REND. SEM. MAT. UNIV. PADOVA, Vol. 110 (2003)

Remarks on a Nonlocal Problem Involving the Dirichlet Energy.

M. CHIPOT (*) - V. VALENTE (**) - G. VERGARA CAFFARELLI (***)

ABSTRACT - We introduce a class of nonlinear parabolic problems of nonlocal type depending on the Dirichlet integral and study the questions of existence, uniqueness and asymptotic behaviour for the solution.

1. Introduction.

Let Ω be a bounded, smooth open subset of $\mathbb{R}^n$, $n \geqslant 1$. For

$$f = f(x) \in L^2(\Omega)\,, \tag{1.1}$$

$$u_0 \in H_0^1(\Omega)\,, \tag{1.2}$$

we would like to consider the problem of finding $u = u(x,t)$ solution to

$$\begin{cases} u_t - a\left(\displaystyle\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f & \text{in } \Omega\times\mathbb{R}^+\,, \\ u(x,0) = u_0(x) \ \text{ in } \Omega\,, \quad u = 0 & \text{on } \partial\Omega\times\mathbb{R}^+\,. \end{cases} \tag{1.3}$$

Here $a = a(s)$ is a continuous function such that

$$0 < m \leqslant a(s) \leqslant M\,. \tag{1.4}$$

(*) Indirizzo dell'A.: Institute of mathematics, Winterthurerstr. 190, CH-8057 Zürich, Switzerland.
(**) Indirizzo dell'A.: Istituto per le Applicazioni del Calcolo, C.N.R., Viale del Policlinico, 137, I-00161 Roma, Italy.
(***) Indirizzo dell'A.: Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Università di Roma «La Sapienza», via A. Scarpa 16, I-00161 Roma, Italy.

for some positive constants m and M and $\partial\Omega$ denotes the boundary of Ω.

Nonlinear nonlocal problems present several interesting features, see for instance [9], [6], [7], [8], [2], [3], [4], [10], [16] for various issues and applications. In particular several equilibria (up to a continuum) can appear. This makes the study of the asymptotic behaviour of these problems interesting and challenging. In addition to the usual properties of nonlocal problems, as we will see, (1.3) admits a Lyapunov function. This will allow us to describe some of its asymptotic behaviour.

The paper is divided as follow. In the next section we prove existence and uniqueness of a solution to (1.3). In section 3 we study the corresponding stationary problem. Finally in the last section we give some results regarding the asymptotic behaviour of (1.3).

2. Existence and uniqueness.

We will need the following Lemma.

LEMMA 2.1. *Let T be a positive number and suppose that a_n is a uniformly bounded sequence of functions such that*

$$a^n(t) \to a^\infty(t) \quad \text{a.e.} \quad t \in (0, T). \tag{2.1}$$

Suppose also that

$$u_0^n \to u_0^\infty, \qquad f^n \to f^\infty \quad in \ L^2(\Omega). \tag{2.2}$$

If u^n is the solution of the problem

$$\begin{gathered} u^n \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega)), \ u_t^n \in L^2(0,T;H^{-1}(\Omega)), \\ \frac{d}{dt}(u^n, v) + a^n(t) \int_\Omega \nabla u^n \nabla v(x)\, dx = (f^n, v), \ \text{in } \mathcal{D}'(0,T) \ \forall v \in H_0^1(\Omega), \\ u^n(\cdot, 0) = u_0^n, \end{gathered} \tag{2.3}$$

then

$$u^n \to u^\infty \quad in \quad L^2(0, T; H_0^1(\Omega)) \tag{2.4}$$

where u^{∞} *is the solution to*

$$u^{\infty} \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega)),\ u_t^{\infty} \in L^2(0,T;H^{-1}(\Omega)),$$

$$\frac{d}{dt}(u^{\infty},v)+a^{\infty}(t)\int_{\Omega}\nabla u^{\infty}\nabla v(x)\,dx=(f^{\infty},v),\ in\ \mathcal{D}'(0,T)\ \forall v\in H_0^1(\Omega), \tag{2.5}$$

$$u^{\infty}(\cdot,0)=u_0^{\infty}.$$

PROOF. The existence and uniqueness of a solution to (2.3), (2.5) follows by a well known result of J. L. Lions (see [11], [5]). We have denoted by (,) the usual scalar product in $L^2(\Omega)$. We refer to [11], [5] for notation. By difference from (2.3), (2.5) we get

$$\frac{d}{dt}(u^n-u^{\infty},\ v)+a^n(t)\int_{\Omega}\nabla(u^n-u^{\infty})\ \nabla v(x)\ dx=(f^n-f^{\infty},\ v)+$$

$$+(a^{\infty}(t)-a^n(t))\int_{\Omega}\nabla u^{\infty}\ \nabla v(x)\ dx,\quad \text{in }\ \mathcal{D}'(0,\ T)\ \ \forall v\in H_0^1(\Omega).$$

Taking $v=(u^n-u^{\infty})$ it comes by the Poincaré inequality for some constant c

$$\frac{1}{2}\frac{d}{dt}|u^n-u^{\infty}|_2^2+a^n\,||\nabla(u^n-u^{\infty})||_2^2=$$

$$=(f^n-f^{\infty},\ u^n-u^{\infty})+(a^{\infty}-a^n)\int_{\Omega}\nabla u^{\infty}\ \nabla(u^n-u^{\infty})(x)\ dx\leqslant$$

$$\leqslant c|f^n-f^{\infty}|_2\,||\nabla(u^n-u^{\infty})||_2+|a^n-a^{\infty}|\,||\nabla u^{\infty}|_2\,||\nabla(u^n-u^{\infty})||_2$$

