On Infinite-Dimensional Grassmannians
and their Quantum Deformations.

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ABSTRACT - An algebraic approach is developed to define and study infinite-dimensional grassmannians. Using this approach a quantum deformation (i.e. a deformation of the coordinate ring) is obtained for both the ind-variety union of all finite-dimensional grassmannians G_∞, and the Sato grassmannian $\widetilde{UGM}$ introduced by Sato in [Sa1], [Sa2]. They are both quantized as homogeneous spaces, that is together with a coaction of a quantum infinite dimensional group. At the end, an infinite-dimensional version of the first theorem of invariant theory is discussed for both the infinite-dimensional special linear group and its quantization.

1. Introduction.

A definition of the infinite-dimensional Sato grassmannian is first introduced by Sato in [Sa1], [Sa2], where he explicitly exhibits the points as infinite-dimensional matrices. Sato proves the remarkable fact that the points of the Sato grassmannian $\widetilde{UGM}$ are in one to one correspondence with the solutions of the KP hierarchy.

A few years later Segal and Wilson [SW], using mainly analytic techniques, explore more deeply this correspondence.

In a later work [PS] Pressley and Segal study more extensively,

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along the same lines, an infinite-dimensional grassmannian closely related to the Sato Grassmannian. In particular they give a stratification and a Plucker embedding of it. They also produce an action of a certain infinite-dimensional linear group realizing it as an infinite-dimensional homogeneous space. Though their definition appears quite different from Sato's one, they essentially describe the same geometrical object, but in a slightly more general setting.

A more geometrical approach to the same subject is taken by Mulase in [Mu1], [Mu2]. He constructs the Sato grassmannian as a scheme of which he gives the functor of points. Also Plaza-Martin [PM] takes the same approach, with special attention given to the physical applications.

Together with the Sato grassmannian, Sato, as well as all the above mentioned authors, introduces what we denote by G_∞, the union of all finite-dimensional grassmannians. G_∞ turns out to be an ind-variety [Ku] and it is dense in various topologies inside $\widetilde{UGM}$. G_∞ is an interesting object in itself. Using the points of G_∞ expressed as infinite wedge products, in [Ka] Kac constructs an infinite-dimensional representation of an infinite-dimensional general linear group and shows the correspondence between points of the infinite-dimensional grassmannian and solutions of the KP hierarchy with algebraic methods.

In the present work we want to study the infinite-dimensional grassmannians $\widetilde{UGM}$ and G_∞ using only algebraic methods, exhibiting explicitly their coordinate rings. This approach turns to be the most natural for our goal, that is to obtain their quantum deformations.

This paper is divided in three parts.

In the first part, §2, we consider the inverse and direct limit of the coordinate rings $k[\delta_{m,n}]$ of the finite-dimensional grassmannian over the algebraically closed field k. Then we give an explicit presentation for the inverse limit $k[\widehat{\delta_\infty}]$ and the direct limit $k[\delta_\infty]$. We also prove that $k[\widehat{\delta_\infty}]$ and $k[\delta_\infty]$ can be in some sense regarded as the homogeneous coordinate rings of G_∞ and $\widetilde{UGM}$. In fact the closed points of $\text{Proj}\,(k[\widehat{\delta_\infty}])$ and of $\text{Proj}\,(k[\delta_\infty])$ turn to be in one-to-one correspondence with the points of G_∞ and $\widetilde{UGM}$ respectively. Both G_∞ and $\widetilde{UGM}$ admit an action of the infinite-dimensional special linear group SL_∞ given by the union of all finite dimensional special linear groups over k. We also show that there is a corresponding coaction of the homogeneous coordinate ring of the ind-variety SL_∞ on both $k[\widehat{\delta_\infty}]$ and $k[\delta_\infty]$. The results in this part are more or less known,

however since we could not locate rigourous proofs for most statements, we included them for completeness.

In the second part of the paper, §3, we repeat these same constructions in the quantum groups setting. We give explicit quantum deformations for both the ind-variety G_∞ and the Sato grassmannian $\widehat{UGM}$. Proceeding in the same way as in §2, we take the inverse and direct limit of the quantum finite-dimensional grassmannian $k_q[\Delta_{m,n}]$ ([Fi1], [TT]). We obtain two non commutative rings, $k_q\widehat{[\Delta_\infty]}$ and $k_q[\Delta_\infty]$ deformations of $k\widehat{[\delta_\infty]}$ and $k[\delta_\infty]$ respectively that we call quantum G_∞ and quantum Sato grassmannian $\widehat{UGM}$. We give an explicit presentation for both of them. G_∞ and $\widehat{UGM}$ are quantized as homogeneous spaces, that is there is a well defined coaction of the quantum special linear infinite-dimensional group $k_q[SL_\infty]$ on them.

In the last part, §4, we examine the following problem of classical invariant theory for the infinite-dimensional case: given the natural right action of the special linear group of order r, $SL_{r,0}(k)$ on the matrix algebra, find the $SL_{r,0}(k)$-invariants. In complete analogy to what happens in the finite-dimensional case, the ring of invariants in the infinite-dimensional case coincides with $k\widehat{[\delta_\infty]}$ the homogeneous coordinate ring for the ind-variety G_∞. We then obtain the corresponding results for the quantum case, generalizing the results in the paper [FH].

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2. The infinite-dimensional grassmannians G_∞ and $\widehat{UGM}$.

Let k be an algebraically closed field of characteristic 0.

Let $G_{(m,n)}$ be the grassmannian of m dimensional subspaces in a vector space of dimension $N = m + n$.

An element of $G_{(m,n)}$ is represented by a $N \times m$ matrix. We will assume (following Sato [Sa2]) that the row indices go from $-m$ to $n-1$ while the column indices go from $-m$ to -1.

Let $k[a_{i,j}]_{m,n}$ be the coordinate ring of the algebra of the $N \times N$ matrices, where we assume that both row and column indices go from $-m$ to $n-1$.

The homogeneous coordinate ring of $G_{(m,n)}$ is isomorphic to the subring of the matrix ring $k[a_{i,j}]_{m,n}$ generated by the determinants $d_{l_0 \ldots l_{m-1}}$ of the minors obtained by taking the columns $-m \ldots -1$ and the rows

$l_0 \ldots l_{m-1}$. We will denote such subring by $k[\delta_{m,n}]$ and the above-mentioned set of determinants by $\delta_{m,n}$.

DEFINITION (2.1).   Let $m' \geqslant m$, $n' \geqslant n$. Define the inverse family of rings:

$$k[\delta_{m',n'}] \xrightarrow{e_{(m',n',m,n)}} k[\delta_{m,n}]$$

$$e_{(m',n',m,n)}(d_{j_0 \ldots j_{m'-1}}) =$$

$$= \begin{cases} d_{l_0 \ldots l_{m-1}} & \text{if } (j_0 \ldots j_{m'-1}) = (-m' \ldots -m-1, l_0 \ldots l_{m-1}) \\ & \text{and } -m \leqslant l_0 < \ldots < l_{m-1} \leqslant n-1 \\ \\ 0 & \text{otherwise} \end{cases}$$

with $-m' \leqslant j_0 < \ldots < j_{m'-1} \leqslant n'-1$.
   We define:

$$k[\widehat{\delta_\infty}] = \varprojlim k[\delta_{m,n}],$$

and denote the induced maps by

$$e_{(m,n)} \colon k[\widehat{\delta_\infty}] \to k[\delta_{m,n}].$$

We observe that the maps $e_{(m',n',m,n)}$ are induced by maps

$$E_{(m',n',m,n)} \colon k[a_{i,j}]_{m',n'} \to k[a_{i,j}]_{m,n}$$

defined by

$$E_{(m',n',m,n)}(a_{i,j}) = a_{i,j}, \quad \forall \ -m \leqslant i \leqslant n-1, \ -m \leqslant j \leqslant -1$$

$$E_{(m',n',m,n)}(a_{i,j}) = 1, \quad \forall \ -m' \leqslant i = j \leqslant -m-1$$

$$E_{(m',n',m,n)}(a_{i,j}) = 0 \ \text{otherwise}.$$

We define:

$$k[M_\infty] = \varprojlim k[a_{i,j}]_{m,n}.$$

REMARK (2.2).   Any element $b \in k[M_\infty]$ is an element of the form $b = \{b_{(m,n)}\}$ such that $b_{(m,n)} \in k[a_{i,j}]_{m,n}$ and for all $m' \geqslant m$, $n' \geqslant n$, $E_{(m',n',m,n)}(b_{(m',n')}) = b_{(m,n)}$. Similarly any element $x \in k[\widehat{\delta_\infty}]$ is of the form $x = \{x_{(m,n)}\}$ such that $x_{(m,n)} \in k[\delta_{i,j}]_{m,n}$ and for all $m' \geqslant m$, $n' \geqslant n$, $e_{(m',n',m,n)}(x_{(m',n')}) = x_{(m,n)}$.

