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Characterisation of finitely generated soluble finite-by-nilpotent groups.

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ABSTRACT - We say that a group G has finite lower central depth if the lower central series of G stabilises after a finite number of steps, that is, G has finite lower central depth if and only if $\gamma_k(G) = \gamma_{k+1}(G)$ for some positive integer k. The least integer k such that $\gamma_k(G) = \gamma_{k+1}(G)$, is called the depth of G. We denote by Ω the class of groups which has finite lower central depth. If k is a positive integer, we denote by Ω_k the class of all groups having finite lower central depth at most k. Let G be a finitely generated soluble group. In this note we prove, G is finite-by-nilpotent if and only if in every infinite set of elements of G there exist two distinct elements x, y such that $\langle x, y \rangle \in \Omega$, and G is finite by a group in wich every two generator subgroup is nilpotent of class at most k if and only if in every infinite set of elements of G there exist two distinct elements x, y such that $\langle x, y \rangle \in \Omega_k$.

1. Introduction and results.

Let $\mathcal{X}$ be a class of groups. We say that a group G has the property $(\mathcal{X}, \infty)$ if every infinite set of elements of G contains two distinct elements x and y such that $\langle x, y \rangle$ is in $\mathcal{X}$.

The idea of characterizing the groups which satisfy the property $(\mathcal{X}, \infty)$ is due to B.H. Neumann who proved in [17] that the property $(\mathcal{A}, \infty)$ is equivalent to being center-by-finite, where $\mathcal{A}$ is the class of abelian groups. This result has initiated a great deal of research. Lennox and Wiegold proved in [14] that a finitely generated soluble group satisfies the property $(\mathcal{N}, \infty)$ if and only if it is finite-by-nilpotent, where $\mathcal{N}$

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is the class of nilpotent groups. Abdollahi and Taeri proved in [3] that a finitely generated soluble group satisfies the property $(\mathcal{N}_k, \infty)$ if and only if it is a finite extension by a group in which any two generator subgroup is nilpotent of class at most k, where k is a positive integer and $\mathcal{N}_k$ is the class of nilpotent groups of class at most k. Further questions of similar nature, with different aspects, have been considered by many authors (see for example [1-5, 7, 8, 9, 13, 14, 15, 16, 17, 20, 21, 22]).

Our notation and terminology are standard, and can be found in [18]: In particular, if X is a subset of G and $x_1, x_2, \dots, x_k \in G$ where k is a positive integer, $Z(G)$ and $\langle X \rangle$, denote respectively the centre of G and the subgroup of G generated by X. The commutator $[x_1, x_2, \dots, x_k]$ is defined by the rules $[x_1, x_2] = x_1^{-1} x_2^{-1} x_1 x_2$, $[x_1, x_2, \dots, x_k] = [[x_1, x_2, \dots, x_{k-1}], x_k]$, $\gamma_k(G)$ is the k-th term of the lower central series of G andwe denote by $\mathcal{N}$ (respectively, $\mathcal{N}_k$) the class of nilpotent groups (respectively, nilpotent of class at most k), $\mathcal{N}_k^{(2)}$ is the class of all groups in which every two generator subgroup is nilpotent of class at most k . We denote by $\mathcal{F}$ the class of finite groups.

We say that a group G has finite lower central depth (or simply, finite depth) if the lower central series of G stabilises after a finite number of steps, that is, G has finite lower central depth if and only if $\gamma_k(G) = \gamma_{k+1}(G)$ for some positive integer k. The least integer k such that $\gamma_k(G) = \gamma_{k+1}(G)$, is called the depth of G. The class of groups which has finite lower central depth will be denoted by Ω. If k is a positive integer, we denote by Ω_k the class of all groups having finite lower central depth at most k. Note that:

$$(\mathcal{N}, \infty) \subseteq (\Omega, \infty)$$

and

$$(\mathcal{N}_k, \infty) \subseteq (\Omega_k, \infty) \subseteq (\Omega, \infty).$$

We prove the following results:

THEOREM 1. *Let G be a finitely generated soluble group. Then the following properties are equivalent*:

(i) *G is finite-by-nilpotent group.*

(ii) *G has the property (Ω, ∞).*

THEOREM 2. *Let G be a finitely generated soluble group. Then the following properties are equivalent*:

(i) *G is finite-by-*$\mathcal{N}_k^{(2)}$.