($|\,|_2$ denotes the usual $L^2(\Omega)$-norm, $|\,|$ the Euclidean norm). Using the Young inequality we get

$$\frac{1}{2}\frac{d}{dt}|u^n-u^{\infty}|_2^2+m||\nabla(u^n-u^{\infty})||_2^2\leqslant$$

$$\leqslant\varepsilon||\nabla(u^n-u^{\infty})||_2^2+C(\varepsilon)\{|f^n-f^{\infty}|_2^2+|a^n-a^{\infty}|^2\,||\nabla u^{\infty}||_2^2\}$$

for some constant $C(\varepsilon)$. Choosing $\varepsilon=\frac{m}{2}$ this leads to

$$\frac{d}{dt}|u^n-u^{\infty}|_2^2+m||\nabla(u^n-u^{\infty})|\,|_2^2\leqslant$$

$$\leqslant C\{|f^n-f^{\infty}|_2^2+|a^n-a^{\infty}|^2\,||\nabla u^{\infty}||_2^2\}.$$

Integrating between 0, T we derive easily

$$m\int_0^T ||\nabla(u^n - u^\infty)||_2^2\, dt \leqslant$$

$$\leqslant |u_0^n - u_0^\infty|_2^2 + CT|f^n - f^\infty|_2^2 + C\int_0^T |a^n - a^\infty|^2\, ||\nabla u^\infty||_2^2\, dt.$$

The result follows by the Lebesgue theorem (to pass to the limit in the last integral). ■

THEOREM 2.1. *For any $T > 0$ there exists a weak solution to* (1.3), *i.e. there exists u such that*

$$u \in L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega)),\ u_t \in L^2(0,T; H^{-1}(\Omega)),$$

$$\frac{d}{dt}(u,v) + a\left(\int_\Omega |\nabla u|^2\, dx\right)\int_\Omega \nabla u\, \nabla v(x)\, dx = (f,v),\ \text{in } \mathcal{D}'(0,T)\ \ \forall v \in H_0^1(\Omega), \tag{2.6}$$

$$u(\cdot, 0) = u_0 .$$

PROOF. Let us set

$$B = \{v \in L^2(0,\, T;\, H_0^1(\Omega));\ \ |w|_{L^2(0,\, T;\, H_0^1(\Omega))} \leqslant C\}$$

where C is a constant that we will fix later on. Let w be fixed in B. There exists a unique u solution to

$$u \in L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega)),\ u_t \in L^2(0,T; H^{-1}(\Omega)),$$

$$\frac{d}{dt}(u,v) + a\left(\int_\Omega |\nabla w|^2\, dx\right)\int_\Omega \nabla u\, \nabla v(x)\, dx = (f,v),\ \text{in } \mathcal{D}'(0,T)\ \ \forall v \in H_0^1(\Omega), \tag{2.7}$$

$$u(\cdot, 0) = u_0.$$

We want show that the mapping

$$w \to u = R(w)$$

admits a fixed point. For that, taking $v = u$ in (2.7), we get

$$\frac{1}{2}\,\frac{d}{dt}\,|u|_2^2 + a(||\nabla w||_2^2)\,||\nabla u||_2^2 = (f,\, u) \leqslant |f|_2\, |u|_2 \leqslant c|f|_2\, ||\nabla u||_2$$

by the Poincaré inequality for some constant c. Using (1.4) we get

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + m\|\nabla u\|_2^2 \leqslant \frac{c^2}{2m}|f|_2^2 + \frac{m}{2}\|\nabla u\|_2^2$$

and hence

$$\frac{d}{dt}|u|_2^2 + m\|\nabla u\|_2^2 \leqslant \frac{c^2}{m}|f|_2^2.$$

Integrating between 0 and T we have

$$|u|_2^2 + m\int_0^T \|\nabla u\|_2^2\, dt \leqslant T\frac{c^2}{m}|f|_2^2 + |u_0|_2^2 \Rightarrow \tag{2.8}$$

$$\Rightarrow |u|^2_{L^2(0,\,T;\,H_0^1(\Omega))} \leqslant T\frac{c^2}{m^2}|f|_2^2 + \frac{1}{m}|u_0|_2^2 \doteq C^2.$$

We have defined C by the above inequality. Note that we choose also

$$|u|_{L^2(0,\,T;\,H_0^1(\Omega))} = \left\{\int_0^T \|\nabla u\|_2^2\, dt\right\}^{1/2}.$$

With C defined in (2.8) we see that $w \to u = R(w)$ maps B into B. We claim now that R is continuous from B into B. Indeed, consider a sequence $w^n \in B$ such that

$$w^n \to w^\infty \quad \text{in } B$$

Denote by u^n the solution to (2.7) where w is replaced by w^n and by u^∞ the solution to (2.7) where w is replaced by w^∞. We have, for a subsequence,

$$\|\nabla(w^{n_k} - w^\infty)\|_2^2 \to 0 \quad \text{a.e.} \quad t \in (0, T)$$

which implies

$$\|\nabla w^{n_k}\|_2^2 \to \|\nabla w^\infty\|_2^2 \quad \text{a.e.} \quad t \in (0, T)$$

and

$$a(\|\nabla w^{n_k}\|_2^2) \to a(\|\nabla w^\infty\|_2^2) \quad \text{a.e.} \quad t \in (0, T).$$