There is a corresponding direct system of inclusions of projective varieties

$$G_{(m, n)} \to G_{(m', n')} \quad \forall\, m' \geqslant m, \quad n' \geqslant n.$$

Define

$$G_\infty = \varinjlim G(m, n).$$

Notice that $G_\infty = \cup G_{(m, n)}$.

We want to view G_∞ as a projective ind-variety.

DEFINITION (2.3). An *ind-variety over k* is a set X together with a filtration:

$$X_0 \subset X_1 \subset X_2 \subset \ldots$$

such that

1) $\bigcup_{n \geqslant 0} X_n = X$

2) Each X_n is a finite-dimensional variety over k such that the inclusion $X_n \subset X_{n+1}$ is a closed immersion.

(See [Ku] for more details).

The ind-variety X is naturally a topological space, $U \subset X$ being *open* in X if and only if, for each n, $U \cap X_n$ is open in X_n. The sheaf of *regular functions on X* is defined by $\mathcal{O}_X := \varprojlim \mathcal{O}_{X_n}$. X is said to be a *locally projective ind-variety* if it admits a filtration such that each X_n is projective. We will say that X is a *projective ind-variety* if it admits a filtration X_n and a line bundle L such that each restriction $L|X_n$ is very ample and the corresponding maps

$$H^0(X_n, L|X_n) \to H^0(X_{n-1}, L|X_{n-1})$$

are surjective. In other words, for each n there are compatible closed immersions $X_n \hookrightarrow \boldsymbol{P}^{N_n} = \boldsymbol{P}(H^0(X_n, L|X_n)^\vee)$ with coordinate rings generated by $H^0(X_n, L|X_n)$ and hence a closed immersion of ind-varieties $X \hookrightarrow \boldsymbol{P}^\infty = \cup \boldsymbol{P}(H^0(X_n, L|X_n)^\vee)$. We define

$$H^0(X, L) := \varprojlim H^0(X_n, L|X_n).$$

Let $S(\boldsymbol{P}^N) = \oplus_{d \geqslant 0} H^0(\boldsymbol{P}^N, \mathcal{O}_{\boldsymbol{P}^N}(d))$ be the homogeneous coordinate ring of $\boldsymbol{P}^N$ and $I(X_n)$ be the homogeneous ideal of $X_n \subset \boldsymbol{P}^{N_n}$, then the homogeneous coordinate ring of $X_n \subset \boldsymbol{P}^{N_n}$ is given by $S(X_n) = S(\boldsymbol{P}^N)/I(X_n)$.

We define the *homogeneous coordinate ring* of the projective ind-variety $X \subset \boldsymbol{P}^\infty$ to be

$$S(X) := \varprojlim S(X_n).$$

THEOREM (2.4). G_∞ *is a projective ind-variety, with homogeneous coordinate ring* $k[\widehat{\delta_\infty}]$.

PROOF. It is well-known that the maps $G_{(m,n)} \to G_{(m',n')}$ are closed immersions (when defined). Let $X_n := G_{(n,n)}$, then we have closed immersions $G_{(m,n)} \hookrightarrow X_{(n+m,n+m)}$. Therefore $X = \cup X_n = \cup G_{(m,n)} = G_\infty$ is an ind-variety. For each $n > 0$ we have the Plücker embeddings $X_n = = G_{n,n} \hookrightarrow \boldsymbol{P}(\wedge^n \boldsymbol{C}^{2n}) = \boldsymbol{P}^{N_n}$. The homogeneous coordinate ring $S(\boldsymbol{P}^{N_n})$ is generated by elements x_I where $I = \{i_1, \ldots, i_n\}$ such that $-n \le i_1 < i_2 < < \ldots < i_n \le n - 1$ and for $n' \ge n$, the closed immersions $\boldsymbol{P}^{N_n} \hookrightarrow \boldsymbol{P}^{N_{n'}}$ correspond to the surjective homomorphisms

$$S(\boldsymbol{P}^{N_{n'}}) \to S(\boldsymbol{P}^{N_n})$$

defined by

$$\varepsilon_{(m',n',m,n)}(x_{i_1 \ldots i_{n'}}) = \begin{cases} x_{i_1 \ldots i_n} & \text{if } (i_1 \ldots i_{n'}) = (-n' \ldots -n-1, i_1 \ldots i_n) \\ & \text{and } -n \le i_1 < \ldots < i_n \le n-1 \\ 0 & \text{otherwise}. \end{cases}$$

The homogeneous coordinate ring of the projective ind-variety $\boldsymbol{P}^\infty = = \bigcup_{n>0} \boldsymbol{P}^{N_n}$ is just generated by $\varprojlim H^0(\boldsymbol{P}^{N_n}, \mathcal{O}_{\boldsymbol{P}^{N_n}}(1))$. The line bundle $L|X_n$ is just the pull-backck of $\mathcal{O}_{\boldsymbol{P}(\wedge^n \boldsymbol{C}^{2n})}(1)$. The immersions $\boldsymbol{P}^{N_n} \to \boldsymbol{P}^{N_{n'}}$ and $X_n \to X_{n'}$ are compatible. The corresponding homogeneous coordinate ring is $S(X_n) = k[\delta_{n,2n}]$. The maps induced by the inclusions $X_n \to X_{n'}$ are just the maps $e_{(n',n',n,n)} \colon k[\delta_{n',2n'}] \to k[\delta_{n,2n}]$. Therefore the homogeneous coordinate ring of X is given by

$$\varprojlim k[\delta_{n,2n}] = \varprojlim k[\delta_{m,n}] = k[\widehat{\delta_\infty}]. \qquad \text{QED}.$$

We now turn our attention to the Sato grassmannian $\widehat{UGM}$ and its relation with G_∞.

DEFINITION (2.5). Let $m' \geq m$, $n' \geq n$. Define a direct family of rings:

$$k[\delta_{m,n}] \xrightarrow{\ r_{(m,n,m',n')}\ } k[\delta_{m',n'}]$$

$$d_{l_0 \ldots l_{m-1}} \longmapsto d_{-m', \ldots -m-1, l_0 \ldots l_{m-1}}$$

for $-m \leq l_0 < \ldots < l_{m-1} \leq n-1$.

It is easy to see using the Plücker relations that this map is well defined. Moreover, the map $e_{(m,n,m',n')}$ is a left inverse for $r_{(m,n,m',n')}$ and in particular $r_{(m,n,m',n')}$ is injective. We define:

$$k[\delta_\infty] = \varinjlim k[\delta_{m,n}].$$

Denote with $r_{(m,n)}$ the induced inclusions $k[\delta_{m,n}] \to k[\delta_\infty]$.

