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Hilbert Functions of Cohen-Macaulay Ideals with Assigned Generators' Degrees.

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ABSTRACT - We give information on the Hilbert function of a Cohen-Macaulay ideal I of the polynomial ring $R = k[x_0, x_1, \ldots, x_r]$ which is minimally generated by t forms of degrees $d_1, \ldots, d_t$. Mainly we deal with the codimension two case in which we show that the Dubreil bound $t \leqslant d_1 + 1$ is a necessary and sufficient condition to have such an ideal and we give a sharp upper bound and lower bound for the Hilbert function. In codimension greater than two we give a characterization for having such an ideal and in codimension 3 we find an Hilbert function which is maximal for these ideals with $d_1 = \ldots = d_t = a$ and we produce a scheme which realizes such a Hilbert function.

Introduction.

Let $R = k[x_0, x_1, \ldots, x_r]$ be the homogeneous polynomial ring over an algebraically closed field k and fix t positive integers $d_1, \ldots, d_t$. It is a very classical question, both of Commutative Algebra and Algebraic Geometry, to try to determinate the Hilbert function of R/I, where I is a homogeneous ideal of R minimally generated by t forms of degrees $d_1, \ldots, d_t$. Of course, one needs some further information on the ideal I. For instance, one point of view can be to ask that the forms defining the ideal are generically chosen. Even in this strong context very few results are known. Clearly, in this case if $t \leqslant r + 1$, we have $htI = t$ and I is a complete intersection and then the Hilbert function of R/I is completely

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determined by $d_1, \dots, d_t$. But when $t > r+1$, we have $htI = r+1$ (so R/I is an Artinian ring) and very little is still known. By results of Stanley [St] and Watanabe [W] the Hilbert function is known if $t = r+2$, since a general Artinian complete intersection has the Strong Lefschetz property. Moreover, the case $r=1$ and $r=2$ was solved, respectively, by Fröberg [F] and by Anick [A]. Other authors studied the case $d_1 = \dots = d_t$ giving information on some part of the Hilbert function or, with some further restriction, on the graded Betti numbers (see, for instance [HL], [Au], [FH], [MM]). In any case, it seems completely unexplored what can be the Hilbert function, or at least bounds for it, for an ideal of height $r+1$ generated by any t forms of degrees $d_1, \dots, d_t$. Now, from the Algebraic Geometry point of view, one is mainly interested with (saturated) ideals of R of height $\leqslant r$. So we can refrase the question of finding the possible Hilbert functions, or bounds for them, for Cohen-Macaulay homogeneous ideals I of R of fixed height h minimally generated by t forms of degrees $d_1, \dots, d_t$. Since, very little is known every result in this field seems interesting. We will deal essentially with the case $r=2$ and in the case $r=3$ when $d_1 = \dots = d_t = a$. Precisely, we set $\mathcal{H}^{(c)}_{d_1, \dots, d_t} = \{H_{R/I}\}_{R/I}$ where $H_{R/I}$ is the Hilbert function of any Artinian reduction of a c-codimensional Cohen-Macaulay ideal I of R minimally generated by t forms of degrees $d_1, \dots, d_t$. We equip this set of an ordering, defining $H_{R/I} \leqslant H_{R/J}$ if $H_{R/I}(n) \leqslant H_{R/J}(n)$ for all n. In this paper we study first the case $c=2$, for which, after observing that $\mathcal{H}^{(c)}_{d_1, \dots, d_t}$ is not empty if and only if $t \leqslant d_1 + 1$ (the Dubreil inequality), we prove that as poset it has both a maximum and a minimum element (Proposition 2.2). Moreover, we produce a scheme which realizes such a maximum and compute explicitely the minimum in the case $d_1, \dots, d_t = a$. In the codimension 3 case we show that $\mathcal{H}^{(3)}_{a;t}$ (i.e. in case when $d_1, \dots, d_t = a$) has a maximal element and again we produce a scheme which realizes such a maximal Hilbert function (Theorem 3.8). We still believe that, indeed, it is also a maximum.

The first section is dedicated to partial intersection schemes which will be used to produce schemes with the required Hilbert functions.

1. Partial intersections: definitions, properties and facts.

Throughout this paper k will denote an algebraically closed field, $\mathbb{P}^r$ the r-dimensional projective space over k, $R = k[x_0, x_1, \dots, x_r] = \bigoplus_{n \in \mathbf{Z}} H^0(\mathcal{O}_{\mathbb{P}^r}(n))$.

If $V \subset \mathbb{P}^r$ is a subscheme, I_V will denote its defining ideal and $H_V(n) = \dim_k R_n - \dim_k (I_V)_n$ its Hilbert function. Moreover, if $V \subset \mathbb{P}^r$ is a c-codimensional aCM scheme with minimal free resolution

$$0 \to \oplus R(-j)^{\alpha_{cj}} \ldots \to \oplus R(-j)^{\alpha_{2j}} \to \oplus R(-j)^{\alpha_{1j}} \to I_V \to 0$$

then the integers $\{\alpha_{ij}\}_j$ will denote the i-th graded Betti numbers.

In this section we recall the construction of the c-codimensional partial intersection schemes made in [RZ] and we collect from there the main facts that will be used in this paper.

Let $(\mathcal{P}, \leqslant)$ be a poset. We denote, for every $H \in \mathcal{P}$,

$$\mathcal{S}_H = \{K \in \mathcal{P} \mid K < H\}, \quad \overline{\mathcal{S}}_H = \{K \in \mathcal{P} \mid K \leqslant H\}.$$

DEFINITION 1.1. A subset $\mathcal{A}$ of the poset $\mathcal{P}$ is said to be a *left segment* if for every $H \in \mathcal{A}$, $\mathcal{S}_H \subseteq \mathcal{A}$. In particular, when $\mathcal{P} = \mathbb{N}^c$ with the order induced by the natural order on $\mathbb{N}$, a finite left segment will be mentioned as a *c-left segment*.

Note that every c-left segment $\mathcal{A}$ has sets of generators but there is a unique minimal set of generators consisting of the maximal elements of $\mathcal{A}$; we will denote it by $G(\mathcal{A})$.

If $\pi_i \colon \mathbb{N}^c \to \mathbb{N}$ will denote the projection to the i-th component, and $\mathcal{A}$ is a c-left segment, we set $v(H) = \sum_{i=1}^{c} \pi_i(H)$ and $a_i = \max\{\pi_i(H) \mid H \in \mathcal{A}\}$, for $1 \leqslant i \leqslant c$. The c-tuple $T = T(\mathcal{A}) = (a_1, \ldots, a_c)$ will be called the *size* of $\mathcal{A}$.