Applying the Lemma 2.1 we conclude that

$$u^{n_k} \to u^{\infty} \quad \text{in } L^2(0, T; H_0^1(\Omega))$$

and the continuity of the map R follows since u^{∞} is the only possible limit of u^n. In order to apply the Schauder fixed point theorem we now just need to prove that $R(B)$ is relatively compact in B. For that we select $a^n \in C^{\infty}([0, T])$, $f^n \in \mathcal{D}(\Omega)$, $u_0^n \in \mathcal{D}(\Omega)$ such that

$$\frac{m}{2} \leqslant a^n \leqslant 2M, \quad a^n \to a(\|\nabla w)\|_2^2) \text{ a.e. } t$$

$$f^n \to f \text{ in } L^2(\Omega), \quad u_0^n \to u_0 \text{ in } H_0^1(\Omega)$$

and we denote by u^n the solution to (2.3). u^n is very smooth and in particular since $u^n(x, t)|_{\partial\Omega} = 0$ for any t, we have

$$u_t^n \in H_0^1(\Omega)\ \forall t\,.$$

This implies also, by the second equation of (2.3) that it holds

$$\Delta u^n \in H_0^1(\Omega) \quad \forall t.$$

Thus taking $v = \Delta u^n$ in (2.3) we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla u^n\|_2^2 + a_n|\Delta u^n|_2^2 = (f^n, \Delta u^n) \leqslant |f^n|_2|\Delta u^n|_2 \Rightarrow$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|\nabla u^n\|_2^2 + \frac{m}{2}|\Delta u^n|_2^2 \leqslant \frac{|f^n|_2^2}{m} + \frac{m}{4}|\Delta u^n|_2^2.$$

It follows easily by integration that

$$\frac{1}{2}\|\nabla u^n\|_2^2 + \frac{m}{4}\int_0^T |\Delta u^n|_2^2\,dt \leqslant T\frac{|f^n|_2^2}{m} + \frac{1}{2}\|\nabla u_0^n\|_2^2 \leqslant C$$

where C is independent of n. It follows that

$$\int_0^T |\Delta u^n|_2^2\,dt \leqslant C$$

for some other constant independent of n and w. Using the Lemma we obtain

$$\int_0^T |\Delta u|_2^2 \, dt \leqslant C \tag{2.9}$$

From (1.3) we derive also

$$|u_t|_2^2 \leqslant C(|\Delta u|_2^2 + |f|_2^2),$$

and

$$|u_t|_{L^2(0, T; L^2(\Omega))} \leqslant C . \tag{2.10}$$

The estimates (2.9)-(2.10) insure that $R(B)$ is relatively compact in B (see [15]) and this concludes the proof. ■

To establish uniqueness, we suppose now that a is a Lipschitz continuous function that is to say for some K it holds that

$$|a(\zeta) - a(\zeta')| \leqslant K|\zeta - \zeta'| \quad \forall \zeta, \zeta' \in \mathbb{R} . \tag{2.11}$$

THEOREM 2.2. *Under the assumptions of Theorem* 2.1 *and if* (2.11) *holds there is a unique weak solution to the problem* (1.3) *i.e. to* (2.6).

PROOF. If u is solution to (2.6) we set

$$\alpha(t) = \int_0^t a\left(\int_\Omega |\nabla u(x, s)|^2 \, dx \right) ds = \int_0^t a(\|\nabla u(., s)\|_2^2) \, ds . \tag{2.12}$$

It is clear that α is a one-to-one mapping from $(0, +\infty)$ into itself. Set

$$w(x, \alpha(t)) = u(x, t). \tag{2.13}$$

Due to (1.4), both α and α^{-1} are Lipschitz continuous so that

$$w(x, t) = u(x, \alpha^{-1}(t)) \in H^1(0, T; H_0^1(\Omega), H^{-1}(\Omega)). \tag{2.14}$$

(See [11], [5] for a definition of this space). If u is solution to (2.6) and

$\varphi \in \mathcal{D}(0, T)$, $v \in H_0^1(\Omega)$, we have

$$(2.15) \qquad -\int_0^T \int_\Omega u(x,t)\, v(x)\, \varphi'(t)\, dx\, dt + \int_0^T a(t) \int_\Omega \nabla u(x,t)\, \nabla v(x)\, \varphi(t)\, dx\, dt = \\ = \int_0^T \int_\Omega f(x)\, v(x)\, \varphi(t)\, dx\, dt,$$

where we have set

$$(2.16) \qquad a(t) = a(\|\nabla u(.\,, t)\|_2^2).$$

This equality holds true for any $\varphi \in H^1(0, T)$. If $\varphi \in \mathcal{D}(0, \alpha(T))$, then $\varphi(\alpha(t)) \in H^1(0, T)$ and using this function in (2.14) we obtain

$$(2.17) \qquad -\int_0^T \int_\Omega \{w(x, \alpha(t))\, v(x) \varphi'(\alpha(t)) + \\ + \nabla w(x, \alpha(t))\, \nabla v(x)\, \varphi(\alpha(t))\}\, a(t)\, dx\, dt = \int_0^T \int_\Omega \frac{f(x)\, v(x)\, \varphi(\alpha(t))\, a(t)}{a(\|\nabla w(.\,, \alpha(t))\|_2^2)}\, dx\, dt.$$

Then, we use the following lemma of change of variables in the integrals:

LEMMA 2.2. *Let*

$$(2.18) \qquad f \in L^1_{\mathrm{loc}}(\mathbb{R}), \quad a \in L^\infty(\mathbb{R}), \quad 0 < m \leqslant a \leqslant M, \quad \alpha(t) = \int_0^t a(s)\, ds,$$

then it holds that

$$(2.19) \qquad \int_0^T f(\alpha(t))\, a(t)\, dt = \int_0^{\alpha(T)} f(t)\, dt.$$

Assuming this Lemma proved we get for $\varphi \in \mathcal{D}(0, \alpha(T))$, $v \in H_0^1(\Omega)$:

$$(2.20) \qquad -\int_0^{\alpha(T)} \int_\Omega \{w(x, t)\, v(x)\, \varphi'(t) + \nabla w(x, t)\, \nabla v(x)\, \varphi(t)\}\, dx\, dt = \\ = \int_0^{\alpha(T)} \int_\Omega \frac{f(x)\, v(x)\, \varphi(t)}{a(\|\nabla w(.\,, t)\|_2^2)}\, dx\, dt,$$

i.e $w(x, t)$ is solution to

$$w \in L^2(0, \alpha(T); H_0^1(\Omega)) \cap C([0, \alpha(T)]; L^2(\Omega)), \ w_t \in L^2(0, \alpha(T); H^{-1}(\Omega)),$$

$$\frac{d}{dt}(w, v) + \int_\Omega \nabla w \, \nabla v(x)\, dx = \frac{(f, v)}{a(\|\nabla w(., t)\|_2^2)}, \tag{2.21}$$

$$\text{in } \mathcal{D}'(0, \alpha(T)) \ \forall v \in H_0^1(\Omega), \quad w(\cdot, 0) = u_0.$$