DEFINITION (2.6). Define Maya diagram of virtual cardinality 0 (or shortly a *Maya diagram*) a strictly increasing sequence $a_\bullet = \{a_i\}$, $i \geq 1$, such that $a_i \in \mathbf{Z}$ and $a_i = i$ for all $i \gg 0$. Define the order $\|a_\bullet\|$ of a Maya diagram to be the smallest number i such that $a_j = j$ for all $j \geq i$. Any sequence $l_* = l_1, \ldots, l_m$ with $l_1 \leq \ldots \leq l_m \leq m$ induces a Maya diagram $a_\bullet = \tilde{l}_*$ of order at most $m + 1$ defined by $a_i = l_i$ for all $1 \leq i \leq m$ and $a_i = i$ for all $i \geq m + 1$. For any Maya diagram $a_\bullet$, let $a_{\leq m}$ denote the ordered set $a_1 < \ldots < a_m$. Clearly if $\|a_\bullet\| \leq m$, then $a_\bullet = \widetilde{a_{\leq m}}$.

Given a Maya diagram $a_\bullet$ of order $m+1$ with $|a_1| \geq -n+1$, we wish to define corresponding elements $d_{a_\bullet} \in k[\delta_\infty]$, and $\widehat{d}_{a_\bullet} \in k\widehat{[\delta_\infty]}$. Define

$$d_{a_\bullet} = r_{(m,n)} d_{a_{\leq m}}$$

where $a_\bullet = \widetilde{a_{\leq m}}$.

$k[\delta_\infty]$ is generated as a ring by the $d_{a_\bullet}$, since it is generated by the images of $k[\delta_{m,n}]$ under $r_{(m,n)}$.

We define a map

$$\varrho_{(m,n)} : k[\delta_\infty] \to k[\delta_{m,n}]$$

$$\varrho_{(m,n)}(d_{a_\bullet}) = \begin{cases} d_{a_{\leq m}} & \text{for all } m \geq \|a_\bullet\|, \ n \geq |a_1| \\ 0 & \text{otherwise.} \end{cases}$$

Then we define

$$\widehat{d}_{a_\bullet} = \{\varrho_{(m,n)} d_{a_\bullet}\} \in k\widehat{[\delta_\infty]}.$$

PROPOSITION (2.7). *a) There is an injection* $I : k[\delta_\infty] \rightarrow k[\widehat{\delta_\infty}]$ *which sends* $d_{a_\bullet}$ *to* $\widehat{d}_{a_\bullet}$.

b) The image of I *is dense in* $k[\widehat{\delta_\infty}]$ *for the inverse limit topology on* $k[\widehat{\delta_\infty}]$ *induced by the discrete topology on each* $k[\delta_{m,n}]$.

PROOF. (a) It suffices to check that for $m' \geqslant m$ and $n' \geqslant n$, one has

$$e_{(m',m,n',n)}(\varrho_{(m',n')}d_{a_\bullet}) = \varrho_{(m,n)}d_{a_\bullet}.$$

(b) Define a fundamental set of neighbourhoods of $0 \in k[\widehat{\delta_\infty}]$ by $U_0 = k[\widehat{\delta_\infty}]$, $U_k = e_{(k,k)}^{-1}(0)$. For any $x = \{x_{(m,n)}\} \in k[\widehat{\delta_\infty}]$, we must define a Cauchy sequence $\{y_k\} = \left\{ \sum_{i=1}^{m_k} b_i \widehat{d}_{a_\bullet^{i,1}} \dots \widehat{d}_{a_\bullet^{i,j_i}} \right\}$ lying in the image of I and converging to x.

Since $e_{(k,k)}(x) \in k[\delta_{k,k}]$ we have

$$e_{(k,k)}(x) = \sum_{i=1}^{m_k} b_i^k d_{L_k^{i,1}} \dots d_{L_k^{i,j_i}}$$

where $L_k^{i,j} = (l_{k;1}^{i,j} \dots l_{k;m}^{i,j})$, $-k \leqslant l_{k;1}^{i,j} < \dots < l_{k;m}^{i,j} \leqslant k-1$. Set

$$y_k = \sum_{i=1}^{m_k} b_i^k \widehat{d}_{\widetilde{L_k^{i,1}}} \dots \widehat{d}_{\widetilde{L_k^{i,j_i}}}.$$

Notice that the summation is finite and hence y_k is in the image of I. For any $k' > k$ we have, $e_{(k,k)}(y_{k'}) = e_{(k,k)}(x)$ i.e. $y_{k'} - x \in U_k$. Hence $\{y_k\}$ is a Cauchy sequence converging to x.   QED.

We now give a presentation of the rings $k[\delta_\infty]$ and $k[\widehat{\delta_\infty}]$.

Define $k[\xi_{a_\bullet}]$ to be the ring generated by the independent variables $\xi_{a_\bullet}$, where $a_\bullet$ is any Maya diagram of virtual cardinality 0.

There is a natural map $\phi : k[\xi_{a_\bullet}] \rightarrow k[\delta_\infty]$ such that $\xi_{a_\bullet} \rightarrow d_{a_\bullet}$. This induces a topology on $k[\xi_{a_\bullet}]$ for which a fundamental set of neighborhoods is given by $V_k := \phi^{-1}I^{-1}U_k$. Let $k[\widehat{\xi_{a_\bullet}}]$ be the completion of $k[\xi_{a_\bullet}]$ with respect to the above topology. In particular the elements of $k[\widehat{\xi_{a_\bullet}}]$ are of the form $\sum b_i \xi_{a_\bullet^{i,1}} \dots \xi_{a_\bullet^{i,k_i}}$ where $b_i \in k$ and the $a_\bullet^{i,k}$ are any Maya diagrams of virtual cardinality 0. The corresponding natural map between completions $\widehat{\phi} : k[\widehat{\xi_{a_\bullet}}] \rightarrow k[\widehat{\delta_\infty}]$ is defined by $\xi_{a_\bullet} \rightarrow \widehat{d}_{a_\bullet}$.

Define $P_{(m,n)}$ to be the ideal of Plücker relations in $k[\delta_{m,n}]$. Let $P = \cup P_{(m,n)}$ be the corresponding ideal in $k[\delta_\infty]$, and similarly $\widehat{P} = \varprojlim P_{(m,n)}$ be the corresponding ideal in $k[\widehat{\delta_\infty}]$.

THEOREM (2.8). *We have ring isomorphisms*

$$i) \ k\widehat{[\delta_\infty]} \cong k\widehat{[\overline{\xi}_{a_\bullet}]}/\widehat{P}$$

$$ii) \ k[\delta_\infty] \cong k[\xi_{a_\bullet}]/P$$

PROOF. (i) The direct limit is an exact functor.

(ii) The inverse limit functor is left exact, and since the inverse system $P_{(m,n)}$ is a surjective system, the corresponding inverse system sequence

$$0 \to \widehat{P} \to k\widehat{[\overline{\xi}_{a_\bullet}]} \to K\widehat{[\delta_\infty]} \to 0$$

is also exact.   QED.

We want now to relate our constructions with the Sato grassmannian. In [Sa2] Sato defines a set of points in an infinite-dimensional projective space. Already theorem (2.8) suggests to view G_∞ as the set of zeros of the ideal $\widehat{P}$ in an infinite-dimensional projective space whose coordinate ring is given by $k\widehat{[\overline{\xi}_{a_\bullet}]}$. We want to make this euristic notion more precise and to relate the ring $k[\delta_\infty]$ with the Sato grassmannian.

Assume that the field k has cardinality strictly greater than $\aleph_0$.