A c-left segment is said to be *degenerate* if $a_i = 1$ for some i.

If $\mathcal{A}$ is a c-left segment, $F(\mathcal{A})$ will denote the set of minimal elements of $\mathbb{N}^c \setminus \mathcal{A}$, i.e.

$$F(\mathcal{A}) = \{H \in \mathbb{N}^c \setminus \mathcal{A} \mid \mathcal{S}_H \subseteq \mathcal{A}\}.$$

Note that, if $H = (m_1, \ldots, m_c) \in F(\mathcal{A})$ and $m_i > 1$, then $H_i = (m_1, \ldots, m_i - 1, \ldots, m_c) \in \mathcal{A}$. Moreover, the elements

$$T_1 = (a_1 + 1, 1, \ldots, 1), \ldots, T_c = (1, 1, \ldots, a_c + 1)$$

always belong to $F(\mathcal{A})$, and we will call them *canonical* c-tuples.

In the sequel we denote the c-tuple $(1, \ldots, 1)$ by I and, for every subset Z of $\mathcal{S}_T$, we denote

$$C_T(Z) = \{T + I - H \mid H \in Z\}.$$

Finally, for every c-left segment $\mathcal{A}$ we define

$$\mathcal{A}^* = C_T(\mathcal{S}_T \setminus \mathcal{A}).$$

Observe that $\mathcal{A}^*$ is a c-left segment.

PROPOSITION 1.2. *If $\mathcal{A}$ is a c-left segment, then*

1. $F(\mathcal{A}) = C_T(G(\mathcal{A}^*)) \cup \{T_1, \dots, T_c\}$,
2. $F(\mathcal{A}^*) = C_T(G(\mathcal{A})) \cup \{T_1^*, \dots, T_c^*\}$.
3. *If $T_i^* \neq T_i$, for some i, then $T_i^* \in C_T(G(\mathcal{A}))$.*

PROOF. See Proposition 1.3 in [RZ]. ■

Fix a c-left segment $\mathcal{A}$ and consider c families of hyperplanes of $\mathbb{P}^r$, $c \leqslant r$,

$$\{A_{1j}\}_{1 \leqslant j \leqslant a_1}, \quad \{A_{2j}\}_{1 \leqslant j \leqslant a_2}, \dots, \quad \{A_{cj}\}_{1 \leqslant j \leqslant a_c}$$

sufficiently generic, in the sense that $A_{1j_1} \cap \dots \cap A_{cj_c}$ are $\prod_{i=1}^{c} a_i$ pairwise distinct linear varieties of codimension c.

For every $H = (j_1, \dots, j_c) \in \mathcal{A}$, we denote by

$$L_H = \bigcap_{h=1}^{c} A_{hj_h}.$$

With this notation we have the following

DEFINITION 1.3. The subscheme of $\mathbb{P}^r$

$$V = \bigcup_{H \in \mathcal{A}} L_H$$

will be called a *c-partial intersection* with respect to the hyperplanes $\{A_{ij}\}$ and support on the c-left segment $\mathcal{A}$.

THEOREM 1.4. *Every c-partial intersection X of $\mathbb{P}^r$ is a reduced aCM subscheme consisting of a union of c-codimensional linear varieties.*

PROOF. See Theorem 1.9 in [RZ]. ■

Here are the main results on c-codimensional partial intersections.

THEOREM 1.5. *If $V \subset \mathbb{P}^r$ is a partial intersection of codimension c with support on $\mathcal{A}$, then the $(r-c+1)$-th difference of its Hilbert function is*

$$\Delta^{r-c+1} H_V(n) = \left| \{ H \in \mathcal{A} \mid v(H) = n + c \} \right| .$$

PROOF. See Theorem 2.1 in [RZ]. ■

Now, if X is a c-codimensional partial intersection with support on $\mathcal{A}$ and with respect to the families of hyperplanes A_{ij} whose defining forms are f_{ij}, to every $H = (m_1, \ldots, m_c) \in \mathcal{A}$ we associate the following form

$$P_H = \prod_{i=1}^{c} \prod_{j=1}^{m_i - 1} f_{ij}.$$

THEOREM 1.6. *Let $V \subset \mathbb{P}^r$ be a partial intersection of codimension c with support $\mathcal{A}$. Then a minimal set of generators for I_V is*

$$\{ P_H \mid H \in F(\mathcal{A}) \}.$$

PROOF. See Theorem 3.1 in [RZ]. ■

COROLLARY 1.7. *Let V be as above then its first graded Betti numbers depend only on $\mathcal{A}$ and they are the following integers*

$$d_H = v(H) - c \quad \forall H \in F(\mathcal{A}).$$

And finally

THEOREM 1.8. *Let $V \subset \mathbb{P}^r$ be a partial intersection of codimension c with support $\mathcal{A}$. Then the last graded Betti numbers of V are*

$$s_H = v(H) \quad \forall H \in G(\mathcal{A}).$$

PROOF. See Theorem 3.4 in [RZ]. ■

We conclude this section by discussing the question we want to deal with in this paper.

Let $d_1 \leqslant \ldots \leqslant d_t$ be t positive integers and $c \leqslant t$; we denote by

$$\mathcal{H}^{(c)}_{d_1, \ldots, d_t} = \{ H_{R/I} \}_{R/I}$$

where I varies on the Artinian ideals of the polynomial ring $R =$

$= k[x_1, \ldots, x_c]$, which are minimally generated by t forms of degrees $d_1, \ldots, d_t$, and $H_{R/I}$ means the Hilbert function of R/I.

Note that the same set $\mathcal{H}^{(c)}_{d_1, \ldots, d_t}$ can be obtained by using ideals I_X of c-codimensional arithmetically Cohen Macaulay schemes of $X \subset \mathbb{P}^n$ and $\Delta^{n+1-c} H_{R/(I_X)}$, where Δ^{n+1-c} denotes the $(n+1-c)$-th difference of the Hilbert function of I_X.

To describe $\mathcal{H}^{(c)}_{d_1, \ldots, d_t}$ is a very hard task in this general setting. Nevertheless, many simpler (but still hard) questions can be posed. For instance, to establish if $\mathcal{H}^{(c)}_{d_1, \ldots, d_t}$ is empty or not or, more generally, to compute its cardinality in terms of the integers $c; d_1 \ldots d_t$.

Moreover, since $\mathcal{H}^{(c)}_{d_1, \ldots, d_t}$ can be ordered by defining $H_{R/I} \leqslant H_{R/J}$ iff $H_{R/I}(n) \leqslant H_{R/J}(n)$ for all $n \in \mathbb{Z}$, (we call such an ordering the *natural partial ordering*) one can ask if $\mathcal{H}^{(c)}_{d_1, \ldots, d_t}$ has a maximum or a minimum element and when the answer is negative one can ask which are the maximal and the minimal elements.