Consider then u_1 and u_2 two solutions of (2.6). Then w_1 and w_2 defined by

$$w_i(x, \alpha(t)) = u_i(x, t), \quad i = 1, 2, \tag{2.22}$$

are solution to (2.21). Taking $v = w_1 - w_2$ in (2.21) written for w_1 and w_2 we get easily by subtraction

$$\frac{1}{2}\frac{d}{dt}|w_1 - w_2|_2^2 + \|\nabla(w_1 - w_2)\|_2^2 =$$

$$= (f, w_1 - w_2)\left\{\frac{1}{a(\|\nabla w_1\|_2^2)} - \frac{1}{a(\|\nabla w_2\|_2^2)}\right\}.$$

Note that $\alpha_i(T)$ are not necessarily the same, but they can be taken arbitrarily like T. Recalling (1.4) we get

$$\frac{1}{2}\frac{d}{dt}|w_1 - w_2|_2^2 + \|\nabla(w_1 - w_2)\|_2^2 \leqslant$$

$$\leqslant |f|_2 \, |w_1 - w_2|_2 \left\{\frac{a(\|\nabla w_2\|_2^2) - a(\|\nabla w_1\|_2^2)}{m^2}\right\}.$$

From (2.11) we deduce

$$|a(\|\nabla w_2\|_2^2) - a(\|\nabla w_1\|_2^2)| \leqslant K\left|\int_\Omega \nabla(w_1 - w_2)\,\nabla(w_1 + w_2)\, dx\right| \leqslant$$

$$\leqslant K\|\nabla(w_1 - w_2)\|_2 \, \||\nabla w_1| + |\nabla w_2|\|_2.$$

Thus we obtain

$$\frac{1}{2}\frac{d}{dt}|w_1 - w_2|_2^2 + \|\nabla(w_1 - w_2)\|_2^2 \leqslant$$

$$\leqslant \frac{|f|_2}{m^2}|w_1 - w_2|_2 K\|\nabla(w_1 - w_2)\|_2 \, \||\nabla w_1| + |\nabla w_2|\|_2.$$

Using the Young inequality we get

$$\frac{1}{2}\frac{d}{dt}|w_1-w_2|_2^2+\|\nabla(w_1-w_2)\|_2^2\leq$$

$$\leq\frac{1}{2}\|\nabla(w_1-w_2)\|_2^2+\frac{|f|_2^2}{2m^4}K^2\,\||\nabla w_1|+|\nabla w_2|\|_2^2\,|w_1-w_2|_2^2.$$

It follows that

$$\frac{d}{dt}|w_1-w_2|_2^2\leq C(t)\,|w_1-w_2|_2^2$$

where $C(t)\in L^1(0,T)$. From the Gronwall inequality we obtain that $w_1=w_2$ and the uniqueness of u follows. ■

Let us now turn to the proof of the Lemma which is more or less classical but that we establish for the reader convenience.

PROOF OF THE LEMMA 2.2. Let $f\in C(\mathbb{R})$, $a_n\in C(\mathbb{R})$ such that

$$\frac{m}{2}\leq a_n\leq 2M \qquad a_n\to a \quad \text{a.e. on } (0,T).$$

One has clearly with $\alpha_n=\int_0^t a_n(s)\,ds$

$$\int_0^T f(\alpha_n(t))\,a_n(t)\,dt=\int_0^{\alpha_n(T)} f(t)\,dt\,. \tag{2.23}$$

Passing to the limit in n we see that (2.19) holds for $f\in C(\mathbb{R})$. Let $f_n\in C(\mathbb{R})$ such that $f_n\to f$ in $L^1_{\text{loc}}(\mathbb{R})$. We have

$$\int_0^T f_n(\alpha(t))\,a(t)\,dt=\int_0^{\alpha(T)} f_n(t)\,dt\,. \tag{2.24}$$

We claim that $f_n(\alpha(t))\,a(t)$ is a Cauchy sequence in $L^1(0,T)$. This follows indeed from

$$\int_0^T |f_n(\alpha(t))\,a(t)-f_m(\alpha(t))a(t)|\,dt=\int_0^{\alpha(T)} |f_n-f_m|\,ds\leq\varepsilon$$

for n, m large enough. Thus, there exists a function $g \in L^1(0, T)$ such that

$$f_n(\alpha(t))\, a(t) \to g(t)$$

in $L^1(0, T)$. Up to a subsequence one can assume that

$$f_n(s) \to f(s) \quad \text{on} \quad [0, \alpha(T)] \setminus N$$

where N is a set of measure 0. Since α^{-1} is Lipschitz continuous, $\alpha^{-1}(N)$ is also of measure 0 and

$$f_n(\alpha(t)) \to f(\alpha(t)) \quad \text{on} \quad [0, T] \setminus \alpha^{-1}(N).$$

Thus $g(t) = f(\alpha(t))\, a(t)$ $a.\ e.$ and passing to the limit in (2.24) the proof of the lemma is complete. ■

3. Steady states.

In this section we study the stationary solutions to (1.3). These steady states can be obtained as minimizers of some energy or as critical points. Let $E(u)$ be the energy defined by

$$E(u) = \frac{1}{2} A\left(\int_\Omega |\nabla u|^2\, dx \right) - (f, u), \tag{3.1}$$

where

$$A(s) = \int_0^s a(\zeta)\, d\zeta. \tag{3.2}$$

We have:

THEOREM 3.1. *The functional $E(u)$ admits a global minimizer in $H_0^1(\Omega)$.*