Consider the directed system given by rings $R_n = k[z_1, \ldots, z_n]$ and homomorphisms of $k[z_1, \ldots, z_{n'}] \to k[z_1, \ldots, z_n]$ (for $n' > n$) defined by $z_i \to z_i$ for all $i \le n$, and $z_i \to 0$ for $i > n$. This corresponds to an affine ind-variety $A^\infty = \cup A^n$ given by the inclusion of affine planes $A^n = \mathrm{Spec}\,(R_n) \hookrightarrow A^{n+1} = \mathrm{Spec}\,(R_{n+1})$. Let

$$\widehat{R} = \varprojlim R_n.$$

LEMMA (2.9). *The set of closed points of A^∞ is in one to one correspondence with* $\mathrm{Spec}_m(\widehat{R})$.

PROOF. Notice that each R_n injects in a natural way in $\widehat{R}$ and set $R = \cup R_n \subset \widehat{R}$. If $m \subset R$ is maximal, $R/m = E$ is a field. By [La] we have $E = k$ and therefore m is generated by $z_i - k_i$ where k_i is the image of $z_i \in R/m$. If m' is a maximal ideal of $\widehat{R}$, and $f : \widehat{R} \to \widehat{R}/m'$, then by the previous observations, the induced maps $f_i : R_i \to \widehat{R}/m'$ have image contained in k. By the universal property of inverse limits then also $f : \widehat{R} \to k$ is determined by $k_i = f(z_i)$. It is clear that in order for f to be defined, one must have $k_i = 0$ for all but finitely many i.   QED.

Recall that $k[\widehat{\xi_{a_\bullet}}] = \varprojlim S(\boldsymbol{P}^{N_n})$. It follows that $\boldsymbol{P}^\infty = \operatorname{Proj}(k[\widehat{\xi_{a_\bullet}}])$ and that the closed points of $\boldsymbol{P}^\infty$ are given by sequences $k_{a_\bullet} \in k$ where $a_\bullet = 0$ for all but finitely many Maya diagrams of virtual cardinality 0 we have $a_\bullet = 0$ and two sequences are considered equivalent if there exists $\lambda \in k^*$ such that $k_{a_\bullet} = \lambda k'_{a_\bullet}$. The assertion can be verified locally on the open cover $\operatorname{Spec}(k[\widehat{\xi_{a_\bullet}}]_{(\xi_{\bar{a}_\bullet})})$ where $k[\widehat{\xi_{a_\bullet}}]_{(\xi_{\bar{a}_\bullet})}$ denotes the subring of elements of degree 0 in the localized ring $k[\widehat{\xi_{a_\bullet}}]_{\xi_{\bar{a}_\bullet}}$. The computation is now analogous to the one above for $\widehat{R}$. Similarly one has that the closed points of $\operatorname{Proj}(k[\xi_{a_\bullet}])$ correspond to all sequences (not necessarily bounded) $k_{a_\bullet} \in$ $\in k$ where $a_\bullet$ runs over all Maya diagrams of virtual cardinality 0 and two sequences are considered equivalent if there exists $\lambda \in k^*$ such that $k_{a_\bullet} = \lambda k'_{a_\bullet}$. It follows that

PROPOSITION (2.10). *Assume that the cardinality of k is strictly greater than $\aleph_0$.*

   i) *The set of closed points of G_∞ is in one to one correspondence with the set of closed points of $\operatorname{Proj}(k[\widehat{\delta_\infty}])$, i.e. with the sequences $\{k_{a_\bullet} \in$ $\in k\}$ satisfying all Plücker relations, where $a_\bullet$ Maya diagram of virtual cardinality 0 and $k_{a_\bullet} = 0$ for all but finitely many Maya diagrams.*

   ii) *The set of closed points of $\operatorname{Proj}(k[\delta_\infty])$ is in one to one correspondence with the sequences $\{k_{a_\bullet} \in k\}$ satisfying all Plücker relations, where $a_\bullet$ Maya diagram of virtual cardinality 0. Moreover we have that those points coincide with $\widehat{UGM}$ the Sato grassmannian (as defined by Sato, [Sa1], [Sa2]).*

REMARK (2.11). Proposition (2.10) shows that the ring $k[\delta_\infty]$ can be regarded as the «coordinate ring» for the Sato grassmannian in the sense that its maximal ideals are in one-to-one correspondence with the points of $\widehat{UGM}$. Theorem (2.9) allows us to interpret the Sato grassmannian as the set of closed points in an infinite-dimensional projective space that are subjected to the relations P.

We now want to define an infinite-dimensional special linear group and show that it has an action on both G_∞ and $\widehat{UGM}$.

DEFINITION (2.12). For all m, n positive integers, define $SL_{(m,n)}(k) \cong SL_N(k)$ as $N \times N$ matrices with determinant 1, whose row and column indicies are between $-m$ and $n-1$. The inclusions $\psi_{(m,n,m',n')}: SL_{m,n}(k) \rightarrow SL_{m',n'}(k)$ are defined for all $m' \geq m$, $n' \geq n$,

$\psi_{(m, n, m', n')}(g) = \mathrm{diag}(Id_{m-m'}, g, Id_{n-n'})$. It is clear that we have an action of $SL_{m,n}(k)$ on $G_{(m,n)}$ for all m, n. We have a corresponding projective system of coordinate rings:

$$k[SL_{m',n'}] \xrightarrow{\phi_{(m',n',m,n)}} k[SL_{m,n}]$$

$$\phi_{(m',n',m,n)}(g_{ij}) = \begin{cases} g_{ij} & \text{if } -m \leqslant i, j \leqslant n-1 \\ 1 & \text{if } -m' \leqslant i = j \leqslant -m-1,\, n \leqslant i = j \leqslant n'-1 \\ 0 & \text{otherwise} . \end{cases}$$

OBSERVATION (2.13).

$$SL_\infty(k) =_{\mathrm{def}} \varinjlim SL_{m,n} = \cup SL_{m,n}(k)$$

is an ind-variety with coordinate ring

$$k[SL_\infty] =_{\mathrm{def}} \varprojlim k[SL_{m,n}].$$

We want now to show that $k[SL_\infty]$ has an Hopf algebra structure. Notice that while in the finite-dimensional case this is an immediate consequence of the fact that the variety $SL_{m,n}(k)$ is a group, in the infinite-dimensional case we need to check the commutativity of certain diagrams.

PROPOSITION (2.14). *Let $(m', n') > (m, n)$. The following are commutative diagrams:*

i)

$$
\begin{array}{ccc}
k[SL_{m',n'}] & \xrightarrow{\phi_{(m',n',m,n)}} & k[SL_{m,n}] \\
\downarrow{\scriptstyle \Delta_{(m',n')}} & & \downarrow{\scriptstyle \Delta_{(m,n)}} \\
k[SL_{m',n'}] \otimes k[SL_{m',n'}] & \xrightarrow{\phi_{(m',n',m,n)} \otimes \phi_{(m',n',m,n)}} & k[SL_{m,n}] \otimes k[SL_{m,n}]
\end{array}
$$

where $\Delta_{(m,n)}$ is the comultiplication in the Hopf algebra $k[SL_{m,n}]$.

ii)

$$k[SL_{m',n'}] \xrightarrow{\phi_{(m',n',m,n)}} k[SL_{m,n}]$$

$$\downarrow{\varepsilon_{(m',n')}} \qquad\qquad \downarrow{\varepsilon_{(m,n)}}$$

$$k \xrightarrow{id} k$$

where $\varepsilon_{(m,n)}$ is the counit in the Hopf algebra $k[SL_{m,n}]$.
  iii)

$$k[SL_{m',n'}] \xrightarrow{\phi_{(m',n',m,n)}} k[SL_{m,n}]$$

$$\downarrow{S_{(m',n')}} \qquad\qquad \downarrow{S_{(m,n)}}$$

$$k[SL_{m',n'}] \xrightarrow{\phi_{(m',n',m,n)}} k[SL_{m,n}]$$

where $S_{(m,n)}$ is the antipode in the Hopf algebra $k[SL_{m,n}]$.