The previous questions will be studied in few particular cases: first in the codimension 2 case and for the codimension 3 case when $d_1 = \ldots = d_t$.

2. The codimension two case.

We first deal with the codimension $c = 2$ for which most of the previous questions can be answered.

In this case $\mathcal{H}^{2}_{d_1, \ldots, d_t}$ is not empty iff $t \leqslant d_1 + 1$ (the Dubreil's inequality). Indeed, in this situation, it is easy to see that $\mathcal{H}^{(2)}_{d_1, \ldots, d_t}$ is in $1-1$ correspondence with the set

$$\mathcal{S}^{(2)}_{d_1, \ldots, d_t} = \left\{ (s_2, \ldots, s_t) \,|\, s_i \leqslant s_{i+1},\ s_i > d_i\ \forall i,\ \sum_i d_i = \sum_i s_i \right\}$$

i.e. the $(t-1)$-tuples which satisfy the Gaeta's conditions. So, if we set $s_i = d_i + x_i$, where x_i is a positive integer, the Gaeta's conditions are equivalent to say $\sum_{i=2}^{t} x_i = d_1$, from which one gets that $\mathcal{S}^{(2)}_{d_1, \ldots, d_t}$ is not empty iff $d_1 \geqslant t-1$.

Of course all these information about 2-codimensional Cohen-Macaulay ideals can be deduced from the Hilbert-Burch theorem.

From now on a $(t-1)$-tuple $(x_2, \ldots, x_t)$ of positive integers such that $\sum_{i=2}^{t} x_i = d_1$, is said a $(t-1)$-partition of d_1. We are interested on the

$(t-1)$-partitions of d_1 which satisfy the condition

$$(*) \quad x_i - x_{i+1} \leqslant d_{i+1} - d_i \qquad \forall i = 2, \dots t\,.$$

Using the correspondence, which associates to $\underline{s} = (s_2, \dots, s_t) \in \mathcal{S}^{(2)}_{d_1, \dots, d_t}$ the element $H_{\underline{s}} \in \mathcal{H}^{(2)}_{d_1, \dots, d_t}$ defined by $H_{\underline{s}}(n) = (n+1) - \sum_{d_i \leqslant n} (n+1-d_i) + \sum_{s_i \leqslant n} (n+1-s_i)$, one can induce an ordering on $\mathcal{S}^{(2)}_{d_1, \dots, d_t}$; precisely, $\underline{s} = (s_2, \dots, s_t) \leqslant \underline{s}' = (s_2', \dots, s_t') \Leftrightarrow H_{\underline{s}} \leqslant H_{\underline{s}'} \Leftrightarrow \sum_{s_i \leqslant n} (n+1-s_i) \leqslant \sum_{s_i' \leqslant n} (n+1-s_i')\, \forall n$.

We need first this simple technical lemma.

LEMMA 2.1. *Let $(M_2, \dots, M_t)$ be the maximum, by lexicographic ordering, in the set of $(t-1)$-partitions of d_1 satisfying condition $(*)$. Then, for every $(t-1)$-partition of d_1 $(x_2, \dots, x_t)$, satisfying condition $(*)$, we have*

$$\sum_{i=2}^{h} M_i \geqslant \sum_{i=2}^{h} x_i \quad \forall\, h = 2, \dots t\,.$$

PROOF. If there is some $h' = 3, \dots t$ such that $\sum_{i=2}^{h'} M_i < \sum_{i=2}^{h'} x_i$ then there exist $j \leqslant h'$ such that $M_j < x_j$; say m the biggest one. On the other hand, since $\sum_{i=2}^{t} M_i = \sum_{i=2}^{t} x_i$ there is some $j > h'$ such that $M_j > x_j$; say n the smallest one. Of course, by construction, $m \leqslant n$. Define, for every $i = 2, \dots, t$,

$$M_i' = \begin{cases} M_i & \forall i \neq m, n \\ M_m + 1 & i = m \\ M_n - 1 & i = n\,. \end{cases}$$

We see that $(M_2', \dots, M_t')$ is a $(t-1)$-partition of d_1 satisfying condition $(*)$. Indeed, we need only to show that $d_m + M_m' \leqslant d_{m+1} + M_{m+1}'$ and $d_{n-1} + M_{n-1}' \leqslant d_n + M_n'$. Now

$d_m + M_m' = d_m + M_m + 1 \leqslant d_m + x_m \leqslant d_{m+1} + x_{m+1} = d_{m+1} + M_{m+1} = d_{m+1} + M_{m+1}'$ (if $m+1 < n$, the case $m+1 = n$ is similar)

$d_{n-1} + M_{n-1}' = d_{n-1} + M_{n-1} = d_{n-1} + x_{n-1} \leqslant d_n + x_n \leqslant d_n + M_n - 1 = d_n + M_n'$ (if $m < n-1$, the case $m = n-1$ is similar).

Now, since $(M_2', \dots, M_t')$ is lexicographically bigger than $(M_2, \dots, M_t)$, we get a contradiction. ∎

PROPOSITION 2.2. *Let* $d_1 \leqslant \ldots \leqslant d_t$ *be* t *positive integers and* $\mathcal{H}^2_{d_1, \ldots, d_t}$ *the set of all Hilbert functions of Artinian ideals of the polynomial ring in two variables which are minimally generated by* t *forms of degrees* $d_1, \ldots, d_t$. *Then* $\mathcal{H}^2_{d_1, \ldots, d_t}$, *as a poset, by the natural partial ordering, has both a maximum and a minimum element. Precisely, if we denote by* $(m_2, \ldots, m_t)$ *and* $(M_2, \ldots, M_t)$, *respectively, the minimum and the maximum of the set of the* $(t-1)$*-partitions of* d_1 *satisfying the condition* $(*)$, *ordered by the lexicographic ordering, then in the poset* $\mathcal{S}^{(2)}_{d_1, \ldots, d_t}$ *the element* $s_i' = d_i + m_i$, *for* $i = 2, \ldots t$, *is the maximum and the element* $s_i'' = d_i + M_i$, *for* $i = 2, \ldots t$, *is the minimum. Hence, the elements* $H_{\underline{s}'}(n) = (n+1) - \sum\limits_{d_i \leqslant n} (n+1-d_i) + \sum\limits_{s_i' \leqslant n} (n+1-s_i')$, $H_{\underline{s}''}(n) = (n+1) - \sum\limits_{d_i \leqslant n} (n+1-d_i) + \sum\limits_{s_i'' \leqslant n} (n+1-s_i'')$ *are, respectively, the maximum and the minimum element in* $\mathcal{H}^2_{d_1, \ldots, d_t}$.