PROOF. First, we see that E is coercive and bounded from below. Indeed, from the Poincaré inequality we have

$$|(f, u)| \leqslant |f|_2\, |u|_2 \leqslant c|f|_2\, \|\nabla u\|_2$$

and hence

$$E(u) = \frac{1}{2} A\left(\int_\Omega |\nabla u|^2\, dx \right) - (f,\, u) \geqslant \frac{m}{2}\, ||\nabla u||_2^2 - c|f|_2\, ||\nabla u||_2. \tag{3.3}$$

This shows the coerciveness of E. E is also bounded from below since

$$\frac{m}{2}\, ||\nabla u||_2^2 - c|f|_2\, ||\nabla u||_2 \geqslant -\frac{c^2}{2m}\, |f|_2^2. \tag{3.4}$$

Let $u_n \in H_0^1(\Omega)$ be a minimizing sequence of E. Due to (3.3) u_n is bounded in $H_0^1(\Omega)$ and we can assume that for some $u_\infty \in H_0^1(\Omega)$ we have

$$u_n \rightharpoonup u_\infty \text{ in } H_0^1(\Omega), \qquad u_n \to u_\infty \text{ in } L^2(\Omega).$$

Due to the weak lower semicontinuity of the norm we deduce

$$\underline{\lim}\, ||\nabla u_n||_2^2 \geqslant ||\nabla u_\infty||_2^2.$$

Consider a subsequence u_{n_k} such that

$$\underline{\lim}\, ||\nabla u_n||_2^2 = \lim_k ||\nabla u_{n_k}||_2^2.$$

Since u_{n_k} is a minimizing sequence we have

$$\inf_{H_0^1(\Omega)} E(u) = \lim_k E(u_{n_k}) = \frac{1}{2} \int_0^{\underline{\lim}\, ||\nabla u_n||_2^2} a(s)\, ds - (f,\, u_\infty) \geqslant$$

$$\geqslant \frac{1}{2} \int_0^{||\nabla u_\infty||_2^2} a(s)\, ds - (f,\, u_\infty) = E(u_\infty).$$

Thus u_∞ is a minimizer of E on $H_0^1(\Omega)$. Note that for any sequence u_n satisfying

$$u_n \rightharpoonup u_\infty \text{ in } H_0^1(\Omega),$$

we have shown that it holds that

$$\underline{\lim}\, E(u_n) = \frac{1}{2} \int_0^{\underline{\lim}\, ||\nabla u_n||_2^2} a(s)\, ds - (f,\, u_\infty) \geqslant E(u_\infty),$$

i.e. E is weakly lower semicontinuous on $H_0^1(\Omega)$. This completes the proof of the theorem. ■

As we will see below E might have several minimizers. First note that if u_∞ is a (local) minimizer of E on $H_0^1(\Omega)$ then u_∞ is a solution of

$$\begin{cases} -a\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Indeed if u_∞ is a (local) minimizer it holds that (Euler equation)

$$\frac{d}{d\lambda}E(u_\infty + \lambda v)\,|_{\lambda=0} = 0 \qquad \forall v \in H_0^1(\Omega).$$

This leads to

$$\frac{d}{d\lambda}\left\{\frac{1}{2}\int_0^{\|\nabla(u_\infty+\lambda v)\|_2^2} a(s)\,ds - (f,\, u_\infty + \lambda v)\right\}\Bigg|_{\lambda=0} = 0,$$

$$\Rightarrow \quad \frac{1}{2}a(\|\nabla u_\infty\|_2^2)\,2\int_\Omega \nabla u_\infty\,\nabla v(x)\,dx - (f,\, v) = 0, \qquad \forall v \in H_0^1(\Omega),$$

i.e. u_∞ is a solution to (3.5) and a stationary point.

Regarding stationary points we have:

THEOREM 3.2. *The mapping* $u \to l(u) = \int_\Omega |\nabla u|^2\,dx$ *is a one-to-one mapping from the set of weak solution to* (3.5) *onto the set of solutions to*

$$a(\mu)^2\mu = l(\varphi) = \int_\Omega |\nabla \varphi|^2\,dx \tag{3.6}$$

where φ *is the weak solution to*

$$\begin{cases} -\Delta\varphi = f & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

PROOF. Let u be a solution of (3.5), we have

$$a(l(u))\,u = \varphi \;\Rightarrow\; l(a(l(u))\,u) = l(\varphi) \;\Rightarrow\; a^2(l(u))\,l(u) = l(\varphi)$$

and $l(u)$ is a solution to (3.6). This shows that l goes from the set of solutions to (3.5) into the set of solutions to (3.6). Let now μ be a solution to (3.6), denoting by u the solution to

$$-a(\mu)\,\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

one has

$$a(\mu)u = \varphi \Rightarrow l(a(\mu)\,u) = l(\varphi)$$

$$\Rightarrow a(\mu)^2\, l(u) = l(\varphi) = a(\mu)^2 \mu \Rightarrow l(u) = \mu$$

and u is a solution to (3.5). Now if we have $l(u_1) = l(u_2)$ then $a(l(u_1)) = a(l(u_2))$ and $u_1 = u_2$. This completes the proof of the theorem. ■

REMARK 3.1. The stationary points are determined (through (3.6)) by the solutions to

$$a(\mu) = \sqrt{\frac{l(\varphi)}{\mu}} \tag{3.8}$$

Of course we can select a so that this equation admits $1, 2, \dots a$ continuum of solutions. The critical points of E are not all global minimizers. Indeed we have:

THEOREM 3.3 (comparison of energies). *Let u_1, u_2 be two solutions of* (3.5) *corresponding to the solutions μ_1, μ_2 of* (3.8). *Assume that*

$$a(\mu) > \sqrt{\frac{l(\varphi)}{\mu}} \quad (\text{resp.} \quad a(\mu) < \sqrt{\frac{l(\varphi)}{\mu}}\,,\ a(\mu) = \sqrt{\frac{l(\varphi)}{\mu}} \quad \forall \mu \in (\mu_1, \mu_2) \tag{3.9}$$

then it holds that

$$E(u_2) > E(u_1) \ (\text{resp.} \quad E(u_2) < E(u_1),\ E(u_2) = E(u_1))$$

PROOF. Recall that u_i is solution to

$$-a(\mu_i)\Delta u_i = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

i.e. we have

$$\mu_i = \|\nabla u_i\|_2^2, \qquad u_i = \frac{\varphi}{a(\mu_i)}\,.$$

Thus it comes

$$E(u_i) = \frac{1}{2}\int_0^{\mu_i} a(s)\,ds - (f,\,u_i) =$$

$$= \frac{1}{2}\int_0^{\mu_i} a(s)\,ds - \left(f,\,\frac{\varphi}{a(\mu_i)}\right) = \frac{1}{2}\int_0^{\mu_i} a(s)\,ds - \frac{l(\varphi)}{a(\mu_i)}\,,$$

since $(f,\,\varphi) = (-\Delta\varphi,\,\varphi) = l(\varphi)$. By subtraction we obtain in the first case above

$$E(u_2) - E(u_1) = \frac{1}{2}\int_{\mu_1}^{\mu_2} a(s)\,ds + \frac{l(\varphi)}{a(\mu_1)} - \frac{l(\varphi)}{a(\mu_2)} = \frac{1}{2}\int_{\mu_1}^{\mu_2} a(s)\,ds +$$

$$+\sqrt{l(\varphi)\,\mu_1} - \sqrt{l(\varphi)\,\mu_2} > \frac{1}{2}\int_{\mu_1}^{\mu_2}\sqrt{\frac{l(\varphi)}{s}}\,ds + \sqrt{l(\varphi)\,\mu_1} - \sqrt{l(\varphi)\,\mu_2} =$$

$$= \sqrt{l(\varphi)\,\mu_2} - \sqrt{l(\varphi)\,\mu_1} + \sqrt{l(\varphi)\,\mu_1} - \sqrt{l(\varphi)\,\mu_2} = 0\,.$$

The other cases could be treated the same way. This completes the proof of the theorem. ■

REMARK 3.2. The functional $E(u)$ is not convex and might have as shown by the above theorem several global minimizers.

4. Asymptotic behaviour.

Let us denote by $H^1(0,\,T;\,L^2(\Omega))$ the space

(4.1) $\quad H^1(0,\,T;\,L^2(\Omega)) = \{v \in L^2(0,\,T;\,L^2(\Omega));\ v_t \in L^2(0,\,T;\,L^2(\Omega))\}.$

LEMMA 4.1. *For* $u \in H^1(0,\,T;\,L^2(\Omega)) \cap C((0,\,T);\,H_0^1(\Omega) \cap H^2(\Omega))$ *it holds that*

(4.2) $$\int_s^t \int_\Omega \{-a(\|\nabla u(.\,,\,t)\|_2^2)\,\Delta u - f\}\,u_t\,dx\,dt = E(u(t)) - E(u(s)).$$

PROOF. By density (see [11], [5]) we can assume that $u \in C^1((0, T); L^2(\Omega))$ and then that $u \in C^1((0, T); H_0^1(\Omega) \cap H^2(\Omega))$ (see [1] p. 74 for a similar argument). Then for such a u we have

$$\int_\Omega \{-a(\|\nabla u(., t)\|_2^2)\Delta u - f\} u_t = \int_\Omega a(\|\nabla u(., t)\|_2^2)\nabla u \nabla u_t - f u_t \, dx =$$

$$= \frac{d}{dt}\frac{1}{2}\int_\Omega A(\|\nabla u(x, t)\|_2^2)\, dx - \int_\Omega f u dx = \frac{d}{dt}E(u(t))$$

where E is defined by (3.1). The result follows by integrating between s and t. ■

LEMMA 4.2. *Let $u(\cdot, t)$ be the unique solution to the problem* (1.3) *or* (2.6) *then when $t \to \infty$ there exists a subsequence t_k such that*

$$u(\cdot, t_k) \rightharpoonup u_\infty \qquad in \ H_0^1(\Omega)$$

where u_∞ is a stationary point.

PROOF. Using the equation (1.3) and the Lemma 4.1 we have (recall also (2.10))

$$-\int_s^t \int_\Omega u_t^2 \, dx \, dt = E(u(t)) - E(u(s)). \tag{4.3}$$

It follows that $E(u(t))$ is nonincreasing and there exists E_∞ such that

$$\lim_{t \to \infty} E(u(t)) = E_\infty. \tag{4.4}$$

Passing to the limit in (4.3) we have

$$\int_0^\infty |u_t|_2^2 \, dt = E(u(s)) - E_\infty, \tag{4.5}$$

and from (3.4)

$$\|\nabla u(., t)\|_2^2 = \int_\Omega |\nabla u(x, t)|^2 dx \leqslant \text{const.} = C_1 \qquad \forall t \geqslant s. \tag{4.6}$$

From (4.5) we have

$$\underline{\lim} \ |u_t|_2^2 = 0 \qquad \text{as } t \to \infty$$

so there is a subsequence which we denote by t_k such that

$$|u_t(\cdot, t_k)|_2^2 \to 0 \quad \text{as } t_k \to \infty . \tag{4.7}$$

We also have from (4.6) that

$$\|\nabla u(\cdot, t_k)\|_2^2 \leq C_1$$

hence passing eventually to a subsequence if $u_k = u(\cdot, t_k)$ we have

$$u_k \rightharpoonup u_\infty \qquad \text{weakly in } H_0^1(\Omega),$$

$$u_k \to u_\infty \qquad \text{strongly in } L^2(\Omega).$$

Now, we want to prove that u_∞ is a stationary point i.e. is a solution to (3.5). Since E is nonincreasing

$$E(u(\cdot, t_k)) = \frac{1}{2} A\left(\int_\Omega |\nabla u_k|^2\, dx \right) - (f, u_k) \to E_\infty$$

$$\Rightarrow \quad \frac{1}{2} A\left(\int_\Omega |\nabla u_k|^2\, dx \right) \to E_\infty + (f, u_\infty)$$

and recalling that A is invertible

$$\int_\Omega |\nabla u_k|^2\, dx \to l_\infty = A^{-1}(2E_\infty + 2(f, u_\infty)). \tag{4.8}$$