PROOF. Direct check.

COROLLARY (2.15). *$k[SL_\infty]$ has an Hopf algebra structure given by:*
  a) comultiplication

$$k[SL_\infty] \xrightarrow{\Delta_\infty} k[SL_\infty]\widehat{\otimes}k[SL_\infty]$$

$$\{a_{(m,n)}\} \longmapsto \{\Delta_{(m,n)}(a_{(m,n)})\}$$

  b) counit

$$k[SL_\infty] \xrightarrow{\varepsilon_\infty} k$$

$$\{a_{(m,n)}\} \longmapsto \varepsilon_{(m,n)}(a_{(m,n)})$$

  c) antipode

$$k[SL_\infty] \xrightarrow{S_\infty} k[SL_\infty]$$

$$\{a_{(m,n)}\} \longmapsto \{S_{(m,n)}(a_{(m,n)})\}$$

where $\widehat{\otimes}$ denotes the completed tensor product and is given by

$$k[SL_\infty]\widehat{\otimes}k[SL_\infty] = \varprojlim k[SL_{m,n}]\otimes k[SL_{m,n}]$$

(see [Ku] for more details).

PROOF. (a) is immediate from proposition (2.14) and from [Ku]. (b), (c) are immediate from proposition (2.14).

The group $SL_\infty(k)$ has an action on both G_∞ and $\widetilde{UGM}$. In order to obtain a quantization of these actions we need to describe the corresponding coactions of $k[SL_\infty]$ on $k[\widehat{\delta_\infty}]$ and $k[\delta_\infty]$.

OBSERVATION (2.16). Since $SL_{m,n}(k)$ acts on $G_{(m,n)}$ we have the coaction:

$$k[\delta_{m,n}] \quad \overset{\lambda_{(m,n)}}{\longrightarrow} \quad k[SL_{m,n}] \otimes k[\delta_{m,n}]$$

$$d_{l_0 \ldots l_{m-1}} \quad \longmapsto \quad \sum_{m \leq k_0 \ldots k_{m-1} \leq n-1} g_{l_0 k_0} \cdots g_{l_{m-1} k_{m-1}} \otimes d_{k_0 \ldots k_{m-1}}$$

One can check the commutativity of the following diagram, for $m' \geq m$, $n' \geq n$:

$$
\begin{array}{ccc}
k[\delta_{m,n}] & \overset{\lambda_{(m,n)}}{\longrightarrow} & k[SL_{m,n}] \otimes k[\delta_{m,n}] \\
\downarrow{\scriptstyle e_{(m',n',m,n)}} & & \downarrow{\scriptstyle \phi_{(m',n',m,n)} \otimes e_{(m',n',m,n)}} \\
k[\delta_{m',n'}] & \overset{\lambda_{(m',n')}}{\longrightarrow} & k[SL_{m',n'}] \otimes k[\delta_{m',n'}]
\end{array}
$$

PROPOSITION (2.17). *There is an coaction of $k[SL_\infty]$ on $k[\widehat{\delta_\infty}]$ and on $k[\delta_\infty]$.*

PROOF. Fix (m_0, n_0). Let $m' > m > m_0$, $n' > n > n_0$. We have a commutative diagram (see observation (2.16)):

$$
\begin{array}{ccc}
k[\delta_{m,n}] & \overset{\lambda_{(m,n,m_0,n_0)}}{\longrightarrow} & k[SL_{m_0,n_0}] \otimes k[\delta_{m,n}] \\
\downarrow{\scriptstyle e_{(m',n',m,n)}} & & \downarrow{\scriptstyle id \otimes e_{(m',n',m,n)}} \\
k[\delta_{m',n'}] & \overset{\lambda_{(m',n',m_0,n_0)}}{\longrightarrow} & k[SL_{m_0,n_0}] \otimes k[\delta_{m',n'}]
\end{array}
$$

$$\lambda_{(m,n,m_0,n_0)}(d_{l_0 \ldots l_{m-1}}) =_{\mathrm{def}} \sum_{m \leq k_0 \ldots k_{m-1} \leq n-1} \tilde{g}_{l_0 k_0} \cdots \tilde{g}_{l_{m-1} k_{m-1}} \otimes d_{k_0 \ldots k_{m-1}}$$

where:

$$\tilde{g}_{ij} = \begin{cases} g_{ij} & \text{if } -m \leq i, j \leq n-1 \\ 1 & \text{if } -m' \leq i = j \leq -m-1, \ n \leq i = j \leq n'-1 \\ 0 & \text{otherwise} \end{cases}$$

Hence there is a map:

$$k[\widehat{\delta_\infty}] \to k[SL_{m_0, n_0}] \otimes \overline{k[\widehat{\delta_\infty}]}.$$

Now taking the inverse limit of $k[SL_{m_0, n_0}]$ on the right side we obtain a coaction of $k[SL_\infty]$ on $k[\widehat{\delta_\infty}]$, that is a map:

$$k[\widehat{\delta_\infty}] \to k[SL_\infty] \widehat{\otimes} \overline{k[\widehat{\delta_\infty}]}$$

where $\widehat{\otimes}$ denotes the completed tensor product (see [Ku]).

By theorem (2.7) (iii) $k[\delta_\infty]$ can be identified with a subring of $k[\widehat{\delta_\infty}]$ and one can check that the given coaction is a well defined coaction when restricted to this subring of $k[\widehat{\delta_\infty}]$.   QED.

3. The quantum infinite-dimensional grassmannians $k_q[G_\infty]$ and $k_q[\widehat{UGM}]$.

We want to obtain a deformation of G_∞ and $\widehat{UGM}$ as quantum homogeneous spaces for a quantum SL_∞. In the language of quantum groups this means that we need to construct deformations of the two rings $k[\widehat{\delta_\infty}]$ and $k[\delta_\infty]$ together with a coaction of a deformation of $k[SL_\infty]$ on them. The naturality of the construction in §2 will allow us to repeat the same arguments used for the commutative case also in the non commutative case with very small changes.

DEFINITION (3.1).   Let $k_q = k[q, q^{-1}]$ and let $k_q\langle a_{i,j}\rangle_{m, n}$ be the free algebra over k_q with a_{ij} as non commutative generators, $-m \leqslant i, j \leqslant n - 1$. Define $k_q[a_{i,j}]_{m, n}$ as the associative k_q-algebra with unit generated by the elements a_{ij}, subject to the relations:

$$a_{ij}a_{kj} = q^{-1}a_{kj}a_{ij}, \quad i < k, \quad a_{ij}a_{kl} = a_{kl}a_{ij}, \quad i < k, j > l \text{ or } i > k, j < l$$

$$a_{ij}a_{il} = q^{-1}a_{il}a_{ij}, \quad j < l, \quad a_{ij}a_{kl} - a_{kl}a_{ij} = (q^{-1} - q)a_{kj}a_{il}, \quad i < k, j < l$$

$k_q[a_{i,j}]_{m, n}$ is a bialgebra with counit and comultiplication:

$$\varepsilon^q_{(m, n)}(a_{ij}) = \delta_{ij} \quad \Delta^q_{(m, n)}(a_{ij}) = \sum a_{ik} \otimes a_{kj}.$$

See [Ma1], [Ma2] for more details.

