in the classes $x_j^{G_F}$,

$$j \in \{0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 14, 15, 16, 17, 19, 23, 24\} \tag{7.12}$$

are G_F-conjugate to an element in H_F.

Furthermore, for $j \in \{20, 21, 22\}$, x_j is G_F-conjugate to the group C_F of F-fixed points in C, where C is an F-stable reductive subgroup of maximal rank of type $W(A_1 \times C_3)^W$. Hence by Proposition 7.3, x_j is G_F-conjugate to an element in H_F or K_F.

The unipotent element $\bar{\gamma} \in K_F$ is G_F-conjugate to x_7. A similar argument shows that $\bar{\gamma}.x_a(1)$, a the simple root fixed by $\bar{\gamma}$ is G_F-conjugate to x_{11}.

By Proposition 7.7, K_F contains a regular unipotent element u of G. Note that there exists a unique G_F-conjugacy class of F-stable reductive groups of maximal rank of type $n_\Xi.W(\Xi)^W$. Since 3 is a bad prime for F_4, $C_G(u)/C_G(u)^\circ$ is a cyclic group of order 3 being generated by u (cf. [24, 1.14(d)]). For $k = 1, 2$, let $g_k \in G$ be such that $F(g_k)g_k^{-1} = u^k$. Then K^{g_k} is the normalizer of an F-stable reductive group of maximal rank of type $n_\Xi.W(\Xi)^W$ and thus G_F-conjugate to K. By construction, $(K^{g_k})_F$ contains the regular unipotent element u^{g_k} fixed by F corresponding to the G_F-conjugacy class $u^k.C_G(u)^\circ$. This shows that K_F contains an element of either of the G_F-conjugacy classes of regular unipotent elements $x_{25}^{G_F}$, $x_{26}^{G_F}$, $x_{27}^{G_F}$ (cf. Lemma 6.2(c)).

A similar argument shows that K_F contains also an element of the conjugacy class $x_{18}^{G_F}$. This yields the claim. $\square$

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