PROOF. Take any element $(s_2, \ldots, s_t)$ in $\mathcal{S}^{(2)}_{d_1, \ldots, d_t}$ and denote $s_i = d_i + x_i$, for $i = 2, \ldots, t$. Of course, $(x_2, \ldots, x_t)$ is a $(t-1)$-partition of d_1 satisfying the condition $(*)$. For every $n \in \mathbb{N}$ we get two integers $i, j \leqslant t$, defined by $s_i'' \leqslant n+1 < s_{i+1}''$ and $s_j \leqslant n+1 < s_{j+1}$. For the minimum case, we need to prove that $\sum\limits_{h=2}^{i} (n+1-d_h-M_h) \leqslant \sum\limits_{h=2}^{j} (n+1-d_h-x_h)$. Now, if $i \leqslant j$, applying Lemma 2.1 we have

$$\sum_{h=2}^{i} (n+1-d_h-M_h) \leqslant \sum_{h=2}^{i} (n+1-d_h-x_h)$$

hence

$$\sum_{h=2}^{i} (n+1-d_h-M_h) \leqslant \sum_{h=2}^{i} (n+1-d_h-x_h) + \sum_{h=i+1}^{j} (n+1-d_h-x_h) = \\ = \sum_{h=2}^{j} (n+1-d_h-x_h).$$

If $i > j$ again by Lemma 2.1 we have

$$\sum_{h=2}^{i} (n+1-d_h-M_h) \leqslant \sum_{h=2}^{i} (n+1-d_h-x_h)$$

hence

$$\sum_{h=2}^{i} (n+1-d_h-M_h) \leqslant \sum_{h=2}^{j} (n+1-d_h-x_h) + \sum_{h=j+1}^{i} (n+1-d_h-x_h);$$

now, since$(n+1-d_h-x_h)<0$ for $h>j$, we get

$$\sum_{h=2}^{i}(n+1-d_h-M_h) \leqslant \sum_{h=2}^{j}(n+1-d_h-x_h).$$

Finally, note that the minimum element in the set of the $(t-1)$-partitions of d_1 satisfying the condition $(*)$, ordered by the lexicographic ordering, is given by $m_i = 1$, for $i = 2, \dots, t-1$, and $m_t = d_1 - t + 2$. This implies that for every $(t-1)$-partitions of d_1 satisfying the condition $(*)$, $(y_2, \dots, y_t)$, one has

$$\sum_{i=2}^{h} m_i \leqslant \sum_{i=2}^{h} y_i \quad \forall\, h = 2, \dots t,$$

i.e. we are in the same situation as in Lemma 2.1 (just reversing the order). Thus, repeating the same argument as in the minimum case, we get for every integer n

$$\sum_{h=2}^{i}(n+1-d_h-m_h) \geqslant \sum_{h=2}^{j}(n+1-d_h-y_h)$$

where i and j are defined as before. ■

COROLLARY 2.3. *The 2-partial intersection $X_{\max}$ whose support is the 2-left segment generated by the $t-1$ elements $\left(d_i - \sum_{h=i+1}^{t} m_h, \sum_{h=i}^{t} m_h\right)$ for $i = 2, \dots, t$, has the maximum Hilbert function in $\mathcal{H}^2_{d_1, \dots, d_t}$; the 2-partial intersection $X_{\min}$ whose support is the 2-left segment generated by the $t-1$ elements $\left(d_i - \sum_{h=i+1}^{t} M_h, \sum_{h=i}^{t} M_h\right)$ for $i = 2, \dots, t$, has the minimum Hilbert function in $\mathcal{H}^2_{d_1, \dots, d_t}$.*

PROOF. The conclusion is a direct consequence of the previous theorem, Corollary 1.7 and Theorem 1.8. ■

REMARK 2.4. We can get also the ideals of the partial intersections in the previous Corollary by lifting suitable Artinian monomial ideals. More precisely every sequence $d_1, \dots, d_t, s_2, \dots, s_t$, satisfying the Gaeta conditions, is realized by the ideal of the maximal minors of the following matrix

$$\begin{bmatrix} x^{s_2-d_1} & y^{s_2-d_2} & \dots & 0 & 0 \\ 0 & x^{s_3-d_2} & y^{s_3-d_3} & 0 & 0 \\ \dots & & & & \\ 0 & \dots & x^{s_{t-1}-d_{t-2}} & y^{s_{t-1}-d_{t-1}} & 0 \\ 0 & \dots & 0 & x^{s_t-d_{t-1}} & y^{s_t-d_t} \end{bmatrix}.$$

COROLLARY 2.5. *If $d_1 = \ldots = d_t = a$ the maximum and the minimum Hilbert function in $\mathcal{H}^2_{a;\,t}$ are given by, respectively,*

$$H_{\max}(n) = \begin{cases} n+1 & \text{if } n < a \\ 2a+1-t-n & \text{if } a \leqslant n < 2a+2-t \\ 0 & \text{if } n > 2a+2-t \end{cases}$$

$$H_{\min}(n) = \begin{cases} n+1 & \text{if } n < a \\ 2a+1-t-n & \text{if } a \leqslant n < a+q \\ 0 & \text{if } n \geqslant a+q \end{cases}$$

where $a = (t-1)\,q + r,\ r < t-1$.

PROOF. To get the conclusion it is enough to observe that in this case we have $m_i = 1$ for $i = 2, \ldots t-1$, $m_t = a-t+2$ and $M_i = q$ for $i = 2, \ldots, t-r$ and $M_j = q+1$ for $i = t-r+1, \ldots, t$. ■

3. The codimension greater than two: ideals generated in one degree.

In this section we approach the problem for codimension c greater than two. A first question is when the set $\mathcal{H}^{(c)}_{d_1, \ldots, d_t}$ is empty. Let us suppose that the integers $d_1 \leqslant d_2 \leqslant \ldots d_t$ are assigned. In this case we denote $p_1 < p_2 < \ldots < p_s$ the distinct elements among the d_i's and we set $\alpha_i :=$ $:= |\{d_j = p_i \mid 1 \leqslant j \leqslant t\}|$. Of course $\sum_{i=1}^{s} \alpha_i = t$. Moreover we set

$$\beta_1 := \binom{p_1+c-1}{c-1},\ \beta_i := (\beta_{i-1} - \alpha_{i-1})^{\langle p_{i-1}\rangle\langle p_{i-1}+1\rangle\ldots\langle p_i-1\rangle} \text{ for } 2 \leqslant i \leqslant s,$$

where the symbol $\langle . \rangle$ denotes the exponential of Macaulay (see the paper of Stanley [St1]).