DEFINITION (3.2).   We define the *quantum determinant obtained by taking rows* $i_1 \dots i_p$, *columns* $j_1 \dots j_p$ as an element $D^{j_1 \dots j_p}_{i_1 \dots i_p} \in k_q\langle a_{i,j}\rangle_{m, n}$

given by:

$$D_{i_1 \ldots i_p}^{j_1 \ldots j_p} = _{\mathrm{def}}$$

$$= _{\mathrm{def}} \sum_{\sigma : (i_1 \ldots i_p) \to (j_1 \ldots j_p)} (-q)^{-l(\sigma)} a_{i_1 \sigma(i_1)} \ldots a_{i_m \sigma(i_p)}, \qquad \begin{array}{l} -m \leqslant i_1 < \ldots < i_p \leqslant n-1 \\[4pt] -m \leqslant j_1 < \ldots < j_p \leqslant n-1 \end{array}$$

where σ runs over all the bijections and $l(\sigma)$ is the length of the permutation σ. p is called the *rank* of $D_{i_1 \ldots i_p}^{j_1 \ldots j_p}$. Its image in $k_q[a_{i,j}]_{m,n}$ is then the usual quantum determinant. We shall write $D_{i_1 \ldots i_p}^{j_1 \ldots j_p}$ for this image also, the context making clear where the element sits. (See [PW] ch. 4 for more details). We will drop the upper indices whenever they coincide with $-p \ldots -1$.

DEFINITION (3.3). Define *the quantum grassmannian ring* $k_q[\Delta_{m,n}]$, as the subring of $k_q[a_{i,j}]_{m,n}$ generated by the quantum determinants $D_{i_0 \ldots i_{m-1}}$ $-m \leqslant i_0 < \ldots < i_{m-1} \leqslant n-1$ (see [Fi1]). We will refer to the set of such determinants with $\Delta_{m,n}$.

An explicit presentation of the ring $k_q[\Delta_{m,n}]$ in terms of generators and relations is given by (see [TT] 3.5, [Fi1], [FH]):

$$q^{-[m-p]} \lambda_J \lambda_I = \lambda_I \lambda_J + \sum_{i=1}^{N} (q^{-1} - q)^i$$

(c)
$$\sum_{(L, L') \in C_i^{i_0 \ldots i_{k_1} \ldots i_{k_p} \ldots i_{m-1}}} (-q)^{-l(\sigma(L)) - l(\sigma(L'))} \lambda_{(L, i_{k_1} \ldots i_{k_p})_{\mathrm{ord}}} \lambda_{(L', i_{k_1} \ldots i_{k_p})_{\mathrm{ord}}}$$

$$I = (i_0 \ldots i_{m-1}) < J = (j_0 \ldots j_{m-1}) \quad I \cap J = \{i_{k_1} \ldots i_{k_p}\}$$

$$i_0 < \ldots < i_{m-1}, \quad j_0 < \ldots < j_{m-1}$$

(y)
$$\sum_{1 \leqslant a_1 < \ldots < a_s \leqslant m+s} (-q)^{-l(z_1 \ldots \hat{z}_{a_1} \ldots \hat{z}_{a_s} \ldots z_{r+s} z_{a_1} \ldots z_{a_s}) - l(z_{a_1} \ldots z_{a_s} l_1 \ldots l_{m-s})}$$

$$\lambda_{z_1 \ldots \hat{z}_{a_1} \ldots \hat{z}_{a_s} \ldots z_{m+s}} \lambda_{(z_{a_1} \ldots z_{a_s} l_1 \ldots l_{m-s})_{\mathrm{ord}}} = 0$$

Each of the relations in the set (y) is computed for any set of fixed indices: $-m \leqslant z_1 < \ldots < z_{m+s} \leqslant n-1$, $-m \leqslant l_1 < \ldots < l_{m-s} \leqslant n-1$.

All the symbols that appear have been defined in [Fi2].

Notice that the relations labeled (c) reduce for $q = 1$ to state the commutativity of the λ_I's while the relations labeled (y) for $q = 1$ become the Young (also called symmetry) relations.

We want now to proceed in analogy with §2 and define the following inverse and direct families.

DEFINITION (3.4).   Let $m' \geqslant m$, $n' \geqslant n$. Define an inverse family of rings:

$$k_q[\varDelta_{m',n'}] \xrightarrow{\ e^q_{(m',n',m,n)}\ } k_q[\varDelta_{m,n}]$$

$$D_{-m'...-m-1 l_0...l_{m-1}} \longmapsto D_{l_0...l_{m-1}}$$

$$D_{l_0...l_{m'-1}} \longmapsto 0 \text{ otherwise}$$

where $-m \leqslant l_0 < ... < l_{m-1} \leqslant n-1$.

We define:

$$k_q\widehat{[\varDelta_\infty]} = \varprojlim k_q[\varDelta_{m,n}].$$

Denote the induced maps

$$e^q_{(m,n)} : k_q\widehat{[\varDelta_\infty]} \to k_q[\varDelta_{m,n}].$$

We observe that the maps $e_{(m',n',m,n)}$ are induced by maps

$$E^q_{(m',n',m,n)} : k_q[a_{i,j}]_{m',n'} \to k_q[a_{i,j}]_{m,n}$$

defined by

$$E^q_{(m',n',m,n)}(a_{ij}) = a_{ij}, \quad \forall \ -m \leqslant i \leqslant n-1, \ -m \leqslant j \leqslant -1$$

$$E^q_{(m',n',m,n)}(a_{ij}) = 1, \quad \forall \ -m' \leqslant i = j \leqslant -m-1$$

$$E^q_{(m',n',m,n)}(a_{ij}) = 0 \text{ otherwise.}$$

We define:

$$k_q[M_\infty] = \varprojlim k_q[a_{i,j}]_{m,n}$$

DEFINITION (3.5).   Let $m' \geqslant m$, $n' \geqslant n$. Define the direct family of rings:

$$k_q[\varDelta_{m,n}] \xrightarrow{\ r^q_{(m,n,m',n')}\ } k_q[\varDelta_{m',n'}]$$

$$D_{l_0...l_{m-1}} \longmapsto D_{-m',...-m-1, l_0...l_{m-1}}$$

for $-m \leqslant l_0 < ... < l_{m-1} \leqslant n-1$. We define:

$$k_q[\varDelta_\infty] = \varinjlim k_q[\varDelta_{m,n}].$$

Denote the induced inclusions $r^q_{(m,n)} : k_q[\varDelta_{m,n}] \to k_q[\varDelta_\infty]$.

OBSERVATION (3.6). Both $r^q_{(m, n, m', n')}$ and $e^q_{(m', n', m, n)}$ are well defined that is they are zero on the relations on the determinants. This can be directly checked.

In analogy to §2, we can define a map:

$$\varrho^q_{(m, n)}: k[\Delta_\infty] \to k[\Delta_{m, n}]$$

$$\varrho^q_{(m, n)}(D_{a_\bullet}) = \begin{cases} D_{a \leqslant m} & \text{for all } m \geqslant \|a_\bullet\|, \ n \geqslant |a_1| \\ 0 & \text{otherwise} \end{cases}$$

Then we define

$$\widehat{D}_{a_\bullet} = \{\varrho^q_{(m, n)} D_{a_\bullet}\} \in k\widehat{[\Delta_\infty]}.$$

Let $k_q\langle \xi_{a_\bullet} \rangle$ to be the non commutative ring generated by the independent variables $\xi_{a_\bullet}$, where $a_\bullet$ is any Maya diagram of virtual cardinality 0.

In analogy with §2 there is a natural map $\phi_q: k\langle \xi_{a_\bullet} \rangle \to k[\Delta_\infty]$ such that $\xi_{a_\bullet} \to D_{a_\bullet}$. This induces a topology on $k\langle \xi_{a_\bullet} \rangle$ for which a fundamental set of neighborhoods is given by $V^q_k := \phi_q^{-1} I_q^{-1} U^q_k$, where U^q_k is a fundamental set of neighbourhoods in $k\widehat{[\Delta_\infty]}$, defined by: $U^q_0 = k\widehat{[\Delta_\infty]}$, $U^q_k = (e^q_{(k, k)})^{-1}(0)$ and I_q is the natural map from $k[\Delta_\infty]$ and $k[\Delta_\infty]$ defined as $I_q(D_{a_\bullet}) = \widehat{D}_{a_\bullet}$.