(ii) *G has the property* (Ω_k, ∞).

By a result of Lennox and Wiegold (Theorem A of [14]), a finitely generated soluble group, has the property $(\mathcal{N}, \infty)$, if and only if it is finite-by-nilpotent. As a consequence of Theorem1, we have the following generalisation of the latter result of Lennox and Wiegold:

COROLLARY 3. *Let* $\mathfrak{X}$ *be a class of groups such that every* $\mathfrak{X}$*-group satisfies minimal condition, and let G be a finitely generated soluble group. Then G has the property* $(\mathfrak{X}\mathcal{N}, \infty)$ *if and only if it is finite-by-nilpotent.*

And by a result of Abdollahi and Taeri [3], a finitely generated soluble group has the property $(\mathcal{N}_k, \infty)$ if and only if G is finite-by-$\mathcal{N}_k^{(2)}$ group. As a consequence of Theorem 2, we have the following generalisation of this result:

COROLLARY 4. *Let* $\mathfrak{X}$ *be a class of groups such that every* $\mathfrak{X}$*-group satisfies minimal condition, and let G be a finitely generated soluble group. Then G has the property* $(\mathfrak{X}\mathcal{N}_k, \infty)$ *if and only if it is finite-by-*$\mathcal{N}_k^{(2)}$.

2. Proofs.

LEMMA 5. *Let G be a finitely generated soluble group which has the property* (Ω, ∞)*. Suppose that G is residually nilpotent.Then G is nilpotent-by-finite.*

PROOF. Let G be a finitely generated soluble group having the property (Ω, ∞), and suppose that G is residually nilpotent. Let X be an infinite subset of G, so there exist two distinct elements x, y in X such that $\langle x, y\rangle$ has finite depth. Hence there exists a positive integer c such that $\gamma_c(\langle x, y\rangle) = \gamma_{c+1}(\langle x, y\rangle)$. Since $\langle x, y\rangle$ is residually nilpotent, we have $\bigcap_{k\in\mathbb{N}^*}\gamma_k(\langle x, y\rangle) = 1 = \gamma_c(\langle x, y\rangle)$. This implies that $\langle x, y\rangle$ is nilpotent, and therefore G has the property $(\mathcal{N}, \infty)$. Now Theorem.A of[14] yields that G is finite-by-nilpotent group. Therefore G is nilpotent-by-finite.

LEMMA 6. *Let G be a finitely generated soluble group which has the property* (Ω, ∞)*. Then G is nilpotent-by-finite.*

PROOF. Let G be a finitely generated soluble group, and suppose that every infinite subset of G contains distinct elements x and y such that $\langle x, y\rangle$ has finite depth. We argue by induction on the derived length d of G. Clearly the hypothesis is inherited by homomorphic images of G, so $\frac{G}{G^{(d-1)}}$ is nilpotent-by-finite. Then G is abelian-by-(nilpotent-by-finite). So G has a finitely generated abelian-by-nilpotent subgroup K. It follows at once from a result of Segal [19], that K has a residually nilpotent normal subgroup of finite index. Thus G has a residually nilpotent normal subgroup N, say, of finite index. By Lemma 5, N is a normal nilpotent-by-finite subgroup of finite index in G. This implies that G is a nilpotent-by- finite group.

OBSERVATION. Let A be a torsion-free abelian normal subgroup of a group G, and let x be an element of finite order in G. Then for every $a \in A$:

$$[a, x, x] = 1 \Rightarrow [a, x] = 1 .$$

In particular $[a,_n x] \neq 1$ for every $a \in A \backslash C_A(x)$ and $n \in \mathbb{N}$.