PROPOSITION 3.1. *Let $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_t$ t positive integers, with $t \geqslant c$. Then*

$$\mathcal{H}^{(c)}_{d_1, \ldots, d_t} \neq \emptyset \Leftrightarrow \alpha_i \leqslant \beta_i \text{ for } 1 \leqslant i \leqslant s.$$

PROOF. Throughout this proof we set $R := k[x_1, \ldots, x_c]$.

Let $I \in \mathcal{H}^{(c)}_{d_1, \ldots, d_t}$; we call $J_{(i)}$ the ideal generated by the homogeneous pieces of I of degree less or equal to p_i, φ_i the Hilbert function of the k-algebra $R/J_{(i)}$ and H the Hilbert function of R/I. Of course, we have $\varphi_i(p_i) = \varphi_{i-1}(p_i) - \alpha_i$. Now we prove that $\varphi_{i-1}(p_i) \leqslant \beta_i$ for $2 \leqslant i \leqslant s$. For $i = 2$, $\varphi_1(p_1) = \beta_1 - \alpha_1$, and using Macaulay Theorem on the maximal growth (see [St1]),

$$\varphi_1(p_1+1) \leqslant (\beta_1 - \alpha_1)\langle p_1 \rangle, \ \varphi_1(p_1+2) \leqslant \varphi_1(p_1+1)^{\langle p_1+1 \rangle} \leqslant$$
$$\leqslant (\beta_1 - \alpha_1)^{\langle p_1 \rangle \langle p_1+1 \rangle}, \ldots, \varphi_1(p_2) \leqslant (\beta_1 - \alpha_1)^{\langle p_1 \rangle \langle p_1+1 \rangle \ldots \langle p_2-1 \rangle} = \beta_2.$$

Now suppose that $\varphi_{i-1}(p_i) \leqslant \beta_i$; then $\varphi_i(p_i) = \varphi_{i-1}(p_i) - \alpha_i \leqslant \beta_i - \alpha_i$ therefore by repeating the previous arguments we obtain that

$$\varphi_i(p_{i+1}) \leqslant (\beta_i - \alpha_i)^{\langle p_i \rangle \langle p_i+1 \rangle \ldots \langle p_{i+1}-1 \rangle} = \beta_{i+1}.$$

Now it is clear that $\alpha_1 \leqslant \beta_1$ since $\beta_1 = \dim_k R_{p_1}$; moreover for $2 \leqslant i \leqslant s$,

$$\alpha_i = \dim_k R_{p_i} - \dim_k R_1 I_{p_i - 1} - H(p_i) = \varphi_{i-1}(p_i) - H(p_i) \leqslant \beta_i - H(p_i) \leqslant \beta_i.$$

Vice versa let us suppose that $d_1, \ldots, d_t$ are integers such that $\alpha_i \leqslant \beta_i$ for $1 \leqslant i \leqslant s$ and $t = \sum_{i=1}^{s} \alpha_i \geqslant c$ (using the same notation). By [St1] there exists a lex-segment ideal $L \subset R$ having α_i minimal generators in degree p_i, for $1 \leqslant i \leqslant s$. Let M be the set of the monomials minimally generating L. Then $M = M_p \cup M_m$ where M_p is the subset of the elements of M which are powers of some x_i and M_m is the subset of the mixed monomials. We set $k := |M_p|$ and $u := |M_m|$; $|M| = t = k + u \geqslant c$, by hypothesis, so $u \geqslant c - k$; let $M'_m = \{m_{k+1}, m_{k+2}, \ldots, m_c\}$ be the subset of M_m of the last $c - k$ monomials in the lexicographic order; we set $P = \{x_i^{\deg m_i} \mid k+1 \leqslant i \leqslant c\}$. Now let us consider the set of monomials $M' := (M \setminus M'_m) \cup P$. We claim that the monomial ideal I, generated by M', belongs to $\mathcal{H}^{(c)}_{d_1, \ldots, d_t}$. First observe that height $I = c$ since, by construction, there is a power of x_i belonging to I for $1 \leqslant i \leqslant c$. So, to conclude the proof, we need to show that M' is a set of minimal generators for I. Using again the lexicographic order, we set $\mu := \max\{m \in M' \mid m$ is not a power$\}$; then $M' = M_1' \cup M_2'$ where $M_1' = \{m \in M' \mid m \leqslant \mu\}$ and $M_2' = M' \setminus M_1'$; since $M_1' \subset M$ and M was a minimal set of generators, every element in M_1' cannot be in the shadow of the previous ones; on the other hand M_2' contains only powers, so, again, every element in M' cannot belong to the shadow of the previous ones. ■

REMARK 3.2. When $\mathcal{H}^{(c)}_{d_1, \ldots, d_t} \neq \emptyset$ it is natural to guess that the minimal value for the Hilbert function is achieved by the ideal generated by t generic forms of degrees $d_1, \ldots, d_t$. For istance the guess is true in the particular case $d_1 = \ldots = d_t = a$ with $nt \geqslant \binom{a+n}{a+1}$, since a result by Hochster and Laksov (see [HL]) says that $H_{R/I}(a+1) = 0$.

In this section we will use the notation about partial intersections introduced in section 1.

Let $A = k[x, y, z]$ and let $n, a \in \mathbb{N}$. We denote by $\mathfrak{I}_{a,n}$ the set of the homogeneous Artinian ideals $I \subset A$ (i.e. A/I is Artinian) where I is minimally generated by n forms of degree a.

We simply denote

$$\mathcal{H}_{a,n} = \{H_{A/I} \mid I \in \mathfrak{I}_{a,n}\}.$$

Fixed two integers $a \geqslant 1$ and $3 \leqslant n \leqslant \binom{a+2}{2}$, in this section we would like to determine a maximal element in $\mathcal{H}_{a,n}$.