Taking $v = u$ and φ solution of (3.7) in (2.6) we get

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + a(\|\nabla u\|_2^2)\|\nabla u\|_2^2 = (f, u),$$

$$\frac{d}{dt}(u, \varphi) + a(\|\nabla u\|_2^2)(f, u) = (f, \varphi) = (-\Delta\varphi, \varphi) = l(\varphi).$$

From (4.8) we deduce that

$$\frac{1}{2}\frac{d}{dt}|u(t_k)|_2^2 = (u_t(\cdot, t_k), u_k) =$$

$$= -a(\|\nabla u_k\|_2^2)|\nabla u_k|^2 + (f, u_k) \to (f, u_\infty) - a(l_\infty)\, l_\infty,$$

$$(u_t(\cdot, t_k), \varphi) = l(\varphi) - a(\|\nabla u_k\|_2^2)(f, u_k) \to l(\varphi) - a(l_\infty)(f, u_\infty).$$

Since u is bounded in $L^2(\Omega)$ and (4.7) holds we get

$$(f, u_\infty) = a(l_\infty) l_\infty, \qquad l(\varphi) = a(l_\infty)(f, u_\infty)$$

$$\Rightarrow \quad l(\varphi) = a^2(l_\infty) l_\infty$$

i.e. l_∞ is a solution to (3.6). For any v belonging to $H_0^1(\Omega)$ it holds that

$$(u_t(\cdot, t_k), v) + a\left(\int_\Omega |\nabla u_k|^2 dx\right)(\nabla u_k, \nabla v) = (f, v).$$

From (4.7), (4.8) passing to the limit we get

$$a(l_\infty)(\nabla u_\infty, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

and we obtain that u_∞ is a stationary point – see Theorem 3.2. ■

REMARK 4.1. The convergence is in fact strong in $H_0^1(\Omega)$ since

$$\int_\Omega |\nabla u_k|^2 dx \to l_\infty = \int_\Omega |\nabla u_\infty|^2 dx \qquad k \to \infty,$$

see (4.8).

We can then show:

THEOREM 4.1. *Let u be the unique solution to the problem* (1.3) *then, if E admits a unique stationary point u_∞, it holds that*

$$u(\cdot, t) \to u_\infty \qquad \text{in } H_0^1(\Omega)$$

when $t \to \infty$.

PROOF. Since $u(t)$ is uniformly bounded in $H_0^1(\Omega)$ for some subsequence it holds that

$$u(\cdot, t_k) \rightharpoonup v_\infty \qquad \text{in } H_0^1(\Omega) - \text{weak}.$$

By Lemma 4.2 – see also Remark 4.1 – we have

$$E(u(t)) \to E(u_\infty)$$

where u_∞ is the global minimizer of E. By the weak lower semicontinuity of E (see Theorem 3.1) we have

$$E(u_\infty) = \lim_{t_k \to \infty} E(u(t_k)) \geqslant E(v_\infty).$$

Since v_∞ is a minimizer we have necessarily $v_\infty = u_\infty$. Since this holds for any subsequence the result follows – see also the Remark 4.1.

REMARK 4.2. If E admits a unique global minimizer u_∞ and if

$$E(u_0) < E(u_i)$$

for any stationary point $u_i \neq u_\infty$, then with the same proof as above we have

$$u(\cdot, t) \to u_\infty \quad \text{in } H_0^1(\Omega)$$

when $t \to \infty$.

We would like to end this section by a result of asymptotic stability. Let c be the constant defined by the Poincaré inequality

$$\|\nabla u\|_2 \leqslant c|\Delta u|_2, \qquad \forall u \in H_0^1(\Omega) \cap H^2(\Omega).$$

We introduce the function $\mathcal{A}(s) = a(s^2)$ which, from (1.4), verifies

$$0 < m \leqslant \mathcal{A}(s) \leqslant M.$$

Then we have

THEOREM 4.2. *Assume $\mathcal{A}$ is a C^1 function and u_∞ is a stationary solution to the problem* (3.5) *which verifies*

$$m - |\mathcal{A}'(\|\nabla u_\infty\|_2)| \frac{c}{m} |f|_2 > \mu > 0, \tag{4.9}$$

then u_∞ is a locally asymptotically stable stationary solution in the sense of Lyapunov. That is to say, there exits $\varrho = \varrho(\mu)$ such that if $\|\nabla(u_0 - u_\infty)\|_2 \leqslant \varrho$ the solution $u(t)$ of the problem (1.3) *verifies*

$$\|\nabla(u(t) - u_\infty)\|_2 \leqslant \|\nabla(u_0 - u_\infty)\|_2 e^{-\frac{\mu}{c^2}t} \qquad \forall t \geqslant 0.$$

PROOF. Let u_∞ be the solution of the stationary problem that verifies (4.9). From (1.3), (3.5), (see also (2.9)-(2.10)), we have:

$$(u - u_\infty)_t - a\left(\int_\Omega |\nabla u|^2 dx\right) \Delta(u - u_\infty) = \left[a\left(\int_\Omega |\nabla u|^2 dx\right) - a\left(\int_\Omega |\nabla u_\infty|^2 dx\right)\right] \Delta u_\infty. \tag{4.10}$$