LEMMA 7. *Let A be a torsion-free abelian normal subgroup of a group G. Suppose that there exists $x \in G\backslash C_G(A)$ such that $\frac{\langle x\rangle C_G(A)}{C_G(A)}$ is of order p, p a prime. Then for every $a \in A\backslash C_A(x)$: $F = \langle x, x^a\rangle \notin \Omega$.*

PROOF. Assume that F is a conterexample, so $\gamma_c(F) = \gamma_{c+1}(F)$ for some positive integer c. Set $A_0 = \langle x^p\rangle(F \cap A)$, and let T be the torsion subgroup of A_0. Since A_0 is finitely generated, T is finite and $T \cap A = 1$.

On the other hand, $\frac{F}{T}$ satisfies the hypothesis of Gruenberg's result, so $\frac{F}{T}$ is residually nilpotent. Hence $\gamma_c(F) \leq T$. But one also has $\gamma_c(F) \leq A$ and thus $\gamma_c(F) = 1$.

This shows that F and thus $F\langle a\rangle$ is nilpotent . Then $[a,_n x] = 1$ for some $n \in \mathbb{N}$, by the precedent Observation we have $[a, x] = 1$, which contradicts the choice of $a \in A\backslash C_A(x)$.

COROLLARY 8. *Let G be a group which has the property (Ω, ∞), and let A be a torsion-free abelian normal subgroup of G. Then $\frac{G}{C_G(A)}$ is torsion-free.*

PROOF. Assume that $\frac{G}{C_G(A)}$ is not torsion-free. Then there exists $x \in G \setminus C_G(A)$ such that $\frac{\langle x \rangle C_G(A)}{C_G(A)}$ is of order p, p a prime.

According to the above Lemma 7, applied to $H = \langle x \rangle A$, $x^H = x^A$ is finite since also H has the property (Ω, ∞). Hence $|A : C_A(x)|$ is finite. As $C_A(x) \leq Z(H)$ we conclude that $\frac{H}{Z(H)}$ is finite. But then by a well-known result also H' is finite.

On the other hand $H' \cap A = 1$ since A is torsion-free. It follows that $H' = 1$ since $\frac{H}{A}$ is cyclic. But then $x \in C_G(A)$, a contradiction.

LEMMA 9. *Let G be a finitely generated group which has the property* (Ω, ∞). *If G is nilpotent-by-finite then it is finite-by-nilpotent.*

PROOF. Let G be a finitely generated group having the property (Ω, ∞), and suppose that G is nilpotent-by-finite. Let N be a maximal nilpotent normal subgroup of G (of finite index), so N contains every nilpotent normal subgroup of G; in particular $Z(G) \leq N$. Then the proof proceeds by induction on the nilpotency class of N. Denote by T the torsion-subgroup of N, so T is a finite normal subgroup of G. By passing to the quotient $\frac{G}{T}$ we may assume that N is torsion-free. This assumption implies that the order of every finite normal subgroup of G is bounded by $\left|\frac{G}{N}\right|$, so there exists a unique maximal one. Hence, after factoring out this maximal finite normal subgroup, one can also assume that 1 is the only finite normal subgroup of G.

For $A = Z(N)$ Corollary 8 implies that $\frac{G}{C_G(A)}$ is torsion-free. On the other hand $\frac{G}{C_G(A)}$ is finite since $N \leq C_G(A)$, so $G = C_G(A)$ and $A = Z(G)$. Hence, to finish the proof it suffices to show that $\frac{G}{A}$ is nilpotent.

By induction there exists $H \leq G$ containing A such that $\frac{H}{A}$ is a finite normal subgroup of $\frac{G}{A}$ and $\frac{G}{H}$ is nilpotent. As A is central and of finite index in H, one gets that H' is finite and thus by the above assumption H is abelian. It follows that $H \leq N$; in particular H is torsion-free.

Let

$$C = C_G\left(\frac{H}{A}\right) = \langle x \in G \backslash [H, x] \leqslant A \rangle$$

and note that $\frac{G}{C}$ is finite. As for $y \in H$ and $c \in C$

$$[y^m, c] = [y, c]^m$$

and $\frac{H}{A}$ is finite while A is torsion-free, one gets $C = C_G(H)$. But now $\frac{G}{C} = \frac{G}{C_G(H)}$ is finite and, again by Corollary 8, $G = C_G(H)$, so $H = A = Z(G)$. Hence $\frac{G}{Z(G)}$ is nilpotent and thus also G.