Let $\overline{n} := \binom{a+2}{2} - n$. We set $b_{-2} := 0$, $b_{-1} := a-1$, $b_i := a - i$ for $0 \leqslant i \leqslant a-2$ and $b_{a-1} := 1$. Note that, since

$$\binom{a+2}{2} - 3 = \sum_{i=-2}^{a-2} b_i,$$

there exist two integers k and h, $-2 \leqslant k \leqslant a-2$, $0 \leqslant h \leqslant b_{k+1} - 1$, such that

$$\overline{n} = \sum_{i=-2}^{k} b_i + h.$$

We set $s := a - 2 - k$; then $0 \leqslant s \leqslant a$. Moreover observe that

$$n = \binom{a+2}{2} - \overline{n} = \sum_{i=1}^{a+1} i - \sum_{i=-1}^{a-2} b_i - h = \sum_{i=1}^{a+1} i - \sum_{i=a-k}^{a} i - (a-1) - h =$$

$$= \sum_{i=1}^{a-k-1} i + (a+1) - (a-1) - h = \binom{a-k}{2} + 2 - h = \binom{s+2}{2} + 2 - h\,.$$

We set also $\overline{s} := a - s$. Now let us consider the 3-left segment $\mathcal{L}_{a,\,n}$ generated by the following elements:

$$(\overline{s},\, a,\, a), (\overline{s}+1,\, h,\, a);$$

$$\{(\overline{s}+1,\, y,\, s+1-y)\,|\, h+1 \leqslant y \leqslant s\};$$

$$\{(x,\, y,\, a+2-x-y)\,|\,\overline{s}+2 \leqslant x \leqslant a,\ 1 \leqslant y \leqslant a+1-x\}.$$

Let $X \subset \mathbb{P}^r$, $r \geqslant 3$, a partial intersection with support on $\mathcal{L}_{a,\,n}$. We would like to show that $\Delta^{r-2} H_X$ is the desired maximal element.

First of all we compute $\Delta^{r-2} H_X$. If $\mathcal{A}$ is a c-left segment and $s \leqslant t$ are two positive integers we set $\mathcal{A}_j := \{(H \in \mathcal{A}\,|\,\pi_1(H) = j\}$, $\mathcal{A}(s,\,t) := \bigcup_{j=s}^{t} \mathcal{A}_j$, and denoted by $\sigma : \mathbb{Z}^c \to \mathbb{Z}^c$ the transformation $\sigma(x_1, x_2, \ldots, x_c) = (x_1 - s + 1, x_2, \ldots, x_c)$ we set also $\overline{\mathcal{A}}(s,\,t) := \sigma(\mathcal{A}(s,\,t))$. Of course $\overline{\mathcal{A}}(s,\,t)$ is a c-left segment. If X is a partial intersection with support on $\mathcal{A}$, let $X(s,\,t)$ be the subscheme of X with support on $\mathcal{A}(s,\,t)$. $X(s,\,t)$ is obviously a partial intersection with support on $\overline{\mathcal{A}}(s,\,t)$.

LEMMA 3.3. *Let $X \subset \mathbb{P}^r$ a c-partial intersection with support on $\mathcal{A}$. Let t be the maximum of the first components of the elements of $\mathcal{A}$, and let $1 \leqslant s \leqslant t$, so $X = X_1 \cup X_2$ where $X_1 := X(1,\,s)$ and $X_2 := X(s+1,\,t)$. Then $\Delta^{r-c+1} H_X(i) = \Delta^{r-c+1} H_{X_1}(i) + \Delta^{r-c+1} H_{X_2}(i-s)$.*

PROOF. By Theorem 1.5,

$$\Delta^{r-c+1} H_X(i) = |\{H \in \mathcal{A}\,|\,v(H) = i+c\}| =$$

$$= |\{H \in \mathcal{A}(1,\,s)\,|\,v(H) = i+c\}| + |\{H \in \mathcal{A}(s+1,\,t)\,|\,v(H) = i+c\}| =$$

$$= |\{H \in \overline{\mathcal{A}}(1,\,s)\,|\,v(H) = i+c\}| + |\{H \in \overline{\mathcal{A}}(s+1,\,t)\,|\,v(H) = i-s+c\}| =$$

$$= \Delta^{r-c+1} H_{X_1}(i) + \Delta^{r-c+1} H_{X_2}(i-s). \qquad \blacksquare$$

Proposition. 3.4. *Let* $a \geqslant 1$ *and* $3 \leqslant n \leqslant \binom{a+2}{2}$; *let* $X \subset \mathbb{P}^r$ *a partial intersection supported on* $\mathcal{L}_{a,n}$. *Then, with above notation,*

$$\Delta^{r-1} H_X(i) = \begin{cases} i+1 & \textit{for } 0 \leqslant i \leqslant a-1 \\ a+1-n & \textit{for } i = a \\ \bar{s} - 2(i-a+1) & \textit{for } a+1 \leqslant i \leqslant a+\bar{s}-1 \\ -\bar{s}-1 & \textit{for } a+\bar{s} \leqslant i \leqslant a+\bar{s}+h-1 \\ -\bar{s} & \textit{for } a+\bar{s}+h \leqslant i \leqslant 2a-1 \\ i-2a-\bar{s}+1 & \textit{for } 2a \leqslant i \leqslant 2a+\bar{s}-2 \\ 0 & \textit{for } i \geqslant 2a+\bar{s}-1\,. \end{cases}$$

Proof. We denote $p_j = \{(x, y, z) \in \mathbb{N}^3 \,|\, x = j\}$. We can decompose X in an union of three disjoint partial intersections X_1, X_2, X_3. X_1 is supported on $\langle(\bar{s}, a, a)\rangle$; X_2 on $\mathcal{L}_{a,n} \cap p_{\bar{s}+1}$; X_3 supported on $\mathcal{L}_{a,n} \cap \left(\bigcup_{j=\bar{s}+2}^{a} p_j\right)$.

X_1 is a complete intersection of type $(\bar{s}, a, a)$. So we have

$$\Delta^{r-1} H_{X_1}(i) = \begin{cases} i+1 & \text{for } 0 \leqslant i \leqslant \bar{s}-1 \\ \bar{s} & \text{for } \bar{s} \leqslant i \leqslant a-1 \\ 2a+\bar{s}-2-2i & \text{for } a \leqslant i \leqslant \bar{s}+a-1 \\ -\bar{s} & \text{for } \bar{s}+a \leqslant i \leqslant 2a-1 \\ i-2a-\bar{s}+1 & \text{for } 2a \leqslant i \leqslant 2a+\bar{s}-2 \\ 0 & \text{for } i \geqslant 2a+\bar{s}-1\,. \end{cases}$$

X_2 is a plane partial intersection such that

$$\Delta^{r-1} H_{X_2}(i) = \begin{cases} 1 & \text{for } 0 \leqslant i \leqslant s-1 \\ h-s & \text{for } i = s \\ 0 & \text{for } s+1 \leqslant i \leqslant a-1 \\ -1 & \text{for } a \leqslant i \leqslant a+h-1 \\ 0 & \text{for } i \geqslant a+h\,. \end{cases}$$

X_3 is a partial intersection such that

$$\Delta^{r-1} H_{X_3}(i) = \begin{cases} i+1 & \text{for } 0 \leqslant i \leqslant s-2 \\ -\binom{s}{2} & \text{for } i = s-1 \\ 0 & \text{for } i \geqslant s. \end{cases}$$

By Lemma 3.3, $\Delta^{r-1} H_X(i) = \Delta^{r-1} H_{X_1}(i) + \Delta^{r-1} H_{X_2}(i - \bar{s}) + \Delta^{r-1} H_{X_3}(i - \mathcal{a}\bar{s} - 1)$, so we are done. ∎

Now we compute the graded Betti numbers of a partial intersection with support on $\mathcal{L}_{a,n}$.