Putting $h = u - u_\infty$ and multiplying the above equation by Δh we get

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla h|^2\,dx - a\left(\int_\Omega |\nabla u|^2\,dx\right)\int_\Omega |\Delta h|^2\,dx =$$

$$= \left[a\left(\int_\Omega |\nabla u|^2\,dx\right) - a\left(\int_\Omega |\nabla u_\infty|^2\,dx\right)\right]\int_\Omega \Delta u_\infty\,\Delta h\,dx.$$

Applying the mean value theorem we deduce

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla h|^2\,dx + a\left(\int_\Omega |\nabla u|^2\,dx\right)\int_\Omega |\Delta h|^2\,dx \leqslant \tag{4.11}$$

$$\leqslant \left|\,\mathcal{a}'(\|\nabla u_\infty\|_2 + \theta(\|\nabla u\|_2 - \|\nabla u_\infty\|_2))\right|\,\big|\,\|\nabla u\|_2 - \|\nabla u_\infty\|_2\,\big|\cdot$$

$$\cdot\left|\int_\Omega \Delta u_\infty\,\Delta h\,dx\right| \leqslant \left|\,\mathcal{a}'(\|\nabla u_\infty\|_2 + \theta(\|\nabla u\|_2 - \|\nabla u_\infty\|_2))\right|\,\|\nabla h\|_2\cdot$$

$$\cdot\left|\int_\Omega \Delta u_\infty \Delta h\,dx\right| \leqslant c\left|\,\mathcal{a}'(\|\nabla u_\infty\|_2 + \theta(\|\nabla u\|_2 - \|\nabla u_\infty\|_2))\right|\,|\Delta u_\infty|_2 |\Delta h|_2^2.$$

We have thus (see (3.5))

$$\frac{1}{2}\frac{d}{dt}\|\nabla h\|_2^2 + \tag{4.12}$$

$$+\left(m - \left|\,\mathcal{a}'(\|\nabla u_\infty\|_2 + \theta(\|\nabla h + \nabla u_\infty\|_2 - \|\nabla u_\infty\|_2))\right|\frac{c}{m}|f|_2\right)|\Delta h|_2^2 \leqslant 0.$$

Now, if $\|\nabla h\|_2 \leqslant \varrho$ it holds that

$$m - \left|\,\mathcal{a}'(\|\nabla u_\infty\|_2 + \theta(\|\nabla h + \nabla u_\infty\|_2 - \|\nabla u_\infty\|_2))\right|\frac{c}{m}|f|_2 > \mu > 0$$

and we have

$$\frac{d}{dt}\left(\|\nabla h\|_2^2\, e^{2\frac{\mu}{c^2}t}\right) \leqslant 0\,. \tag{4.13}$$

This completes the proof. ∎

REMARK 4.3. If $\mathcal{a}$ is a Lipschitz continuous function with Lipschitz constant K and

$$m - K\frac{c}{m}\,|f|_2 > \mu > 0 \tag{4.14}$$

then the stationary problem has a unique solution and hence for this solution we have global asymptotic stability. Indeed, if u^1_∞ and u^2_∞ are two solutions of the problem (3.5), repeating the steps of the proof of Theorem 4.2 we get (see (4.12))

$$\left(m - K\frac{c}{m}\,|f|_2\right)\,|\Delta(u^1_\infty - u^2_\infty)\,|^2_2 \leqslant 0 \tag{4.15}$$

which leads to $u^1_\infty = u^2_\infty$.

Acknowledgements. The first author would like to acknowledge the support of the Swiss National Science foundation under the contract # 20-67618.02. The work of the second author has been supported by the European Community Program «Improving the Human Research Potential and Socio-Economic Knowledge Base» under the contract HPRN-CT-2002-00284. The work of the third author has been supported by Progetto Cofinanziato 2002 «Modelli Matematici per la Scienza dei Materiali».

REFERENCES

[1] T. CAZENAVE - A. HARAUX, *An Introduction to Semilinear Evolution Equations*, Oxford Science Publications (1998).

[2] N.-H. CHANG - M. CHIPOT, *On some mixed boundary value problems with nonlocal diffusion.* (To appear in Adv. in Math. Sci. and Appl.).

[3] N.-H. CHANG - M. CHIPOT, *On some model diffusion problems with a nonlocal lower order term.* (To appear in the Chinese Ann. of Math.).

[4] N.-H. CHANG - M. CHIPOT, *Nonlinear nonlocal evolution problems.* (To appear in RACSAM).

[5] M. CHIPOT, *Elements of Nonlinear Analysis*, Birkhäuser Advanced Text (2000).

[6] M. CHIPOT - B. LOVAT, *Existence and uniqueness results for a class of nonlocal elliptic and parabolic problems*, DCDIS, Series A, 8 (2001), # 1, pp. 35-51.

[7] M. CHIPOT - B. LOVAT, *On the asymptotic behaviour of some nonlocal problems*, Positivity, 3 (1999), pp. 65-81.

[8] M. Chipot, L. Molinet: *Asymptotic behavior of some nonlocal diffusion problems.* Applicable Analysis, 80 (2001) p. 273-315.
[9] M. Chipot - J. F. Rodrigues, *On a class of nonlocal nonlinear elliptic problems*, Math. Mod. and Num. Anal., **26** (1992), pp. 447-468.
[10] M. Chipot - M. Siegwart, *On the asymptotic behaviour of some nonlocal mixed boundary value problems.* To appear.
[11] R. Dautray - J. L. Lions, *Mathematical analysis and numerical methods for science and technology*, Springer Verlag (1990).
[12] D. Gilbarg - N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer Verlag, New York (1983).
[13] A. Haraux, *Systèmes Dynamiques Dissipatifs et Applications*, Masson Paris (1991).
[14] D. Kinderlehrer - G. Stampacchia, *An introduction to variational inequalities and their applications*, Acad. Press (1980).
[15] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod Gauthiers-Villards, Paris (1969).
[16] M. Siegwart, *Thesis*, University of Zürich (to appear).

Manoscritto pervenuto in redazione il 28 marzo 2003.