Let $k\widehat{\langle \xi_{a_\bullet} \rangle}$ be the completion of $k\langle \xi_{a_\bullet} \rangle$ with respect to the above topology. In particular the elements of $k\widehat{\langle \xi_{a_\bullet} \rangle}$ are of the form $\sum b_i \xi_{a^{i, 1}_\bullet} \ldots \xi_{a^{i, k_i}_\bullet}$ where $b_i \in k$ and the $a^{i, k}_\bullet$ are any Maya diagrams of virtual cardinality 0. The corresponding natural map between completions $\widehat{\phi^q}: k\widehat{\langle \xi_{a_\bullet} \rangle} \to k\widehat{[\Delta_\infty]}$ is defined by $\xi_{a_\bullet} \to \widehat{D}_{a_\bullet}$.

Define $P_{(m, n), q}$ to be the two-sided ideal generated by the relations (c) and (y) in $k_q[\Delta_{m, n}]$. Let $P_q = \cup P_{(m, n), q}$ be the corresponding two-sided ideal in $k[\Delta_\infty]$, and similarly $\widehat{P}_q = \varprojlim P_{(m, n)}$ be the corresponding ideal in $k_q\widehat{[\Delta_\infty]}$. As in the commutative case we have the following theorem.

THEOREM (3.7). Quantum deformation of the ind-variety G_∞ and the Sato grassmannian $\widehat{UGM}$.

$$k_q\widehat{[\Delta_\infty]} \cong k_q\widehat{\langle \xi_{a_\bullet} \rangle}/\widehat{P}_q$$

$$k_q[\Delta_\infty] \cong k_q\langle \xi_{a_\bullet} \rangle/P_q.$$

PROOF. Same as (2.9).

REMARK (3.8).   If we specialize q to 1 we obtain theorem (2.9), that is a presentation of the commutative rings $k[\delta_\infty]$ and $k\widehat{[\delta_\infty]}$. Since by (2.11) $\mathrm{Spec}_m(k[\delta_\infty])$ coincides with G_∞ and $\mathrm{Spec}_m(k\widehat{[\delta_\infty]})$ with $\widehat{UGM}$ we refer to the rings $k_q[\varDelta_\infty]$ and $k_q\widehat{[\varDelta_\infty]}$ as *quantum ind-variety G_∞* and *quantum Sato grassmannian* respectively.

Let's now proceed to show that the quantum ind-variety G_∞ and the quantum Sato grassmannian are quantum homogeneous spaces.

DEFINITION (3.9).   In complete analogy with §2 define

$$k_q[SL_{m,n}] = k_q[a_{i,j}]_{m,n}/(D^{-m\ldots n-1}_{-m\ldots n-1} - 1).$$

$k_q[SL_{m,n}]$ is a quantum group, that is an Hopf algebra, with antipode:

$$S^q_{(m,n)}(a_{ij}) = (-q)^{i-j} D^{-m\ldots\hat{j}\ldots n-1}_{-m\ldots\hat{i}\ldots n-1}.$$

The coalgebra structure (i.e. the comultiplication and counit maps) is naturally inherited from the matrix bialgebra $k_q[a_{i,j}]_{m,n}$. For more details see [Ma1], [Ma2].

Then define the inverse system:

$$k_q[SL_{m',n'}] \overset{\phi^q_{(m',n',m,n)}}{\longrightarrow} k_q[SL_{m,n}]$$

$$\phi^q_{(m',n',m,n)}(g_{ij}) = \begin{cases} g_{ij} & \text{if } -m \leqslant i,j \leqslant n-1 \\ 1 & \text{if } -m' \leqslant i = j \leqslant m'-1,\ n \leqslant i = j \leqslant n'-1 \\ 0 & \text{otherwise} \end{cases}$$

One can directly check that these maps are well defined.

Define *quantum infinite-dimensional special linear group*:

$$k_q[SL_\infty] =_{\mathrm{def}} \varprojlim k_q[SL_{m,n}]$$

Notice that for $q = 1$ this coincides with the coordinate ring of the ind-variety $SL_\infty(k)$.

We now intend to show that $k_q[SL_\infty]$ is a quantum group that is it admits an Hopf algebra structure. This will be proved in the same exact way we did for $k[SL_\infty]$ in §2.

PROPOSITION (3.10).   *Let $(m',n') > (m,n)$. The following diagrams are commutative:*

i)

$$k_q[SL_{m',n'}] \xrightarrow{\phi^q_{(m',n',m,n)}} k_q[SL_{m,n}]$$

$$\downarrow{\Delta^q_{(m',n')}} \qquad\qquad\qquad\qquad \downarrow{\Delta^q_{(m,n)}}$$

$$k_q[SL_{m',n'}] \otimes k_q[SL_{m',n'}] \xrightarrow{\phi_{(m',n',m,n)} \otimes \phi^q_{(m',n',m,n)}} k_q[SL_{m,n}] \otimes k_q[SL_{m,n}]$$

where $\Delta^q_{(m,n)}$ is the comultiplication in $k_q[SL_{m,n}]$.

ii)

$$k_q[SL_{m',n'}] \xrightarrow{\phi_{(m',n',m,n)}} k_q[SL_{m,n}]$$

$$\downarrow{\varepsilon^q_{(m',n')}} \qquad\qquad\qquad \downarrow{\varepsilon^q_{(m,n)}}$$

$$k_q \xrightarrow{id} k$$

where $\varepsilon^q_{(m,n)}$ is the counit in $k_q[SL_{m,n}]$.

iii)

$$k_q[SL_{m',n'}] \xrightarrow{\phi_{(m',n',m,n)}} k_q[SL_{m,n}]$$

$$\downarrow{S^q_{(m',n')}} \qquad\qquad\qquad \downarrow{S^q_{(m,n)}}$$

$$k_q[SL_{m',n'}] \xrightarrow{\phi_{(m',n',m,n)}} k_q[SL_{m,n}]$$

where $S_{(m,n)}$ is the antipode in $k_q[SL_{m,n}]$.

PROOF. Direct check. Notice that here the check involves also the non commutative relations among the generators.

COROLLARY (3.11). $k_q[SL_\infty]$ has an Hopf algebra structure given by:

a) *comultiplication*

$$k_q[SL_\infty] \xrightarrow{\Delta^q_\infty} k_q[SL_\infty] \bar{\otimes} k_q[SL_\infty]$$

$$\{a_{(m,n)}\} \mapsto \{\Delta^q_{(m,n)}(a_{(m,n)})\}$$

b) *counit*

$$k_q[SL_\infty] \xrightarrow{\varepsilon^q_\infty} k_q$$

$$\{a_{(m,n)}\} \mapsto \varepsilon^q_{(m,n)}(a_{(m,n)})$$

c) *antipode*

$$k_q[SL_\infty] \xrightarrow{S_\infty^q} k_q[SL_\infty]$$
$$\{a_{(m,n)}\} \mapsto \{S_{(m,n)}^q(a_{(m,n)})\}$$

where $\widehat{\otimes}$ *denotes the completed tensor product (see* [Ku] *for more details).*

Now we are ready to show that $k[\widehat{\Delta_\infty}]$ and $k[\Delta_\infty]$ are quantum homogeneous spaces.