Proposition 3.5. *Let $a \geqslant 1$ and $3 \leqslant n \leqslant \binom{a+2}{2}$; let $X \subset \mathbb{P}^r$ a partial intersection with support on $\mathcal{L}_{a,n}$, and let $R := H^0_*(\mathcal{O}_{\mathbb{P}^r})$. Then a graded minimal free resolution for I_X, the saturated homogeneous ideal of X, is of the type*:

$$0 \to R(-(a+2))^{n-s-3} \oplus R(-(2a-s+h+1)) \oplus R(-(3a-s)) \to$$

$$\to R(-(a+1))^{2n-s-6} \oplus R(-(2a-s)) \oplus R(-(2a-s+1)) \oplus$$

$$\oplus R(-(2a-s+h)) \oplus R(-2a) \to R(-a)^n \to I_X \to 0$$

Proof. By [RZ], Proposition 3.4, we can compute the last graded Betti numbers of I_X directly from the generators of $\mathcal{L}_{a,n}$. To compute the first graded Betti numbers, we have to find, using notation in section 1, the set $F(\mathcal{L}_{a,n})$. An easy verification shows that the elements in $F(\mathcal{L}_{a,n})$ are

$$(a+1, 1, 1), (1, a+1, 1), (1, 1, a+1);$$

$$\{(\bar{s}+1, y, s+2-y) \mid h+1 \leqslant y \leqslant s+1\};$$

$$\{(x, y, a+3-x-y) \mid \bar{s}+2 \leqslant x \leqslant a, 1 \leqslant y \leqslant a+2-x\}.$$

Finally we can compute the second graded Betti numbers, as it is well known, through the first and last ones and the Hilbert function. ∎

If $\mathcal{A}$ is a 3-left segment, we set

$$m_i(\mathcal{A}) := \max\{\pi_i(H) \mid H \in \mathcal{A}\}; \quad \text{for } 1 \leqslant i \leqslant 3,$$

and $I := (1, 1, 1)$. Of course $m_i(\mathcal{L}_{a,n}) = a$, for $1 \leqslant i \leqslant 3$. Now we consider $\mathcal{L}^*_{a,n}$; then $m_1(\mathcal{L}^*_{a,n}) = a - \bar{s} = s$ and $m_i(\mathcal{L}^*_{a,n}) = a$ for $i = 2, 3$. So if we set $T := (a, a, a)$ and $U := (s, a, a)$ we have

$$\mathcal{L}^{**}_{a,n} = C_U\big(\bar{S}_U \setminus C_T(\bar{S}_T \setminus \mathcal{L}_{a,n})\big).$$

Now let $\sigma : \mathbb{Z}^3 \to \mathbb{Z}^3$ be the transformation $\sigma(x, y, z) = (x - \bar{s}, y, z)$.

Lemma 3.6. *With the previous notation*

$$\mathcal{L}^{**}_{a,n} = \sigma(\mathcal{L}_{a,n}) \cap \mathbb{N}^3.$$

Proof. $H \in \mathcal{L}^{**}_{a,n} \Leftrightarrow U + I - H \in \mathcal{A}\bar{S}_U \setminus C_T(\bar{S}_T \setminus \mathcal{L}_{a,n}) \Leftrightarrow U + I - H \leqslant U$ and $U + I - H \notin C_T(\bar{S}_T \setminus \mathcal{L}_{a,n}) \Leftrightarrow H \geqslant I$ and $T + I - (U + I - H) \notin \bar{S}_T \setminus \mathcal{L}_{a,n} \Leftrightarrow H \in \mathbb{N}^3$ and $H + T - U \notin \bar{S}_T \setminus \mathcal{L}_{a,n} \Leftrightarrow H \in \mathbb{N}^3$ and $H + (\bar{s}, 0, 0) \in \mathcal{L}_{a,n} \Leftrightarrow H \in \sigma(\mathcal{L}_{a,n}) \cap \mathbb{N}^3$. ■

Lemma 3.7. *Let $X \subset \mathbb{P}^r$ a scheme with support on $\mathcal{L}_{a,n}$. Then X is CI-linked in two steps to a scheme Y whose $(r-2)-th$ difference of the Hilbert function is*

$$\Delta^{r-2} H_Y(i) = \begin{cases} \dbinom{i+1}{2} & \textit{for } 0 \leqslant i \leqslant s-1 \\ h & \textit{for } s \leqslant i \leqslant a-1 \\ a+h-1-i & \textit{for } a \leqslant i \leqslant a+h-2 \end{cases}$$

through complete intersections of type (a, a, a) in the first step, and of type (s, a, a) in the second step.

Proof. We link at first X in a partial intersection complete intersection of type (a, a, a) to a scheme Y', and then we link Y' in a partial intersection complete intersection of type (s, a, a) to a scheme Y. Then Y is a partial intersection with support on $\mathcal{L}^{**}_{a,n}$. So we can compute the $(r-2)$-th difference of the Hilbert function of Y, as in the proof of Proposition 3.4, by considering $Y = Y_1 \cup Y_2$, Y_1 with support on $\mathcal{L}^{**}_{a,n} \cap p_1$ and Y_2 with support on $\mathcal{L}^{**}_{a,n} \cap \left(\bigcup_{j=2}^{s} p_j\right)$; then, using Lemma 3.3, $\Delta^{r-2} H_Y(i) =$

$= \Delta^{r-2} H_{Y_1}(i) + \Delta^{r-2} H_{Y_2}(i-1)$. So we have

i	0	1	2	$\cdots$	$s-1$	s	$\cdots$	$a-1$	a	$a+1$	$\cdots$	$a+h-2$	$a+h-1$
$\Delta^{r-2}H_{Y_1}(i)$	1	2	3	$\cdots$	s	h	$\cdots$	h	$h-1$	$h-2$	$\cdots$	1	0
$\Delta^{r-2}H_{Y_2}(i-1)$		1	3	$\cdots$	$\binom{s}{2}$	0	$\cdots$	0	0	0	$\cdots$	0	0
$\Delta^{r-2}H_Y(i)$	1	3	6	$\cdots$	$\binom{s+1}{2}$	h	$\cdots$	h	$h-1$	$h-2$	$\cdots$	1	0

■

Now we prove the main theorem of this section.