If G is finite-by-nilpotent, it is nilpotent-by-finite and the proof completes.

Proof of Theorem 1. Let G be a finitely generated soluble group. If G is finite-by-nilpotent, then G hasa finite subgroup N such that $\frac{G}{N}$ is nilpotent, so there exists a positive integer c such that $\gamma_c(G)$ is contained in N. Thus

$$N \geqslant \gamma_c(G) \geqslant \gamma_{c+1}(G) \geqslant \dots$$

But N is finite, therefore there exist $k \geqslant c$, such that $\gamma_k(G) = \gamma_{k+1}(G)$, so G is in Ω, and G has the property (Ω, ∞).

Now suppose that every infinite subset of G contains distinct elements x and y such that $\langle x, y \rangle$ has finite depth. By Lemma 5 and Lemma 9, G is finite-by-nilpotent group.

Proof of Corollary 3. Suppose that in every infinite subset of G there exist distinct elements x, y such that $\langle x, y \rangle$ is $\mathcal{X}$-by-nilpotent. Thus $\langle x, y \rangle$ has a normal subgroup N such that N is in $\mathcal{X}$ and $\frac{\langle x, y \rangle}{N}$ is nilpotent. Thus there exists a positive integer c such that $\gamma_c(\langle x, y \rangle)$ is contained in N. Therefore

$$N \geqslant \gamma_c(\langle x, y \rangle) \geqslant \gamma_{c+1}(\langle x, y \rangle) \geqslant \dots$$

But N satisfies minimal condition, therefore there exists a positive integer k, $k \geqslant c$, such that $\gamma_k(\langle x, y \rangle) = \gamma_{k+1}(\langle x, y \rangle)$. Then $\langle x, y \rangle$ has finite depth, and G has the property (Ω, ∞). Therefore, by Theorem 1, G is finite-by-nilpotent.

PROOF OF THEOREM 2. Let G be a finitely generated soluble group. If G is finite-by-$\mathcal{N}_k^{(2)}$, by the Theorem in [3], G has the property $(\mathcal{N}_k, \infty)$, so it has the properety (Ω_k, ∞).

Suppose now that in every infinite subset of G there exist distinct elements x, y such that $\langle x, y\rangle$ is in Ω_k. Then, by Theorem 1, G is a finite-by-nilpotent group. Let N be a finite subgroup of G such that $\frac{G}{N}$ is nilpotent, and X be an infinite subset of $\frac{G}{N}$. Then there exist distinct elements xN, yN of X such that $\langle x, y\rangle$ is in Ω_k. Hence $\langle xN, yN\rangle$ is in $\Omega_k \cap \mathcal{N}$, since $\Omega_k \cap \mathcal{N} = N_k$, therefore $\frac{G}{N}$ is in $(\mathcal{N}_k^{(2)}, \infty)$, by the Theorem in [3], $\frac{G}{N}$ is finite-by-$\mathcal{N}_k^{(2)}$. Therefore G is finite-by-(finite-by-$\mathcal{N}_k^{(2)}$), and so G is finite-by- $\mathcal{N}_k^{(2)}$.

PROOF OF COROLLARY 4. Suppose that in every infinite subset of G there exist distinct elements x, y such that $\langle x, y\rangle$ is $\mathcal{X}$-by-$\mathcal{N}_k$, so $\langle x, y\rangle$ has a normal subgroup N such that N is in $\mathcal{X}$ and $\frac{\langle x, y\rangle}{N}$ is in $\mathcal{N}_k$. But, by Corollary 3, G is a finitely generated finite-by-nilpotent group, so G satisfies maximal condition. Therefore N is a finitely generated soluble group which satisfies minimal condition, so N is finite and G has the property $(\mathcal{F}\mathcal{N}_k, \infty)$, by the Theorem of [5], G is finite-by-$\mathcal{N}_k^{(2)}$.

We can apply Corollary 3 and Corollary 4 in case where $\mathcal{X}$ is the class of finite groups or the class of Černikov groups.

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