We have the following coaction (see [Fi1]):

$$k_q[\Delta_{m,n}] \xrightarrow{\lambda_{(m,n)}^q} k_q[SL_{m,n}] \otimes k_q[\Delta_{m,n}]$$
$$D_{l_0 \ldots l_{m-1}} \mapsto \sum_{m \le k_0 \ldots k_{m-1} \le n-1} g_{l_0 k_0} \cdots g_{l_{m-1} k_{m-1}} \otimes D_{k_0 \ldots k_{m-1}}$$

One can check the commutativity of the following diagram, for $m' \ge m$, $n' \ge n$:

$$
\begin{array}{ccc}
k_q[\Delta_{m,n}] & \xrightarrow{\lambda_{(m,n)}^q} & k_q[SL_{m,n}] \otimes k[\delta_{m,n}] \\
\downarrow{e_{(m',n',m,n)}^q} & & \downarrow{\phi_{(m',n',m,n)}^q \otimes e_{(m',n',m,n)}^q} \\
k_q[\Delta_{m',n'}] & \xrightarrow{\lambda_{(m',n')}^q} & k_q[SL_{m',n'}] \otimes k_q[\Delta_{m',n'}]
\end{array}
$$

PROPOSITION (3.12). *There is a coaction of* $k_q[SL_\infty]$ *on* $k_q[\widehat{\Delta_\infty}]$ *and on* $k_q[\Delta_\infty]$.

PROOF. Same as (2.18).

4. Infinite-dimensional invariant theory for SL_∞ and its quantum deformation $k_q[SL_\infty]$.

The first fundamental theorem of invariant theory for the special linear group $SL_{m,0}(k)$ (see (2.12) for the notation) states that given the right action of $SL_{m,0}(k)$ on the matrix algebra $k[b_{i,j}]_{m,n}$, where $m < n$, $-m \le i \le n-1$, $-m \le j \le -1$:

$$k[b_{i,j}]_{m,n} \times SL_{m,0}(k) \rightarrow k[b_{i,j}]_{m,n}$$
$$(b_{ij}, g) \mapsto \sum b_{ik} g_{kj}$$

the subring of invariants $k[b_{i,j}]_{m,n}^{SL_{m,0}(k)}$ coincides with the subring generated by the determinants of rank m in $k[b_{i,j}]_{m,n}$. We want to generalize this result to the infinite-dimensional case.

OBSERVATION. (4.1). There is a natural right action of $SL_{m,0}(k)$ on $M_{m,n}$ the set of matrices with row indices from $-m$ to $n-1$ and column indices from -1 to $-m$.

$$M_{m,n} \times SL_{m,0}(k) \quad \rightarrow \quad M_{m,n}$$
$$A, g \qquad \mapsto \qquad Ag$$

This action gives rise to the following coaction:

$$k[b_{i,j}]_{m,n} \quad \overset{\varrho_{(m,n)}}{\longrightarrow} \quad k[b_{i,j}]_{m,n} \otimes k[SL_{m,0}]$$
$$b_{ij} \qquad \mapsto \qquad \sum b_{ik} \otimes g_{kj}$$

An element $x \in k[b_{i,j}]_{m,n}$ is said to be coinvariant under this coaction if $\varrho_{(m,n)}(x) = x \otimes 1$. The first theorem of coinvariant theory equivalently states that the subring of coinvariants under the coaction $\varrho_{(m,n)}$, $k[b_{i,j}]_{m,n}^{k[SL_{m,0}]}$ coincides with $k[\delta_{m,n}]$.

PROPOSITION (4.2). *There is a coaction* ϱ_∞ *of* $k[SL_{\infty,0}] := \varprojlim k[SL_{m,0}]$ *on* $k[M_\infty]$.

PROOF. Fix an index m_0 and for $m, n \geq m_0$ define the map:

$$k[b_{i,j}]_{m,n} \quad \overset{\varrho_{(m,n,m_0)}}{\longrightarrow} \quad k[b_{i,j}]_{m,n} \otimes k[SL_{m_0,0}]$$
$$b_{ij} \qquad \mapsto \qquad \sum_{k=-m_0}^{n-1} b_{ik} \otimes g_{kj} + \sum_{k=-m'}^{-m_0-1} b_{ik} \otimes \delta_{kj} g_{kj}$$

where $\delta_{kj} = 1$ if $k = j$ and 0 otherwise.

For any indices $m' \geq m \geq m_0$, $n' \geq n \geq n_0$ we have the commutative diagram:

$$
\begin{array}{ccc}
k[b_{i,j}]_{m,n} & \overset{\varrho_{(m,n,m_0)}}{\longrightarrow} & k[b_{i,j}] \otimes k[SL_{m_0,0}] \\
\downarrow {\scriptstyle e_{(m',n',m,n)}} & & \downarrow {\scriptstyle e_{(m',n',m,n)} \otimes e_{(m',n',m,n)}^q} \\
k[b_{i,j}]_{m',n'} & \overset{\varrho_{(m',n',m_0)}}{\longrightarrow} & k[b_{i,j}] \otimes k_q[\Delta_{m',n'}]
\end{array}
$$

This gives us a map:

$$k[M_\infty] \overset{\varrho_{m_0}}{\to} k[M_\infty] \otimes k[SL_{m_0,0}]$$

Going to the inverse limit we obtain a map:

$$k[M_\infty] \overset{\varrho_\infty}{\to} k[M_\infty] \widehat{\otimes} k[SL_{\infty,0}]$$

which is the required coaction.   QED.

We remark that the natural inclusion $k[SL_{\infty,0}] \to k[SL_\infty]$ is not an isomorphism, however there exist non canonical isomorphisms between these two rings.

Let $k[M_\infty]^{k[SL_\infty,0]}$ denote the subring of $k[SL_{\infty,0}]$-coinvariants, that is of those elements X such that $\varrho_\infty(X) = X \otimes 1$. It is easy to see that $x = \{x_{(m,n)}\} \in k[M_\infty]$ is $k[SL_{\infty,0}]$-coinvariant if and only if each $x_{(m,n)} \in k[a_{i,j}]_{m,n}$ is $k[SL_{m,0}]$-coinvariant, i.e. if $\varrho_{m,n}(x_{m,n}) = a_{m,n} \otimes 1$.

THEOREM (4.3).   The first fundamental theorem of coinvariant theory for $SL_{\infty,0}(k)$.

$$k[M_\infty]^{k[SL_\infty,0]} = k[\widehat{\delta_\infty}]$$

PROOF. The fact that $k[\widehat{\delta_\infty}] \subset k[M_\infty]^{k[SL_\infty,0]}$ can be shown by checking directly that the generators of $k[\widehat{\delta_\infty}]$ are coinvariant, that is:

$$\varrho_\infty(\widehat{d}_{a_\bullet}) = \widehat{d}_{a_\bullet} \otimes 1 .$$

For the other inclusion, let $x \in k[M_\infty]^{k[SL_\infty,0]}$. We need to prove that x can be written as:

$$x = \sum x_i \widehat{d}_{a_\bullet^{i,1}} \dots \widehat{d}_{a_\bullet^{i,ik}}$$

This can be done using exactly the same argument as in (2.7)(b).

We now turn to examine the quantum case.

In [FH] we prove that there is a well defined coaction:

$$
\begin{array}{ccc}
k_q[b_{i,j}]_{m,n} & \overset{\varrho^q_{m,n}}{\to} & k_q[b_{i,j}]_{m,n} \otimes k_q[SL_{m,0}] \\
b_{ij} & \mapsto & \sum b_{ik} \otimes g_{kj}
\end{array}
$$

and that:

$$k_q[b_{i,j}]_{m,n}^{k_q[SL_{m,0}]} = k_q[\Delta_{m,n}]$$

THEOREM (4.4). The first fundamental theorem of quantum coinvariant theory for $k_q[SL_{\infty,0}]$.

There is a natural right coaction ϱ_∞ of $k_q[SL_{\infty,0}]$ on $k_q[M_\infty]$. Under this coaction the ring of coinvariants coincides with the quantum infinite-dimensional grassmannian $k_q\widehat{[\Delta_\infty]}$ i.e.

$$k_q[M_\infty]^{k_q[SL_{\infty,0}]} = k_q\widehat{[\Delta_\infty]}.$$

PROOF. Same as (4.2) and (4.3).

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