THEOREM 3.8. *Let $a \geq 1$ and $3 \leq n \leq \binom{a+2}{2}$. Let $X \subset \mathbb{P}^r$ a partial intersection with support on $\mathcal{L}_{a,n}$. Then $\Delta^{r-2} H_X$ is a maximal element in $\mathcal{H}_{a,n}$.*

PROOF. We set $\mu := \Delta^{r-2} H_X$. By contradiction let $\varphi \in \mathcal{H}_{a,n}$, $\varphi > \mu$. Let $k := \min\{i \in \mathbb{N}_0 \mid \varphi(i) > \mu(i)\}$. Since $\varphi \in \mathcal{H}_{a,n}$, $\varphi(i) = \mu(i)$ for $0 \leq i \leq a$; moreover $\mu(i) = 0$ for $i \geq 2a + \bar{s} - 2$, so $a + 1 \leq k \leq 2a + \bar{s} - 2$. Let $Y \subset \subset \mathbb{P}^r$ a scheme such that $\Delta^{r-2} H_Y = \varphi$ and $I_Y \in \mathfrak{I}_{a,n}$. We can link Y in a complete intersection Z of type (a, a, a) to a scheme Y'. Then $\Delta^{r-2} H_{Y'}(i) = \Delta^{r-2} H_Z(i) - \varphi(3a - 3 - i)$. Therefore $\Delta^{r-2} H_{Y'}(s) = = \Delta^{r-2} H_Z(s) - \varphi(2a + \bar{s} - 3)$, and since $\varphi(2a + \bar{s} - 3) \geq \mu(2a + \bar{s} - 3) > 0$ we have that $\Delta^{r-2} H_{Y'}(s) < \binom{s+1}{2}$, i.e. there is an hypersurface of degree s containing Y', so we can link Y' in a complete intersection W of type (s, a, a) to another scheme Y''. By Lemma 3.7, X is CI-linked in two steps to a partial intersection X'' with support on $\mathcal{L}^{**}_{a,n}$ through complete intersections of type (a, a, a) and (s, a, a) too. We set $\mu'' := \Delta^{r-2} H_{X''}$ and $\varphi'' := \Delta^{r-2} H_{Y''}$.

Then

$$\varphi''(i) = \Delta^{r-2} H_W(i) - \Delta^{r-2} H_{Y'}(2a + s - 3 - i) =$$

$$= \Delta^{r-2} H_W(i) - (\Delta^{r-2} H_Z(3a - 3 - (2a + s - 3 - i)) -$$

$$= \Delta^{r-2} H_Y(3a - 3 - (2a + s - 3 - i))) =$$

$$= \Delta^{r-2} H_W(i) - \Delta^{r-2} H_Z(a - s + i) + \varphi(a - s + i);$$

so we have

$$\varphi''(s+k-1-a)=\Delta^{r-2}H_W(s+k-1-a)-\Delta^{r-2}H_Z(k-1)+\varphi(k-1)=$$
$$=\Delta^{r-2}H_W(s+k-1-a)-\Delta^{r-2}H_Z(k-1)+\mu(k-1)=\mu(s+k-1-a)$$

and

$$\varphi(s+k-a)=\Delta^{r-2}H_W(s+k-a)-\Delta^{r-2}H_Z(k)+\varphi(k)>$$
$$>\Delta^{r-2}H_W(s+k-a)-\Delta^{r-2}H_Z(k)+\mu(k)=\mu(s+k-a).$$

Note that $s+k-1-a\geqslant s$. So if $s\leqslant s+k-1-a\leqslant a-2$ then $\varphi''(s+k-1-a)=\mu''(s+k-1-a)=h$ and $\varphi''(s+k-a)>\mu''(s+k-a)=h=\varphi''(s+k-1-a)^{\langle s+k-1-a\rangle}$, (since $s+k-1-a\geqslant s\geqslant h$) a contradiction, since φ'' is an O-sequence.

If $a-1\leqslant s+k-1-a\leqslant a+h-2$ then

$$\varphi''(s+k-1-a)^{\langle s+k-1-a\rangle}=$$
$$=\varphi''(s+k-1-a)=\mu''(s+k-1-a)=\mu''(s+k-a)+1\leqslant\varphi''(s+k-a),$$

and since φ'' is an O sequence, we get that $\varphi''(s+k-1-a)^{\langle s+k-1-a\rangle}=\varphi''(s+k-a)$ i.e. φ'' has a maximal growth in $s+k-a$; but this is again a contradiction, because it means, by [BGM], that the hypersurfaces of degree less or equal to $s+k-a\geqslant a$, containing Y'', have a fixed component of dimension 1, while Y'' is contained in a complete intersection of type (s,a,a). ■

EXAMPLE 3.9. Let $a=8$, $n=20$; so $\overline{n}=25$ and $25=0+7+8+7+3$ so $k=1$, $h=3$, $s=5$, $s=3$. $\mathcal{L}_{8,20}$ is generated by the following 3-tuples:

$$\begin{array}{cccc}(3,8,8) & (4,3,8) & (4,4,2) & (4,5,1)\\ (5,1,4) & (5,2,3) & (5,3,2) & (5,4,1)\\ (6,1,3) & (6,2,2) & (6,3,1) & \\ (7,1,2) & (7,2,1) & (8,1,1) & \end{array}.$$

If $X\subset\mathbb{P}^r$, $r\geqslant 3$, is a partial intersection with support on $\mathcal{L}_{8,20}$, $\Delta^{r-2}H_X$ is

$$\begin{array}{cccccccccccccccccccc}0&1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16&17&\dots\\ 1&3&6&10&15&21&28&36&25&24&21&17&13&9&6&3&1&0&\rightarrow\end{array}$$

The minimal graded resolution of I_X is

$$0 \to R(-19) \oplus R(-15) \oplus R(-10)^{12} \to R(-16) \oplus R(-14) \oplus R(-12) \oplus$$
$$\oplus R(-11) \oplus R(-9)^{29} \to R(-8)^{20} \to I_X \to 0\,.$$

We finish with the following

QUESTION 3.10. Is $\Delta^{r-2} H_X$, where $X \subset \mathbb{P}^r$ is a partial intersection with support on $\mathcal{L}_{a,n}$, the maximum element in $\mathcal{H}_{a,n}$?

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